

SECOND EDITION

# THE MULTIBODY SYSTEMS APPROACH TO VEHICLE DYNAMICS



MIKE BLUNDELL AND DAMIAN HARTY

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# The Multibody Systems Approach to Vehicle Dynamics

Second Edition

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# Preface

This book, the second edition, is intended to bridge a gap between the subject of classical vehicle dynamics and the general-purpose computer-based discipline multibody systems (MBS) analysis. Whilst there are several textbooks that focus entirely on the subject, and mathematical foundations, of vehicle dynamics and other more recent texts dealing with MBS there are none yet that link the two subjects in a comprehensive manner.

After 10 years a second edition of this book is indeed timely. Since the first edition there have been notable developments in the understanding and use of active systems, tyre modelling and the use of MBS software.

MBS analysis became established as a tool for engineering designers during the 1980s in a similar manner to the growth in finite element analysis technology during the previous decade. A number of computer programs were developed and marketed to the engineering industry, such as MSC ADAMS™ (*Automatic Dynamic Analysis of Mechanical Systems*), which in this edition still forms the basis for many of the examples provided. During the 1990s MBS became firmly established as part of the vehicle design and development process. It is inevitable that the engineer working on problems involving vehicle ride and handling in a modern automotive environment will be required to interface with the use of MBS to simulate vehicle motion. During the last 10 years several other MBS programmes have become more established, most notably SIMPACK which appropriately receives more coverage in this edition.

The book is aimed at a wide audience including not only undergraduate, postgraduate and research students working in this area, but also practising engineers in industry requiring a reference text dealing with the major relevant areas within the discipline.

The book was originally planned as an individual effort on the part of Mike Blundell drawing on past experience consulting on and researching into the application of MBS to solve a class of problems in the area of vehicle dynamics. From the start it was clear that a major challenge in preparing a book on this subject would be to provide meaningful comment on not only the modelling techniques but also the vast range of simulation outputs and responses that can be generated. Deciding whether a vehicle has good or bad handling characteristics is often a matter of human judgement based on the response or feel of the vehicle, or how easy the vehicle is to drive through certain manoeuvres. To a large extent automotive manufacturers still rely on track measurements and the instincts of experienced test engineers as to whether the design has produced a vehicle with the required handling qualities. To address this problem the book has been co-authored by Damian Harty. At the time of writing the first edition Damian was the Chief Engineer — Dynamics at Prodrive. In the 10 years since the first edition he continued in that role and after a few years working as a Senior Research fellow at Coventry University he moved to his current position with Polaris where he enjoys

the additional challenge of modelling vehicles on wide ranging terrain. With experience not only in the area of computer simulation but also the in the practical development and testing of vehicles on the proving ground Damian continues to help in documenting the realistic application of MBS in vehicle development.

Chapter 1 is intended to document the emergence of MBS and provide an overview of its role in vehicle design and development. Previous work by contributors including Olley, Segel, Milliken, Crolla and Sharp is identified providing a historical perspective on the subject during the latter part of the twentieth century.

Chapter 2 is included for completeness and covers the underlying formulations in kinematics and dynamics required for a good understanding of MBS formulations. A three-dimensional vector approach is used to develop the theory, this being the most suitable method for developing the rigid body equations of motion and constraint formulations described later.

Chapter 3 covers the modelling, analysis and postprocessing capabilities of a typical simulation software. There are many commercial programs to choose from including not only MSC ADAMS but also other software packages such as DADS and SIMPACK. The descriptions provided in Chapter 3 are based in the main on MSC ADAMS; the main reason for this choice being that the two authors have between them 25 years of experience working with the software. The fact that the software is also well established in automotive companies and academic institutions worldwide is also a factor. It is not intended in Chapter 3 to provide an MSC ADAMS primer. There is extensive user documentation and training material available in this area from the program vendors MSC Software. The information included in Chapter 3 is therefore limited to that needed to introduce a new reader to the subject and to provide a supporting reference for the vehicle modelling and analysis methodologies described in the following chapters. As discussed, the emergence of SIMPACK and its growing use by the automotive community has led to additional examples to illustrate the modelling approaches with that software.

Existing users of MSC ADAMS will note that the modelling examples provided in Chapter 3 are based on a text-based format of model inputs, known in MSC ADAMS as solver data sets. This was the original method used to develop MSC ADAMS models and has subsequently been replaced by a powerful graphical user interface (GUI) known as ADAMS/View™ that allows model parameterisation, and design optimisation studies. The ADAMS/View environment is also the basis for customised versions of MSC ADAMS such as ADAMS/Car™ that are becoming established in industry and are also discussed in Chapter 3. The use of text-based data sets has been adopted here for a number of reasons. The first of these is that the GUI of a modern simulation program such as MSC ADAMS is subject to extensive and ongoing development. Any attempt to describe such a facility in a textbook such as this would become outdated after a short passage of time. As mentioned, the software developers provide their own user documentation covering this in any case. It is also clear that the text-based formulations translate more readily to book format and are also useful for demonstrating the underlying techniques in planning a model, preparing model schematics and establishing the degrees of freedom in a system

model. These techniques are needed to interpret the models and data sets that are described in later chapters and appendices. It is also hoped that by treating the software at this fundamental level the dependence of the book on any one software package is reduced and that the methods and principles will be adaptable for practitioners using alternative software. Examples of the later ADAMS/View command file format are included in Chapters 6 and 8 for completeness.

Chapter 4 addresses the modelling and analysis of the suspension system. An attempt has been made to bridge the gap between the textbook treatment of suspension systems and the MBS approach to building and simulating suspension models. As such a number of case studies have been included to demonstrate the application of the models and their use in the vehicle design process. The chapter concludes with an extensive case study comparing a full set of analytical calculations, using the vector-based methods introduced in Chapter 2, with the output produced from MSC ADAMS. It is intended that this exercise will demonstrate to readers the underlying computations in process when running an MBS simulation.

Chapter 5 addresses the tyre force and moment generating characteristics and the subsequent modelling of these in an MBS simulation. As a major area of importance it deserves to be the largest chapter in the book. Examples are provided of tyre test data and the derived parameters for established tyre models. The chapter concludes with a case study using an MBS virtual tyre test machine to interrogate and compare tyre models and data sets. Since the first edition new tyre models such as the FTire model from Gipser and the TAME Tire model from Michelin have become established and therefore receive a more extended coverage in this edition.

Chapter 6 describes the modelling and assembly of the rest of the vehicle, including the anti-roll bars and steering systems. Near the beginning a range of simplified suspension modelling strategies for the full vehicle is described. This forms the basis for subsequent discussion involving the representation of the road springs and steering system in simple models that do not include a model of the suspension linkages. The chapter includes a consideration of modelling driver inputs to the steering system using several control methodologies and concludes with a case study comparing the performance of several full vehicle modelling strategies for a vehicle handling manoeuvre.

Chapter 7 deals with the simulation output and interpretation of results. An overview of vehicle dynamics for travel on a curved path is included. The classical treatment of understeer/oversteer based on steady state cornering is presented followed by an alternative treatment that considers yaw rate and lateral acceleration gains. The subjective/objective problem is discussed with consideration of steering feel and roll angle as subjective modifiers. The chapter concludes with a consideration of the use of analytical models with a signal-to-noise approach.

Chapter 8 concludes with a review of the use of active systems to modify the dynamics in modern passenger cars. The use of electronic control in systems such as active suspension and variable damping, brake-based systems, active steering systems, active camber systems and active torque distribution is described. A final summary matches the application of these systems with driving styles described as normal, spirited or the execution of emergency manoeuvres.

Appendix A contains a full set of vehicle model schematics and a complete set of vehicle data that can be used to build suspension models and full vehicle models of varying complexity. The data provided in Appendix A were used for many of the case studies presented throughout the book.

Appendix B contains example Fortran Tire subroutines to supplement the description of the tyre modelling process given in Chapter 5. A subroutine is included that uses a general interpolation approach using a cubic spline fit through measured tyre test data. The second subroutine is based on Version 3 of the Magic Formula and has an embedded set of tyre parameters based on the tyre data described in Chapter 5. A final subroutine ‘The Harty Model’ was developed by Damian at Prodrive and is provided for readers who would like to experiment with a new tyre model that uses a reduced set of model parameters and can represent combined slip in the tyre contact patch.

In conclusion it seems to the authors there still remain two camps for addressing the vehicle dynamics problem. In one is the practical ride and handling expert. The second camp contains theoretical vehicle dynamics experts. This book is aimed at the reader who, like the authors, seeks to live between the two camps and move forward the process of vehicle design, taking full advantage of the widespread availability of convenient digital computing.

There is, however, an enormous difficulty in achieving this end. Lewis Carroll, in *Alice through the Looking Glass*, describes an encounter between Alice and a certain Mr H Dumpty:

*‘When I use a word’, Humpty Dumpty said, in rather a scornful tone, ‘it means just what I choose it to mean—neither more nor less’.*

*‘The question is’, said Alice, ‘whether you can make words mean so many different things’.*

There is a similar difficulty between practical and theoretical vehicle dynamicists and even between different individuals of the same persuasion. The same word is used, often without definition, to mean just what the speaker chooses. There is no universal solution to the problem save for a thoughtful and attentive style of discussion and enquiry, taking pains to establish the meanings of even apparently obvious terms such as ‘camber’ — motorcycles do not have any camber by some definitions (vehicle-body-referenced) and yet to zero the camber forces in a motorcycle tyre is clearly folly. A glossary is included in Appendix C, not as some declaration of correctness but as an illumination for the text. In this edition a new appendix has been added. Appendix D lists some of the test procedures defined by the International Standards Organisation that are used to validate the handling performance of a new vehicle.

**Mike Blundell, Damian Harty**

*April 2014*

# Acknowledgements

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## Mike Blundell

In developing my sections of this book I am indebted to my colleagues and students at Coventry University who have provided encouragement and material that I have been able to use. In particular I thank Barry Bolland and Peter Griffiths for their input to Chapter 2 and Bryan Phillips for his help with Chapter 5. I am also grateful to many within the vehicle dynamics community who have made a contribution including Roger Williams, Jim Forbes, Adrian Griffiths, Colin Lucas, John Janevic and Grahame Walter. I am especially grateful to the late David Crolla. He was an inspiration to me as my career took me into the area of vehicle dynamics and my mentor during the preparation of the first edition of this book. I will never forget him. Finally I thank the staff at Elsevier Science for their patience and help throughout the years it has taken to bring both editions of this book to print.

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## Damian Harty

Mike's gracious invitation to join him and infectious enthusiasm for both the topic and this project has kept me buoyed. At the time of the first edition I acknowledged Robin Sharp, Doug and Bill Milliken for keeping me grounded and rigorous when it is tempting just to play in cars and jump to conclusions. Bill Milliken in particular made a significant contribution to the discipline for an astonishingly long period; Bill passed away in 2012 after a fruitful and remarkable 101-year life that included driving at speed up the hill at Goodwood in 2002 and again in 2007. For those unfamiliar with the clarity of his thinking and the vivacity of a life lived to the full, his autobiography 'Equations of Motion' is an excellent read. Bill's legacy persists with his son Doug continuing to run Milliken Research Associates in Buffalo, NY, USA.

During our work on the first edition of this book the late David Crolla was an ever-present voice of reason keeping this text focused on its *raison d'être*—the useful fusion of practical and theoretical vehicle dynamics. Professional colleagues who have used banter, barracking and sometimes even rational discussion to help me progress my thinking are too numerous to mention—apart from Duncan Riding, whom I have to single out as being exceptionally encouraging. I hope I show my gratitude in person and on a regular basis to all of them and invite them to kick me if I do not. Someone who must be mentioned is Isaac Newton; his original and definitive brilliance at describing my world amazes me everyday. As Mike, I thank the staff at Elsevier Science for their saintly patience.

Finally, I would just like to say I am very sorry for all the vehicles I have damaged while 'testing' them. I really am.

# Nomenclature

$a_1, a_2, a_3$	Distances for six-mass approximation
$a, b$	Distance from CM to front and rear axles, respectively
$a_{11} \dots a_{22}$	Elements of a matrix (generic)
$\{a_I\}_1$	Unit vector at marker I resolved parallel to frame 1 (GRF)
$\{a_J\}_1$	Unit vector at marker J resolved parallel to frame 1 (GRF)
$a_x$	Longitudinal acceleration (Wenzel model)
$a_y$	Lateral acceleration (Wenzel model)
$b$	Longitudinal distance of body mass centre from front axle
$c$	Damping coefficient
$c$	Longitudinal distance of body mass centre from rear axle
$c$	Specific heat capacity of brake rotor
$d$	Wire diameter
$\left(\frac{dB}{dF_z}\right)$	Variation in scaling factor with load (Harty Model)
$\{d_{IJ}\}_1$	Position vector of marker I relative to J resolved parallel to frame 1 (GRF)
$e_1$	Path error
$f$	Natural frequency (Hz)
$g$	Gravitational acceleration
$h$	Brake rotor convection coefficient
$h$	Height of body mass centre above roll axis
$i$	Square root of $-1$
$k$	Path curvature
$k$	Radius of gyration
$k$	Stiffness
$k$	Spring constant in hysteretic model
$k$	Tyre spring constant
$k_1, k_2$	Front and rear ride rates, respectively
$k_s$	Spring stiffness
$k_w$	Stiffness of equivalent spring at the wheel centre
$l$	Length of pendulum
$m$	Mass of a body
$m\{g\}_1$	Weight force vector for a part resolved parallel to frame 1 (GRF)
$m_t$	Mass of tyre
$n$	Number of active coils
$n$	Number of friction surfaces (pads)
$p$	Brake pressure
$q_j$	Set of part generalised coordinates
$r$	Yaw rate
$r_1, r_2, r_3$	Coupler constraint rotations
$\{r_I\}_1$	Position vector of marker I relative to frame 1 resolved parallel to frame 1 (GRF)
$\{r_J\}_1$	Position vector of marker J relative to frame 1 resolved parallel to frame 1 (GRF)



$r_u$	Unladen radius
$r_l$	Laden radius
$r_w$	Wheel radius
$s_1, s_2, s_3$	Coupler constraint scale factors
$t_f$	Front track
$t_r$	Rear track
$v_{cog}$	Centre of gravity (Wenzel model)
$v_x$	Longitudinal velocity (Wenzel model)
$v_y$	Lateral velocity (Wenzel model)
$x$	Generic variable for describing tanh function
$x_i, y_i, z_i$	Coordinates of each of the six masses in the six-mass approximation
$x_i, y_i, z_i$	Components of the $i$ th eigenvector
$x(t)$	Function of time (generic)
$x_{CM}, y_{CM}, z_{CM}$	Coordinates of body centre of mass
$\{x_I\}_1$	Unit vector along x-axis of marker I resolved parallel to frame 1 (GRF)
$\{y_I\}_1$	Unit vector along y-axis of marker I resolved parallel to frame 1 (GRF)
$\{x_J\}_1$	Unit vector along x-axis of marker J resolved parallel to frame 1 (GRF)
$\{y_J\}_1$	Unit vector along y-axis of marker J resolved parallel to frame 1 (GRF)
$y_s$	Asymptotic value at large slip (Magic Formula)
$z$	Auxiliary state variable
$z$	Heave displacement variable
$\{z_I\}_1$	Unit vector along z-axis of marker I resolved parallel to frame 1 (GRF)
$\{z_J\}_1$	Unit vector along z-axis of marker J resolved parallel to frame 1 (GRF)
$A$	Area
$A$	Linear acceleration
$A, B, C$	Intermediate terms in a cubic equation
$A$	Scaling for solution form of a differential equation (generic)
$A$	Step height
$A_c$	Convective area of brake disc
$[A_{1n}]$	Euler matrix for part n
$\{A_n\}_1$	Acceleration vector for part n resolved parallel to frame 1 (GRF)
$A^P$	Centripetal acceleration
$\{A^P_{PQ}\}_1$	Centripetal acceleration vector P relative to Q referred to frame 1 (GRF)
$\{A^t_{PQ}\}_1$	Transverse acceleration vector P relative to Q referred to frame 1 (GRF)
$\{A^c_{PQ}\}_1$	Coriolis acceleration vector P relative to Q referred to frame 1 (GRF)
$\{A^s_{PQ}\}_1$	Sliding acceleration vector P relative to Q referred to frame 1 (GRF)
$A_{vehicle}$	Acceleration of vehicle

$A_X$	Longitudinal curvature factor
$A_y$	Lateral acceleration
$AyG$	Lateral acceleration gain
$B$	Load scaling factor (Harty Model)
$B$	Stiffness factor (Magic Formula)
$[B]$	Transformation matrix from frame $O_e$ to $O_n$
$BK_{id}$	Bottom kingpin marker
$BM$	Bump movement
$B_T$	Brake torque
$C$	Shape factor (Magic Formula)
$[C]$	Compliance matrix
$C_{D0}$	Drag coefficient at zero aerodynamic yaw angle
$C_{D\beta}$	Drag coefficient sensitivity to aerodynamic yaw angle
$C_F$	Front axle cornering stiffness
$C_\gamma$	Camber coefficient
$C_{L0}$	Coefficient of lift at zero angle of attack
$C_{L\alpha}$	Variation in coefficient of lift with angle of attack
$C_{MX}$	Overturning moment coefficient
$C_r$	Rolling resistance moment coefficient
$C_R$	Rear axle cornering stiffness
$C_S$	Tyre longitudinal stiffness
$C_p$	Process capability
$CP$	Centre of pressure
$C_\alpha$	Tyre lateral stiffness due to slip angle
$C_{\alpha f}$	Front tyre lateral stiffness due to slip angle
$C_{\alpha r}$	Rear tyre lateral stiffness due to slip angle
$C_\gamma$	Tyre lateral stiffness due to camber angle
$D$	Clipped camber scale constant
$D$	Mean coil diameter
$D$	Peak value (Magic Formula)
$DZ$	Displacement variable (generic)
$DM(I,J)$	Magnitude of displacement of I marker relative to J marker
$DX(I,J)$	Displacement in X-direction of I marker relative to J marker parallel to GRF
$DY(I,J)$	Displacement in Y-direction of I marker relative to J marker parallel to GRF
$DZ(I,J)$	Displacement in Z-direction of I marker relative to J marker parallel to GRF
$E$	Camber clip curvature constant
$E$	Young's modulus of elasticity
$E$	Curvature factor (Magic Formula)
$F$	Aerodynamics force
$F$	Applied force
$F$	Force generated by hysteretic model
$F$	Spring force
$F_{hyst}$	Amplitude of hysteretic force
$F_{hyst}$	Final outcome from sequence of hysteretic calculations
$\{F_{nA}\}_1$	Applied force vector on part n resolved parallel to frame 1 (GRF)

$\{\mathbf{F}_n\mathbf{C}\}_1$	Constraint force vector on part n resolved parallel to frame 1 (GRF)
$\mathbf{F}_{\text{FRC}}$	Lateral force reacted by front roll centre
$\mathbf{F}_{\text{RRC}}$	Lateral force reacted by rear roll centre
$\mathbf{F}_x$	Frictional force
$\mathbf{F}_x$	Longitudinal tractive or braking tyre force
$\mathbf{F}_{x1}$	Friction moderated longitudinal load in moderate slip
$\mathbf{F}_{x2}$	Friction moderated longitudinal load in deep slip
$\mathbf{F}_y$	Lateral tyre force
$\mathbf{F}_{y1}$	Friction moderated lateral load at moderate slip angles
$\mathbf{F}_{y2}$	Friction moderated lateral load at deep slip angles
$\mathbf{F}'_y$	Lagged (relaxed) side force
$\mathbf{F}_{y\alpha}$	Lateral load due to slip angle
$\mathbf{F}_{y\alpha'}$	Friction moderated side force due to slip angle
$\mathbf{F}_{y\gamma}$	Lateral load due to camber/inclination angle
$\mathbf{F}_{y\gamma'}$	Friction moderated side force due to camber/inclination angle
$\frac{\mathbf{F}_y}{\mu\mathbf{F}_z}$	Lateral capacity fraction
$\mathbf{F}_z$	Normal force
$\mathbf{F}_z$	Vertical tyre force
$\mathbf{F}_z$	Time varying tyre load
$\mathbf{F}_{z0}$	Static corner load
$\mathbf{F}_{zc}$	Vertical tyre force due to damping
$\mathbf{F}_{zk}$	Vertical tyre force due to stiffness
$\{\mathbf{F}_A\}_1 \{\mathbf{F}_B\}_1 \dots$	Applied force vectors at points A, B,... resolved parallel to frame 1 (GRF)
$[\mathbf{F}_E]$	Elastic compliance matrix (concept suspension)
$\mathbf{F}_D$	Drag force
$\mathbf{F}_G$	Fixed ground marker
$\mathbf{G}$	Shear modulus
$\mathbf{G}_C$	Gravitational constant
$\mathbf{G}_O$	Ground level offset
$\mathbf{GRF}$	Ground reference frame
$\{\mathbf{H}\}_1$	Angular momentum vector for a body
$\mathbf{H}(\omega)$	Transfer function
$\mathbf{HTC}$	Half track change
$\mathbf{I}$	Mass moment of inertia
$\mathbf{I}$	Second moment of area
$\mathbf{I}_2$	Pitch inertia of vehicle
$\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$	Principal mass moments of inertia of a body
$\mathbf{I}_{\text{wheel}}$	Mass moment of inertia of road wheel in the rolling direction
$\mathbf{I}_{xx}, \mathbf{I}_{yy}, \mathbf{I}_{zz}, \mathbf{I}_{xy}, \mathbf{I}_{yz}, \mathbf{I}_{xz}$	Components of inertia tensor
$\mathbf{IC}_Y$	Y-coordinate of instant centre
$\mathbf{IC}_Z$	Z-coordinate of instant centre
$[\mathbf{I}_n]$	Inertia tensor for a part
$\mathbf{J}$	Polar second moment of area
$\mathbf{J}_z$	Vehicle body yaw inertia (Wenzel model)
$\mathbf{K}$	Drive torque controller constant
$\mathbf{K}$	Spring stiffness

<b>K</b>	Stability factor
<b>K</b>	Understeer gradient
<b>K<sub>z</sub></b>	Tyre radial stiffness
<b>K<sub>T</sub></b>	Torsional stiffness
<b>K<sub>Ts</sub></b>	Roll stiffness due to springs
<b>K<sub>Tr</sub></b>	Roll stiffness due to anti-roll bar
<b>L</b>	Contact patch length
<b>L</b>	Length
<b>L</b>	Wheelbase
<b>{L}<sub>1</sub></b>	Linear momentum vector for a particle or body
<b>L<sub>PFZ2</sub></b>	Pneumatic lead scaling factor with load squared
<b>L<sub>PFZ</sub></b>	Pneumatic lead scaling factor with load
<b>L<sub>PC</sub></b>	Pneumatic lead at reference load
<b>LPRF</b>	Local part reference frame
<b>L<sub>R</sub></b>	Tyre relaxation length
<b>M<sub>FRC</sub></b>	Moment reacted by front roll centre
<b>{M<sub>nA</sub>}<sub>e</sub></b>	Applied moment vector on part n resolved parallel to frame e
<b>{M<sub>nC</sub>}<sub>e</sub></b>	Constraint moment vector on part n resolved parallel to frame e
<b>M<sub>s</sub></b>	Equivalent roll moment due to springs
<b>M<sub>x</sub></b>	Tyre overturning moment
<b>M<sub>Xγκ</sub></b>	Overturning moment due to longitudinal forces
<b>M<sub>y</sub></b>	Moment about y-axis
<b>M<sub>y</sub></b>	Tyre rolling resistance moment
<b>M<sub>z</sub></b>	Tyre self aligning moment
<b>M<sub>zα</sub></b>	Friction moderated side force due to slip angle
<b>M<sub>zγ</sub></b>	Friction moderated side force due to camber/inclination angle
<b>M<sub>Zγκ</sub></b>	Aligning moment due to longitudinal forces
<b>MRF</b>	Marker reference frame
<b>M<sub>RRC</sub></b>	Moment reacted by rear roll centre
<b>N<sub>r</sub></b>	Vehicle yaw moment with respect to yaw rate
<b>[N<sub>t</sub>]</b>	Norsieck vector
<b>N<sub>vy</sub></b>	Vehicle yaw moment with respect to lateral velocity
<b>O<sub>1</sub></b>	Frame 1 (GRF)
<b>O<sub>e</sub></b>	Euler axis frame
<b>O<sub>i</sub></b>	Reference frame for part i
<b>O<sub>j</sub></b>	Reference frame for part j
<b>O<sub>n</sub></b>	Frame for part n
<b>O<sub>P</sub></b>	Lateral offset of contact patch
<b>P<sub>0</sub></b>	Initial tyre pressure at zero load
<b><math>\bar{P}</math></b>	Average footprint pressure
<b>{P<sub>nr</sub>}<sub>1</sub></b>	Rotational momenta vector for part n resolved parallel to frame 1 (GRF)
<b>{P<sub>nt</sub>}<sub>1</sub></b>	Translational momenta vector for part n resolved parallel to frame 1 (GRF)
<b>P<sub>t</sub></b>	Constant power acceleration
<b>P<sub>Δz</sub></b>	Change in nominal pressure
<b>P<sub>Δz</sub></b>	Pressure due to tyre vertical deflection
<b>QG</b>	Position vector of a marker relative to the GRF

<b>QP</b>	Position vector of a marker relative to the LPRF
<b>R</b>	Radius (generic)
<b>R</b>	Radius of turn
<b>R</b>	Fraction of roll moment distributed between front and rear axles
<b>R<sub>1</sub></b>	Unloaded tyre radius
<b>R<sub>2</sub></b>	Tyre carcass radius
<b>R<sub>d</sub></b>	Radius to centre of brake pad
<b>R<sub>e</sub></b>	Effective rolling radius
<b>{R<sub>i</sub>}<sub>1</sub></b>	Position vector of frame i on part i resolved parallel to frame 1 (GRF)
<b>{R<sub>j</sub>}<sub>1</sub></b>	Position vector of frame j on part j resolved parallel to frame 1 (GRF)
<b>R<sub>l</sub></b>	Loaded tyre radius
<b>{R<sub>n</sub>}<sub>1</sub></b>	Position vector for part n resolved parallel to frame 1 (GRF)
<b>{R<sub>p</sub>}<sub>1</sub></b>	Position vector of tyre contact point P relative to frame 1, referenced to frame 1
<b>R<sub>u</sub></b>	Unloaded tyre radius
<b>{R<sub>w</sub>}<sub>1</sub></b>	Position vector of wheel centre relative to frame 1, referenced to frame 1
<b>{R<sub>AG</sub>}<sub>n</sub></b>	Position vector of point A relative to mass centre G resolved parallel to frame n
<b>{R<sub>BG</sub>}<sub>n</sub></b>	Position vector of point B relative to mass centre G resolved parallel to frame n
<b>RC<sub>front</sub></b>	Front roll centre
<b>RC<sub>rear</sub></b>	Rear roll centre
<b>RCY</b>	Y-coordinate of roll centre
<b>RCZ</b>	Z-coordinate of roll centre
<b>R<sub>Z</sub></b>	Reference load (Harty Model)
<b>S</b>	Distance travelled
<b>SA</b>	Spindle axis reference point
<b>S<sub>CX</sub></b>	Critical slip ratio
<b>S<sub>e</sub></b>	Error variation
<b>Sh</b>	Horizontal shift (Magic Formula)
<b>Sv</b>	Vertical shift (Magic Formula)
<b>S<sub>L</sub></b>	Longitudinal slip ratio
<b>S<sub>L</sub><sup>*</sup></b>	Critical value of longitudinal slip
<b>SN</b>	Signal-to-noise ratio
<b>S<sub>T</sub></b>	Total variation
<b>S<sub>α</sub></b>	Lateral slip ratio
<b>S<sub>Lα</sub></b>	Comprehensive slip ratio
<b>S<sub>α</sub><sup>*</sup></b>	Critical slip angle
<b>S<sub>κ</sub></b>	Variation due to linear effect
<b>T</b>	Camber clipping threshold fraction
<b>T</b>	Kinetic energy for a part
<b>T</b>	Temperature
<b>T</b>	Torque
<b>T<sub>B</sub></b>	Brake torque
<b>T<sub>env</sub></b>	Environmental temperature

<b>TPFZ2</b>	Pneumatic trail scaling factor with load squared
<b>TPFZ</b>	Pneumatic trail scaling factor with load
<b>TPC</b>	Pneumatic trail scaling constant
<b>T<sub>S</sub></b>	Spin up torque
<b>T<sub>0</sub></b>	Initial brake rotor temperature
<b>{T<sub>A</sub>}<sub>1</sub> {T<sub>B</sub>}<sub>1</sub>...</b>	Applied torque vectors at points A, B,... resolved parallel to frame 1 (GRF)
<b>TK</b>	Top kingpin marker
<b>TR</b>	Suspension trail
<b>{U<sub>r</sub>}</b>	Unit vector normal to road surface at tyre contact point
<b>{U<sub>s</sub>}</b>	Unit vector acting along spin axis of tyre
<b>UCF</b>	Units consistency factor
<b>US</b>	Understeer
<b>V</b>	Forward velocity
<b>V<sub>0</sub></b>	Initial tyre volume at zero load
<b>V<sub>a</sub></b>	Actual forward velocity
<b>V<sub>e</sub></b>	Error variance
<b>V<sub>g</sub></b>	Ground plane velocity
<b>V<sub>lowlimit</sub></b>	Limiting velocity
<b>{V<sub>n</sub>}<sub>1</sub></b>	Velocity vector for part n resolved parallel to frame 1 (GRF)
<b>{V<sub>p</sub>}<sub>1</sub></b>	Velocity vector of tyre contact point P referenced to frame 1
<b>V<sub>s</sub></b>	Desired simulation velocity
<b>V<sub>x</sub></b>	Sliding velocity
<b>V<sub>xc</sub></b>	Longitudinal slip velocity of tyre contact point
<b>V<sub>y</sub></b>	Lateral slip velocity of tyre contact point
<b>V<sub>z</sub></b>	Vertical velocity of tyre contact point
<b>V<sub>ref</sub></b>	Reference velocity in hysteretic model
<b>VR(I,J)</b>	Radial line of sight velocity of I marker relative to J marker
<b>VZ</b>	Velocity variable (generic)
<b>V<sub>Δz</sub></b>	Reduced tyre cavity volume
<b>W</b>	Tyre width
<b>WB</b>	Wheelbase marker
<b>WC</b>	Wheel centre marker
<b>WF</b>	Wheel front marker
<b>WR</b>	Wheel recession
<b>XP</b>	Position vector of a point in a marker xz-plane
<b>{X<sub>sae</sub>}<sub>1</sub></b>	Unit vector acting at tyre contact point in X <sub>sae</sub> direction referenced to frame 1
<b>Y<sub>r</sub></b>	Vehicle side force with respect to yaw rate
<b>Y<sub>vy</sub></b>	Vehicle side force with respect to lateral velocity
<b>YRG</b>	Yaw rate gain
<b>{Y<sub>sae</sub>}<sub>1</sub></b>	Unit vector acting at tyre contact point in Y <sub>sae</sub> direction referenced to frame 1
<b>{Z<sub>sae</sub>}<sub>1</sub></b>	Unit vector acting at tyre contact point in Z <sub>sae</sub> direction referenced to frame 1
<b>ZP</b>	Position vector of a point on a marker z-axis
<b>α</b>	Angle of attack
<b>α</b>	Tyre slip angle

$\alpha_{CY}$	Critical slip angle (Harty Model)
$\{\alpha_n\}_1$	Angular acceleration vector for part n resolved parallel to frame 1 (GRF)
$\alpha_f$	Front axle slip angle
$\alpha_r$	Rear axle slip angle
$\beta$	Aerodynamic yaw angle (or body slip angle surrogate)
$\beta$	Side slip angle
$\dot{\beta}$	Rate of change of side slip angle (Beta Dot)
$\delta$	Steer or toe angle
$\delta_o$	Steer angle of outer wheel
$\delta_i$	Steer angle of inner wheel
$\delta_{mean}$	Average steer angle of inner and outer wheels
$\gamma$	Camber angle
$\{\gamma_n\}_e$	Set of Euler angles for part n
$\zeta$	Damping ratio
$\kappa$	Longitudinal slip (Pacjeka)
$\kappa$	Sensitivity of process
$\theta$	2nd Euler angle rotation
$\theta$	Pendulum displacement variable
$\theta$	Pitch displacement variable
$\theta_1$	Orientation of the first principal axis within a plane of symmetry
$\lambda$	Eigenvalue (generic)
$\{\lambda\}_1$	Reaction force vector resolved parallel to frame 1 (GRF)
$\lambda_d$	Magnitude of reaction force for constraint d
$\lambda_p$	Magnitude of reaction force for constraint p
$\lambda_\alpha$	Magnitude of reaction force for constraint $\alpha$
$\mu$	Friction coefficient
$\mu_o$	Tyre to road coefficient of static friction
$\mu_1$	Tyre to road coefficient of sliding friction
$\eta$	Signal-to-noise ratio
$\eta$	Hysteresis constant/loss factor
$\rho$	Density
$\sigma$	Standard deviation
$\sigma_d$	Standard deviation of attribute d
$\Phi$	3rd Euler angle rotation
$\psi$	1st Euler angle rotation
$\psi$	Compass heading angle
$\dot{\psi}$	Yaw rate (Wenzel model)
$\omega$	Angular frequency (rads s <sup>-1</sup> )
$\omega$	Yaw rate
$\omega_d$	Damped natural frequency
$\omega_d$	Demanded yaw rate
$\omega_{err}$	Yaw rate error
$\omega_{fns}$	Front axle no-slip yaw rate
$\omega_{friction}$	Yaw rate from limiting friction
$\omega_{geom}$	Yaw rate from geometry
$\omega_n$	Undamped natural frequency
$\{\omega_e\}_1$	Angular velocity vector for part n resolved parallel to frame e



$\{\omega_n\}_1$	Angular velocity vector for part n resolved parallel to frame 1 (GRF)
$\omega_0$	Angular velocity of free rolling wheel
$\omega_D$	Angular velocity of driven wheel
$\Delta_d$	Allowable range for attribute d
$\Delta x$	Change in longitudinal position of wheel (concept suspension)
$\Delta y$	Change in lateral position (half/track) of wheel (concept suspension)
$\Delta z$	Deformation of tyre
$\Delta V$	Change in tyre cavity volume
$\Delta \varepsilon$	Change in steer angle (toe in/out) of wheel (concept suspension)
$\Delta \gamma$	Change in camber angle of wheel (concept suspension)
$\{\Phi_d\}_1$	Vector constraint equation resolved parallel to frame 1 (GRF)
$\Phi_d$	Scalar constraint expression for constraint d
$\Phi_p$	Scalar constraint expression for constraint p
$\Phi_\alpha$	Scalar constraint expression for constraint $\alpha$
$\dot{\Omega}_{\text{wheel}}$	Angular acceleration of road wheel

# Introduction

# 1

*To undertake a numerical process without knowledge of the difference between good and bad numbers is folly.*

Damian Harty, 2014

## 1.1 Overview

In 1969, man travelled to the moon and back, using mathematics invented by Kepler, Newton and Einstein to calculate trajectories hundreds of thousands of miles long and spacecraft with less on-board computing power than a mobile telephone.<sup>1</sup> With today's computing power and the mathematical frameworks handed down to us by Newton and Lagrange, it is scarcely credible that the motor car, itself over 100 years old, can exercise so many minds and still show scope for improvement. Yet we are still repeating errors in the dynamic design of our vehicles that were made in the 1960s. Every car will spin – rotate excessively about a vertical axis until it is no longer pointing where it is going – if driven through an emergency lane change at highway speed without skilled correction from the driver or from computer-controlled stability systems (or both).

Legislation now demands embedded electronic control of brake systems to retain control and stability of vehicles although the driver still retains executive control in terms of choosing speed and path. However, with the exception of braking performance, road vehicle manufacturers are not currently forced by legislation to achieve a measurable standard of vehicle handling and stability. International standards exist that outline procedures for proving ground tests with new vehicles but these are nothing more than recommendations. Vehicle manufacturers make use of many of the tests but in the main will develop and test vehicles using in-company experience and knowledge to define the test programme.

In the absence of legislated standards, vehicle manufacturers are driven by market forces. Journalists report favourably on vehicles they enjoy driving – whether or

<sup>1</sup>The Apollo Guidance Computer (AGC) had 2 kB of memory, 32 kB of non-rewritable flash drive and 1 MHz clock speed. A typical smart phone at the time of writing has 1000 kB of memory, 32 million kB of rewritable flash drive and a clock speed of 1000 MHz.

not these are safe in the hands of the general public — and the legal profession seeks every opportunity to blur the distinction between bad driving and poor vehicle design. Matters are further complicated by market pressures driving vehicle designs to be too tall for their width — city cars and sport-utility vehicles have this disadvantage in common.

The growth in media attention and reporting to the public is undoubtedly significant. When the first edition was written, the most well-publicised example of this was the reported rollover of the Mercedes A Class (top left image in [Figure 1.1](#)) during testing by the Motoring Press. The test involves a slalom type manoeuvre and became popularly known as the ‘Elk Test’ or ‘Moose Test’. With the arrival of YouTube in 2005, everyone is a journalist. A short search netted three more examples of well-performed tests illustrating other vehicles in disarray. No manufacturer wants videos like these going viral.

It seems to the authors that there are two camps for addressing the vehicle dynamics problem. In the first camp are the practical ride and handling experts. Skilled at the driving task and able to project themselves into the minds of a variety of different possible purchasers of the vehicle, they are able to quickly take an established vehicle design and adjust its character to make it acceptable for the market into which it will be launched. Rarely, though, are experts from this camp called upon to work in advance on the concept or detail of the vehicle design.

The second camp contains theoretical vehicle dynamics experts. They are skilled academics in the mould of Leonard Segel who in 1956 published his ‘Theoretical prediction and experimental substantiation of the responses of the automobile to steering control’ (Segel, 1956).



**FIGURE 1.1**

Two-wheel lift abounds. (Top left — Mercedes courtesy of Auto Motor und Sport). There are two very similar tests being used, one is the ISO3888 Lane Change and the other is an ADAC avoidance test.



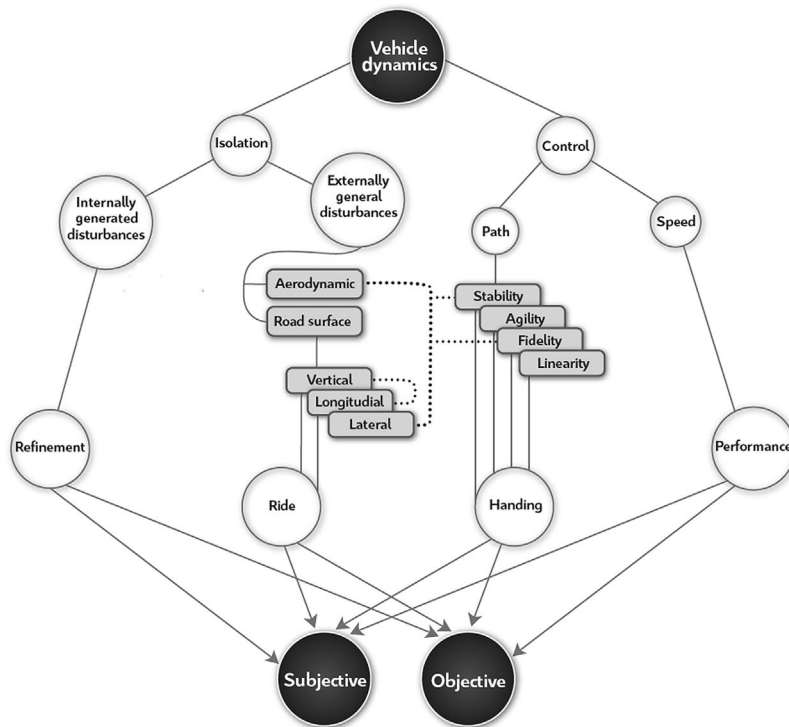
What vehicle dynamics looks like to outsiders.

These two camps, a little like England and the USA, are ‘separated by a common language’.<sup>2</sup> They use similar terms very differently (Figure 1.2) and can often have contemptuous relationships in a given organisation.

## 1.2 What is vehicle dynamics?

Isolation is about separating the driver from disturbances occurring as a result of the vehicle operation. This, too, breaks into two topics; disturbances the vehicle generates itself (engine vibration and noise, for example) and those imposed upon it by the outside world. The former category is captured by the umbrella term

<sup>2</sup>Often attributed to George Bernard Shaw, this quote cannot actually be found in his writings anywhere according to Wikipedia.

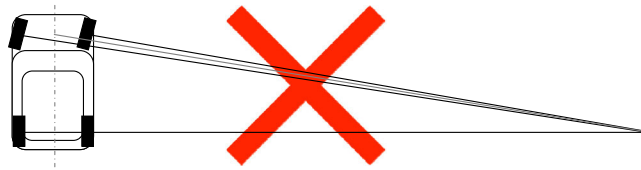
**FIGURE 1.3**

Vehicle dynamics interactions.

‘refinement’. The disturbances in the latter category are primarily road undulations and aerodynamic interaction of the vehicle with its surroundings — crosswinds, wakes of structures and wakes of other vehicles. The behaviour of the vehicle in response to road undulations is referred to as ‘ride’ and could conceivably be grouped with refinement, though it rarely is in practice.

There is some substantial crossover in aerodynamic behaviour between isolation and control, since control implies the rejection of disturbances (‘fidelity’) and an absence of their amplification (‘stability’). Similarly, one response to road disturbances is a change in the vertical load supported by the tyre; this has a strong influence on the lateral force the tyre is generating at any given instant in time and is thus crucial for both fidelity and stability. It can be seen with some little reflection that one of the difficulties of vehicle dynamics work is not the complexity of the individual effects being considered but rather the complexity of their interactions.

Control is concerned largely with the behaviour of the vehicle in response to driver demands. The driver continuously varies both path curvature and speed, subject to the limits of the vehicle capabilities, in order to follow an arbitrary course.

**FIGURE 1.4**

Geometric approximations of vehicle behaviour are incorrect.

Speed variation is governed by vehicle mass and tractive power availability at all but the lowest speed, and is easily understood. Within the performance task, issues such as unintended driveline oscillations and tractive force variation with driver demand may interact strongly with the path of the vehicle.

The adjustment of path curvature at a given speed is altogether more interesting. In a passenger car, the driver has a steering wheel, which for clarity will be referred to as a *handwheel*<sup>3</sup> throughout the book. The handwheel is a ‘yaw rate’ demand — a demand for rotational velocity of the vehicle when viewed from above. The combination of a yaw rate and a forward velocity vector that rotates with the vehicle gives rise to a curved path. There is a maximum path curvature available in normal driving, which is the turning circle, available only at the lowest speeds.

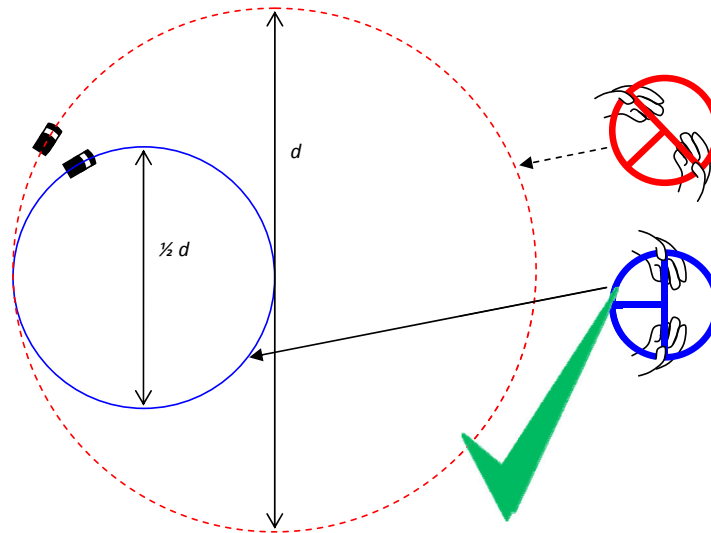
It is generically true that the vehicle does not behave in a ‘geometric’ manner and its radius of turn cannot usefully be predicted by considering the angle of the front wheels relative to the rear wheels, except below around 30 mph.

The geometric view (Figure 1.4) becomes increasingly inaccurate as speed increases and can lead to an over-estimate of vehicle responses by a factor of up to four at European highway speeds. The lower-than-geometric response of the car is a consequence of pneumatic tyres and modern vehicle engineering practice; it is not necessarily the ‘unengineered’ behaviour of all vehicle layouts.

In normal circumstances (that is to say in day-to-day road use) the driver moves the handwheel slowly and is well within the limits of the vehicle capability. The vehicle has no difficulty responding to the demanded yaw rate. If the driver increases yaw rate demand slightly then the vehicle will increase its yaw rate by a proportional amount (Figure 1.5). This property is referred to as ‘linearity’; the vehicle is described as ‘linear’. For the driver, the behaviour of the vehicle is quite instinctive. A discussion of the analysis and interpretation of vehicle non-geometric behaviour, linearity and departure from linearity is given in Chapter 7.

In the linear region, the behaviour of the vehicle can be represented as a connected series of ‘steady state’ events. Steady state is the condition in which, if the handwheel remains stationary then all the vehicle states — speed, yaw rate, path

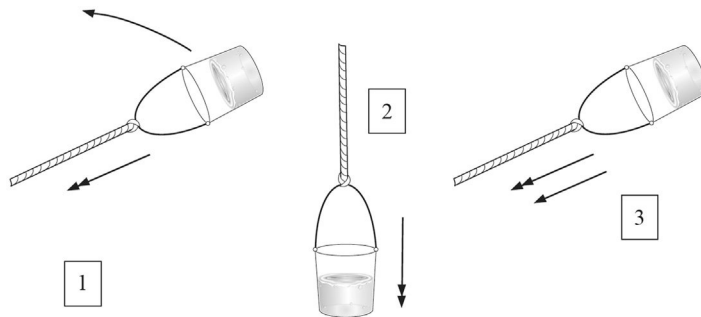
<sup>3</sup>‘Steering Wheel’ could mean a roadwheel that is steered, or a wheel held by the driver. In generic discussions including vehicles other than four wheeled passenger cars (motorcycles, tilting tricycles, etc.) ‘steering wheel’ contains too much ambiguity; therefore ‘handwheel’ is preferred since it adds precision.

**FIGURE 1.5**

Linearity: More handwheel input results in proportionally more yaw rate.

curvature and so on — remain constant and is more fully defined in Chapter 7. The steady state condition is easy to represent using an equilibrium analogy, constructed with the help of so-called ‘centrifugal force’. It should be noted that this fictitious force is invented solely for convenience of calculation of the analogous equilibrium state or the calculation of forces in an accelerating frame of reference. When a vehicle is travelling on a curved path it is not in equilibrium.

The curved path of the vehicle requires some lateral acceleration. Correctly, the lateral acceleration on a cornering vehicle is a centripetal acceleration — ‘centre seeking’. Note that speed is not the same as velocity; travelling in a curved path with a constant speed implies a changing direction and therefore a changing velocity. Even the centripetal acceleration definition causes some problems since everyone

**FIGURE 1.6**

Thought experiment comparing centripetal acceleration with linear acceleration.



'knows' that they are flung to the outside of a car if unrestrained and so there is much lax talk of centrifugal forces — 'centre fleeing'. To clarify this issue, a brief thought experiment is required. Imagine a bucket of water on a rope being swung around by a subject (Figure 1.6). If the subject looks at the bucket then the water is apparently pressed into the bucket by the mythical 'centrifugal force' (presuming the bucket is being swung fast enough). If the swinging is halted and the bucket simply suspended by the rope then the water is held in the bucket by the downward gravitational field of the earth — the weight of the water pulls it into the bucket. Imagine now a different scenario in which the bucket (on a frictionless plane) is pulled horizontally towards the observer at a constant acceleration in a linear fashion. It's best not to complicate the experiment by worrying about what will happen when the bucket reaches the subject. It is this third scenario and not the second that is useful in constructing the cornering case. If both the first and third cases are imagined in a zero gravity environment, they still work — the water will stay in the bucket. Note that for the third scenario — what we might call the 'inertial' case as against the gravitational case in the second scenario — the acceleration is towards the open end of the bucket. This is also true for the first scenario, in which the bucket is swung; the acceleration is towards the open end of the bucket and is towards the subject — i.e. it is centripetal. That the water stays in the bucket is simply a consequence of the way the bucket applies the centripetal force to the water. Thus the tyres on a car exert a force towards the centre of a turn and the body mass is accelerated by those forces centripetally — in a curved path.

An accelerometer in the car is effectively a load cell that would be between the bucket and the water in the scenarios here and so it measures the centripetal force applied between the calibrated mass within the accelerometer (the water) and its support in the casing (the bucket). The so-called centrifugal force is one half of an action–reaction pair within the system but a free-body diagram of the bucket and rope in all three cases shows tension in the rope as an externally applied force when considering the rope as a separate free body. Only in case 2 is the bucket actually in equilibrium, with the addition of the gravitational force on the bucket and water. Therefore an accelerometer (or an observer) in the vehicle apparently senses a centrifugal force while theoretical vehicle dynamicists talk always of centripetal acceleration. Changing the sign on the inertial force, so that it is now a d'Alembert force, appears to solve the apparent confusion. This can be misleading as we now have the impression that the analysis of the cornering vehicle is a static equilibrium problem. The water is not in equilibrium when travelling in a curved path, and neither is a car.

Centripetal forces accelerate the vehicle towards the centre of the turn. This acceleration, perpendicular to the forward velocity vector, is often referred to as 'lateral' acceleration, since the vehicle broadly points in the direction of the forward velocity vector (see Chapter 7 for a more precise description of the body attitude). It can be seen that the relationship between centripetal acceleration,  $A^p$ , yaw rate,  $\omega$ , forward velocity,  $V$  and radius of turn,  $R$  is given by:

$$A^p = V^2/R = \omega V = \omega^2 R \quad (1.1)$$

The absolute limit for lateral acceleration, and hence yaw rate, is set by the friction available between the tyres and the road surface. Competition tyres ('racing slicks') have a coefficient of friction substantially in excess of unity<sup>4</sup> and, together with large aerodynamic downforces, allow a lateral acceleration in the region of  $30 \text{ m/s}^2$ , with yaw rates correspondingly over  $40 \text{ deg/s}$  for a speed of  $40 \text{ m/s}$  ( $90 \text{ mph}$ ). For more typical road vehicles, limit lateral accelerations rarely exceed  $9 \text{ m/s}^2$ , with yaw rates correspondingly down to around  $12 \text{ deg/s}$  at the same speed. However, for the tyre behaviour to remain substantially linear for a road car, the lateral accelerations must be generally less than about  $3 \text{ m/s}^2$ , so yaw rates are down to a mere  $4 \text{ deg/s}$  at the same speed.

While apparently a small fraction of the capability of the vehicle, there is much evidence to suggest that the driving population as a whole rarely exceed the linearity limits of the vehicle at speed and only the most confident exceed them at lower speeds (Lechmer and Perrin, 1993). The 100 Car Naturalistic Driving Study categorises any event greater than  $4 \text{ m/s}^2$  as a 'near accident' or 'dangerous occurrence' (Dingus et al., 2004).

When racing or during emergency manoeuvres on the road — typically attempting to avoid an accident — the vehicle becomes strongly 'non-linear'. The handwheel is moved rapidly and the vehicle generally has difficulty in responding accurately to the handwheel. This is the arena called 'Transient Handling' and is correctly the object of many studies during the product design process. In contrast to the steady state condition, all the vehicle states fluctuate rapidly and the expressions above are modified. Steady state and transient behaviour are connected. While good steady state behaviour is connected with good transient behaviour, it is not in itself sufficient (Sharp, 2000).

Transient handling studies concentrate on capturing, analysing and understanding the yaw moments applied to the vehicle and its response to them. Those moments are dominated by the lateral and longitudinal forces from the tyres. For road cars, additional aerodynamic contributions are a small modifier but for racing, the aerodynamic behaviour rises in importance.

The generation of tyre forces is frequently the biggest source of confusion in vehicle dynamics, since both lateral and longitudinal mechanisms are neither obvious nor intuitive. Tyres are dealt with in some depth by Pacejka (2012) in a companion volume in this series and also have some further coverage in Chapter 5.

The tyres generate lateral forces by two mechanisms, 'camber' and 'slip angle'. Camber is the angle at which the tyre is presented to the road when viewed from the front. There exists some confusion when referring to and measuring camber angle; for clarity within this text camber angle is measured with respect to the road<sup>5</sup> unless

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<sup>4</sup>Some people are troubled by this idea but the authors have never been able to understand why. In any case, the performance of top fuel dragsters should leave no doubt that friction coefficients handsomely above unity exist.

<sup>5</sup>This is referred to as 'inclination angle' in some contexts.

explicitly defined as being relative to the vehicle body. It is the angle with respect to the road that generates a side force. Thus a motorcycle runs a large camber angle when cornering but runs no camber angle with respect to the vehicle body.

Slip angle is the angle at which the moving tyre is presented to the road when viewed in plan. It is important to note that slip angle only exists when the vehicle is in motion. At a standstill (and at speeds under about 10 mph) the lateral stiffness of the tyres generate the forces that constrain the vehicle to its intended path. As speed rises above walking pace, the tyres have a falling static lateral stiffness until above about 5 m/s (about 10 mph) they have effectively none; an applied lateral force, such as a wind load, will move the vehicle sideways from its intended path. It is important to note that the presence of a slip angle does not necessarily imply sliding behaviour at the contact patch.

Slip angle forces are typically more than 20 times camber forces for a particular angle, and are thus the more important aspect for vehicle dynamics. The lateral forces induced by the angles are strongly modified by the vertical loads on the tyres at each moment in time.

The tyres generate longitudinal forces by spinning at a speed different to their 'free-rolling' speed. The free-rolling speed is the speed at which the wheel and tyre would spin if no brake or drive forces are applied to them. The difference in speed is described as 'slip ratio', which is unfortunate since it is confusingly similar to slip angle. It is expressed as a percentage, so for example a tyre turning with a 5% slip ratio will perform 105 revolutions to travel the same distance as a free-rolling tyre performing 100 revolutions. In doing so, it will impart a tractive force to the vehicle. A -5% slip ratio would imply 95 revolutions of the same wheel and the presence of a braking force.

Managing lateral tyre forces by controlling slip and camber angles is the work of the suspension linkage. For the front wheels, the driver has the ability to vary the slip angle using the handwheel. Managing the vertical loads on the tyres is the function of the suspension 'calibration' (springs, dampers and any active devices, if present). Chapter 4 deals with suspension analysis in some detail. Management of longitudinal forces is the role of the vehicle driveline and braking system, including anti-lock braking system or brake intervention systems, dealt with in Chapter 8.

A vehicle travelling in a straight line has a yaw velocity of zero and a centripetal acceleration of zero. When travelling in a steady curve, the centripetal acceleration is not zero and the yaw rate is not zero but both are constant and are related as described in [Eqn \(1.1\)](#). In performing the transition from straight running to a curved path there must be a period of yaw acceleration in order to acquire the yaw velocity that matches the centripetal acceleration. The yaw acceleration is induced and controlled by yaw moments acting on the vehicle yaw inertia.

Transient handling therefore implies the variation of yaw moments applied to the vehicle. Those moments are applied by aerodynamic behaviour and the force-generating qualities of the tyres at a distance from the vehicle's centre of mass. No other mechanisms exist for generating a meaningful yaw moment on the vehicle; while gyroscopic torques associated with camber changes exist, they are small. For

road vehicles, the aerodynamic modifications are generally small. Multibody system methods allow the convenient exploration of aspects of the vehicle design that influence those qualities of the tyres. Chapter 6 addresses different methods of modelling those aspects of the vehicle and their relative merits.

The modelling of the tyre forces and moments at the tyre to road contact patch is one of the most complex issues in vehicle handling simulation. The models used are typically not causal<sup>6</sup> but are rather empirical formulations used to represent the tyre force and moment curves typically found through laboratory or road based rig testing of a tyre.

Like all models, tyre models have a spectrum of complexity ranging from simple vertical spring rates with hysteretic damping through to fully causal. Chapter 5 explores this spectrum in some depth. Examples of tyre models used for vehicle handling discussed in this book include:

1. A sophisticated empirical point-follower tyre model known as the 'Magic Formula'. This tyre model has been developed by Pacejka and his associates (Bakker et al., 1986, 1989; Pacejka and Bakker, 1993; Pacejka, 2012) and is known to give usefully accurate representation of measured tyre characteristics. The model uses modified trigonometric functions to represent the shape of curves that plot tyre forces and moments as functions of longitudinal slip or slip angle. For 25 years the work of Pacejka has been well known throughout the vehicle dynamics community. The result of this is a tyre model that is now widely used both by industry and academic institutions and is undergoing continual improvement and development. The complexity of the model does however mean that well over 50 parameters are needed to define a tyre model and that software must be obtained or developed to derive the parameters from measured test data. The latest versions of the model (6.1 at the time of writing) include the effects of inflation pressure and can accommodate motorcycle and passenger car tyres within the same model.
2. An alternative modelling approach is to use a straightforward interpolation model. This was the original tyre modelling method used in MSC ADAMS (Ryan, 1990). This methodology is still used by some companies but has, to a large extent, been superseded by more recent parameter based models. The method is included here as a useful benchmark for the comparison of other tyre models in Chapter 5.
3. Another point-follower tyre model is provided for readers as a source listing in Appendix B. This model (Blundell, 2003) has been developed by Harty and has

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<sup>6</sup>A Causal model is one in which the causes for every effect produced are explicitly described. An example might be a finite element model of a spring, which infers the behaviour of the whole spring from constituent stress/strain relationships. An empirical model is one assembled entirely from observations but making no attempt to quantify or describe the mechanisms causing the effect. An example might be the linear elastic equation of a spring, which requires and produces no description of the stress or strain state of the metal.

the advantage of requiring only a limited number of input parameters compared to the Pacejka model. The implementation is more complete, however, than the interpolation model and includes representation of the following:

- a. Comprehensive slip
- b. Load dependency
- c. Camber thrust
- d. Post limit

It has been found that the Harty model is robust when modelling limit behaviour including for example problems involving low grip or prolonged wheelspin.

The same model is suitable for representing motorcycle and passenger car tyre behaviour when different parameters are employed.

4. An elaborate 'semi-causal' model of the tyre structure known as FTire (Gipser, 1999), sold commercially including carcass enveloping and contact patch velocity details. Based on an elegant formulation, conceptually a blend of finite element-style simultaneous solutions of small segments with a modal-component style aggregate behaviour representation, it boasts some impressive abilities to resolve pressure and velocity distributions in the contact patch. It is gaining ground in durability prediction in particular.
5. A further model, which is less elaborate than (4) structurally but incorporates thermal aspects in order to be able to migrate from cold to hot and from fresh to worn during the course of a simulation. Of particular interest in motorsport where the thermal modification of tyres is most marked, the so-called Thermal And MEchanical (TAME Tire) representation (Hague, 2010), complements the FTire approach and is more appropriate for certain applications

These five approaches to tyre modelling represent a comprehensive overview of the breadth of modelling practice now in use and are a substantial expansion over what was common when the first edition was written.

In order to progress from travelling in a straight line to travelling in a curved path, the following sequence of events is suggested:

1. The driver turns the handwheel, applying a slip angle at the front wheels.
2. After a delay associated with the front tyre relaxation lengths (see Chapter 5), side force is applied at the front of the vehicle. Lateral and yaw accelerations exist.
3. The body yaws (rotates in plan), applying a slip angle at the rear wheels.
4. After a delay associated with the rear tyre relaxation lengths, side force is applied at the rear of the vehicle. Lateral acceleration is increased; yaw acceleration is reduced to zero.

In the real world, the driver intervenes and the events run into one another rather than being discrete as suggested here, but it is a useful sequence for discussion purposes. A similar sequence of events describes the return to straight-line travel. Any yaw rate adjustments made by the driver follow similar sequences, too.

During the period of yaw acceleration (stages 2 and 3 above) there exists the need for an excess of lateral forces from the front tyres when compared to the rear in order

to deliver the required yaw moment. At the end of this period, that excess must disappear. Side force requirements for the rear tyre are thus increasing while those for the front tyres are steady or decreasing. To understand the significance of this fact, some further understanding of tyre behaviour is necessary.

To a first approximation, camber forces may be neglected from tyre behaviour for vehicles that do not roll (lean) freely. Slip angle is the dominant side force generation mechanism. It is important to note that a tyre will adjust its slip angle to support the required side force, and not the other way around; this is a frequent source of difficulty in comprehending vehicle dynamic behaviour. All tyres display a slip angle at which the maximum side force is generated, sometimes referred to as the ‘critical slip angle’. If a force is required which is greater than that which can be generated at the critical slip angle, the tyre will run up to and then beyond the critical slip angle. Beyond the critical slip angle the side force falls off with slip angle, and so an increasing amount of surplus lateral force is available to accelerate the growth of slip angle once the critical slip angle is passed.

Returning to the vehicle, the side force requirement for the rear tyres is increasing while that for the front tyres is steady or decreasing. The rear tyres will be experiencing a growing slip angle, while the fronts experience a steady or reducing one. If at this time the rear tyres exceed their critical slip angle, their ability to remove yaw moment is lost. The only possible way for yaw moment to be removed is by a reduction in the front tyre forces. If the yaw moment persists then yaw acceleration persists. With increasing yaw velocity, the slip angle at the front axle is reduced while that at the rear axle is increased further, further removing the rear tyres’ ability to remove yaw moment from the vehicle. If the front tyres are past their critical slip angle, too, the normal stabilisation mechanism is reversed. The result is an accelerating spin that departs rapidly from the driver’s control. The modelling and interpretation of such events is dealt with in Chapter 7.

The behaviour of the driver is important to the system performance as a whole. The driver is called on to act as a yaw rate manager, acting on the vehicle controls as part of a closed-loop feedback system to impart the yaw moments required to control the yaw rate of the vehicle. At critical times, the workload of the driver may exceed his or her capability, resulting in a loss of control.

The goal of vehicle dynamics work is to maintain the vehicle behaviour within the bounds that can be comprehended by and controlled by the driver. The increasing use of electronic systems in vehicles drives the application of standards for the safe engineering of such critical standards, the most common of which is the ISO26262 standard for automobiles. So-called ‘controllability’ is a key factor in this and multi-body simulations have an important role in assessing the controllability of many situations, as discussed later in Chapter 8.

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### 1.3 Why analyse?

In any real product-engineering programme, particularly in the ground vehicle industry, there are always time constraints. The need to introduce a new product to

retain market share or to preserve competitive advantage drives increasingly tight timetabling for product-engineering tasks. In the western world, and to a growing extent in the developing world, tastes are becoming ever more refined such that the demand for both quality of design and quality of construction is increasing all the while. Unlike a few decades ago, there are few genuinely bad products available.

It seems, therefore, that demands for better products are at odds with demands for compressed engineering timetables. This is true; the resolution of this conflict lies in improving the efficiency of the engineering process. It is here that predictive methods hold out some promise.

Predictive methods notionally allow several good things:

- Improved comprehension and ranking of design variables
- Rapid experimentation with design configurations
- Genuine optimisation of numerical response variables

Therefore the use of predictive methods is crucial for staying ‘ahead of the game’ in vehicle engineering.

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## 1.4 Classical methods

These methods are taught formally in universities as part of the syllabus. While they can be daunting at first sight, they are elegant and can prove tremendously illuminating in forming a holistic framework for what can easily be a bewildering arena. A quick tour through a classically formulated vehicle model is given in Chapter 7 and an implementation in MS-Excel is available to download with the book.

The best practitioners of the art recommend the use of a body-centred state-space formulation. While full of simplifications, useful insights can be gained by studying a two degree-of-freedom model for typical passenger cars. With a reasonable increase in sophistication but well worth the effort is the elaboration to three degrees of freedom (four states) to include the influence of suspension roll.

Such classical models help the analyst discern ‘the wood for the trees’ — they easily bring forth, for example, the influence of suspension steer derivatives on straight-line stability. In this they contrast strongly with ‘literal’ linkage models, in which all the problems of real vehicles (the lack of isolation of single effects) obscures their ranking and comprehension.

Although the task of deriving the equations of motion and arranging the terms for subsequent solution is laborious and may be error-prone, the proponents of the method point quite correctly to the increased comprehension of the problem to which it leads.

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## 1.5 Analytical process

It is clear that the tasks undertaken have expanded to fit the time available. Despite remorselessly increasing computing power, analysis tasks are still taking as long to



complete as they always have done. The increased computing power available is an irresistible temptation to add complexity to predictive models (Harty, 1999).

Complex models require more data to define them. This data takes time to acquire. More importantly, it must exist. Early in the design cycle, it is easy to fall into the ‘paralysis of analysis’; nothing can be analysed accurately until it is defined to a level of accuracy matching the complexity of the modelling technique. More than the model itself, the process within which it fits must be suited to the tasks at hand. *The model is not the product!*

There is nothing new in the authors’ observations; Sharp (1991) comments

*Models do not possess intrinsic value. They are for solving problems. They should be thought of in relation to the problem or range of problems which they are intended to solve. The ideal model is that with minimum complexity which is capable of solving the problems of concern with an acceptable risk of the solution being “wrong”. This acceptable risk is not quantifiable and it must remain a matter of judgement. However, it is clear that diminishing returns are obtained for model elaboration.*

Any method of analysis must be part of a structured process if it is to produce useful results in a timely manner. Interesting results that are too late to influence product design are of little use in modern concurrent<sup>7</sup> engineering practice. Rapid results that are so flawed as to produce poor engineering decisions are also of little use. The use of predictive methods within vehicle design for addressing dynamic issues with the vehicle should follow a pattern not dissimilar to that in [Figure 1.7](#), whatever the problem or the vehicle.

### 1.5.1 Aspiration

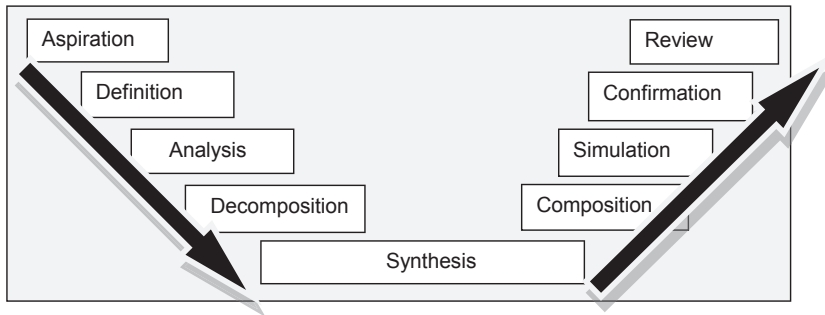
The method properly starts with the recognition of the end goals. In some organisations, confusion surrounds this part of the process, with obfuscation between targets, objectives and goals; the terms are used differently between organisations and frequently with some differences between individuals in the same organisation. Cutting through this confusion requires time and energy but is vital.

### 1.5.2 Definition

After definition, a clear description of ‘success’ and ‘failure’ must exist; without it the rest of the activities are, at best, wasteful dissipation. Aspirations are frequently set in terms of subjective comparisons – ‘Ride Comfort better than best in class’.

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<sup>7</sup>Concurrent = Taking place at the same time. ‘Concurrent Engineering’ was a fashionable phrase in the recent past and refers to the practice of considering functional, cost and manufacturing issues together rather than the historically derived ‘sequential’ approach. It was also referred to as ‘Simultaneous Engineering’ for a while, though the segmented connotations of simultaneous were considered unhelpful and so the ‘concurrent’ epithet was adopted.

**FIGURE 1.7**

'V' Process for product design.

To usefully feed these into an analytical process, they must be capable of being quantified — i.e. of having numbers associated with them. Without numbers, it is impossible to address the task using analytical methods and the analytical process should be halted. This is not to say that product development cannot continue but that *to persist with a numerical process in the absence of a knowledge of the difference between good and bad numbers is folly*. Analysts and Development staff must be involved closely with each other and agree on the type of numerical data that defines success, how it is calculated predictively and how it is measured on a real vehicle. Commonly, some form of 'benchmarking' study — a measurement exercise to quantify the current best performers — is associated with this stage. The activity to find the benchmark is a useful shakedown for the proposed measurement processes and is generally a fruitful education for those involved.

### 1.5.3 Analysis

When success and failure have been defined for the system as a whole, the individual parts of the system must be considered. There is generally more than one way to reach a system solution by combining individual subsystems or elements. During this stage of the process, some decisions must be taken about what combination is preferred. It may be, for example, that in seeking a certain level of vehicle performance there is a choice between increasing power output and reducing weight in order to achieve a given power-to-weight ratio. That choice will be influenced by such simple things as cost (saving weight may be more expensive than simply selecting a larger engine from the corporate library) or by more abstruse notions (the need to be seen to be 'environmentally friendly', perhaps). The task of analysis is to illuminate that choice. The analysis carried out must be sufficiently accurate but not excessively so. 'Simple models smartly used' is the order of the day for analysis work. The analysis may consider many possible combinations in order to recommend a favoured combination. This activity is sometimes referred to by the authors as 'mapping the design space' — producing guidance for those who wish to make design decisions based on wider considerations and who wish to comprehend the

consequences of their decisions. *The most cost-effective activity at this stage is accurately recalling and comprehending what has gone before.* Since the first edition of the book, ‘design of experiments’ and ‘signal to noise’ software has matured tremendously and with tools such as Minitab, Mode Frontier, Heeds, Altair Hyperstudy or MSC Insight, this mapping process can be undertaken more diligently than ever before. Chapter 7 discusses these new possibilities in more detail.

### **1.5.4 Decomposition**

Once the analytical stage is complete, it is time for design decisions to be made. The whole entity must be decomposed into its constituent parts, each of which has design goals associated with it – cost, performance, weight, etc. It is at this time the first real design decisions are made that shape the product – section properties, geometries, manufacturing process and so on. Those decisions are to be made in the light of the preceding analysis. Many organisations still begin this part of the process too early and as a consequence paint themselves into a corner.

### **1.5.5 Synthesis**

Once the design is decomposed into manageable portions, the task of synthesising (creating) the design begins. During this phase, analytical tools are used to support individual activities and verify the conformance of the proposed design with the intended design goals. An example of this might be the use of kinematic simulation to verify that the suspension geometry characteristics are those required. Discerning the requirement itself is the function of the earlier decomposition phase.

### **1.5.6 Composition**

The reassembly of the separate portions of the design, each of which by now has a high level of confidence at reaching its individual design goals.

### **1.5.7 Simulation**

Before production commences in volume, confidence is needed that the design will be appropriate to go to market. It is often referred to as ‘prototyping’ and can be real or virtual. The distinguishing feature of this stage is that it is very high fidelity, as distinct from the analysis stage that was no more accurate than necessary.

Prototype vehicles, produced from non-representative tools and/or processes, are physical simulations instead of mathematical ones. The increasing use of ‘virtual’ prototyping obviates these physical prototypes except for those where an understanding of the man/machine interaction is necessary. One of several arenas where this remains true is the dynamics task.

Predictive models that have been a long time in preparation can be used to assess, virtually and in some detail, the behaviour of the whole design. Models prepared

during the Synthesis activity are taken and re-used. It is in this arena that great strides have been made in terms of processing power, model re-use and inter-package integration over the last decade. Unfortunately, in the minds of some, these super elaborate models are all that is useful and anything less is simply worthless, passé and old fashioned. These are valuable models and have a crucial part to play in the process, but without a well-shaped concept design they are unwieldy white elephants. When used unthinkingly they become part of a rather ‘Victorian’ process in which designs are completed and then inspected for fitness-for-purposes; in the event of them being unfit, an iterative loop is commenced. Far more rapid would have been to have understood the question and to proceed toward the solution in a linear, non-looping fashion.

### 1.5.8 Confirmation

Sign-off testing is to be carried out on real vehicles that are as representative as possible. This stage should reveal no surprises, as changes at this stage are expensive.

### 1.5.9 Review

Once the design is successfully signed off, a stage that is frequently omitted is the review. What was done well? What could have been better? What technology do we wish we had then that might be available to us now? Since the most cost-effective analysis activity is to recall accurately and comprehend what has gone before, a well-documented review activity saves time and money in the next vehicle programme.

The process described is not definitive, nor is it intended to be prescriptive. It should, however, illustrate the difference between ‘analysis’ and ‘simulation’ and clearly distinguish them.

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## 1.6 Computational methods

Whether the equations of motion have been derived by hand or delegated to a commercial software package, the primary goal when considering vehicle dynamics is to be able to predict the time-domain solution to those equations. Another important type of solution, the eigensolution, is discussed in Chapter 3.

Once the equations of motion have been assembled, they are integrated numerically. This is a specialised field in its own right. There are many publications in the field and it is an area rife with difficulties and pitfalls for the unwary. However, in order to successfully use the commercially available software products, some comprehension of the difficulties involved are necessary for users. Chapter 3 deals with some of the more common difficulties with some examples for the reader. By far the most dangerous type of difficulty is the ‘plausible but wrong’ solution.

Commercial analysts must studiously guard against the ‘garbage in, gospel out’ mentality that pervades the engineering industry at present.

The equations can be solved in a fairly direct fashion as assembled by the commercial package pre-processor or they can be subject to further symbolic manipulation before numerical solution. So-called ‘symbolic’ codes offer some tremendous computational efficiency benefits and are being hailed by many as the future of multibody system analysis since they allow real-time computation of reasonably complex models without excessive computing power. The prospect of a real-time multibody system of the vehicle solved on-board in order to generate reference signals for the generation-after-next vehicle control systems seems genuine.

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## 1.7 Computer-based tools

Multibody systems analysis software has become so easy to use that many users lack even a basic awareness of the methods they are using. This chapter charts the background and development to the current generation of multibody systems analysis programs. While the freedom from the purgatory of formulating one’s own equations of motion is a blessing, it is partly that purgatory that aids the analyst’s final understanding of the problem. Chapter 2, Kinematics and Dynamics of Rigid Bodies, is intended as a reference and also as a ‘launch pad’ for the enthusiastic readers to be able to teach themselves the process of so-called ‘classical’ modelling.

Crolla (1995) identifies the main types of computer-based tools, which can be used for vehicle dynamic simulation, and categorises these as:

1. Purpose designed simulation codes
2. Multibody simulation packages, which are numerical
3. Multibody simulation packages, which are algebraic (symbolic)
4. Toolkits such as MATLAB

One of the major conclusions that Crolla draws is that it is still generally the case that the ride and handling performance of a vehicle will be developed and refined mainly through subjective assessments. Most importantly he suggests that in concentrating on sophistication and precision in modelling, practising vehicle dynamicists may have got the balance wrong. This is an important issue that reinforces the main approach in this book, which is to encourage the application of models that lead to positive decisions and inputs to the vehicle design process.

Crolla’s paper also provides an interesting historical review that highlights an important meeting at IMechE headquarters in 1956, ‘Research in automobile stability and control and tyre performance’. The author states that in the field of vehicle dynamics the papers presented at this meeting are now regarded as seminal and are referred to in the USA as simply ‘The IME Papers’.

One of the authors at that meeting, Segel, can be considered to be a pioneer in the field of vehicle dynamics. His paper (Segel, 1956) is one of the first examples where classical mechanics has been applied to an automobile in the study of lateral rigid

body motion resulting from steering inputs. The paper describes work carried out on a Buick vehicle for General Motors and is based on transferable experience of aircraft stability gained at the Flight Research Department, Cornell Aeronautical Laboratory. The main thrust of the project was the development of a mathematical vehicle model that included the formulation of lateral tyre forces and the experimental verification using instrumented vehicle tests.

In 1993, almost 40 years after embarking on this early work in vehicle dynamics, Segel again visited the IMechE to present a comprehensive review paper (Segel, 1993), 'An overview of developments in road vehicle dynamics: past, present and future'. In it he provides a historical review that considers the development of vehicle dynamics theory in three distinct phases:

Period 1 – Invention of the car to early 1930s

Period 2 – Early 1930s to 1953

Period 3 – 1953 to the then present (1993)

In describing the start of Period 3 Segel references his early 'IME paper' (Segel, 1956). In terms of preparing a review of work in the area of vehicle dynamics there is an important point made in the paper regarding the rapid expansion in literature that makes any comprehensive summary and critique difficult. This is highlighted by his example of the 1992 FISITA Congress where a total of 70 papers were presented under the general title of 'Total Vehicle Dynamics'. In 2013, searching the SAE website alone using 'vehicle dynamics' (in quotes, so as to preclude matches to only the word vehicle or the word dynamics) produces 2692 papers. There are no fewer than 98 standards produced by the same search.

Following Segel's historical classification of the vehicle dynamics discipline to date, the authors of this text suggest that we have now entered a fourth era that may be characterised by the use of engineering analysis software as something of a 'commodity', bought and sold and often used without a great deal of formal comprehension. In these circumstances there is a need for the software to be absolutely watertight (currently not possible to guarantee) or else for a small number of experts – 'champions' – within organisations to ensure the 'commodity' users are not drifting off the rails, to use a horribly mixed metaphor. This mode of operation is already becoming established within the analysis groups of large automotive companies where analysts make use of customised software programs such as ADAMS/Car, the Simpack Vehicle Wizard or the Dymola Vehicle Dynamics library. These programs have two distinct types of usage. At one level the software is used by an 'expert' with the experience, knowledge and skill and to customise the models generated, the types of simulation to be performed and the format in which selected results will be presented. A larger group of 'standard' users are then able to use the program to carry out suspension or full vehicle simulations assuming little or no knowledge of multibody systems formulations and solution methods. Standard users in many organisations are not full-time analysts but designers or development engineers whose remit includes using analytical tools to inform their opinions according to protocols imposed by the organisation.

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## 1.8 Commercial computer packages

Before the evolution of multibody system programs, engineers analysed the behaviour of mechanisms such as cam-followers and four-bar linkages on the basis of pure kinematic behaviour. Graphical methods were often used to obtain solutions. Chace (1985) summarises the early programs that led to the development of the MSC ADAMS program. One of the first programs (Cooper et al., 1965) was KAM (Kinematic Analysis Method) capable of performing displacement, velocity and acceleration analysis and solving reaction forces for a limited set of linkages and suspension models. Another early program (Knappe, 1965) was COMMEND (Computer-Orientated Mechanical Engineering Design), which was used for planar problems.

By 1969, Chace (1969, 1970) and Korybalski (Chace and Korybalski, 1970) had completed the original version of DAMN (Dynamic Analysis of Mechanical Networks). This was historically the first general program to solve time histories for systems undergoing large displacement dynamic motion. This work led in 1971 to a new program DRAM (Dynamic Response of Articulated Machinery) that was further enhanced by Angel (Chace and Angel, 1977).

Orlandea published two *American Society of Mechanical Engineers* (ASME) papers (Orlandea et al., 1976a, 1976b). These were a development of the earlier two-dimensional programs to a three-dimensional code but without some of the impact capability contained in DRAM at that time. This program went on to form the core of MSC ADAMS.

General-purpose programs have been developed with a view to commercial gain and as such are able to address a much larger set of problems across a wide range of engineering industries.

A number of other systems based on commercial software have at times been developed specifically for automotive vehicle modelling applications. Several of the larger vehicle manufacturers have at some time integrated MSC ADAMS into their own in-house vehicle design systems. Early examples of these were the AMIGO system at Audi (Hudi, 1988), and MOGESSA at Volkswagen (Terlinden et al., 1987). The WOODS system based on user-defined worksheets was another system at that time in this case developed by German consultants for Ford in the UK (Kaminski, 1990). Ford's Global vehicle modelling activities have since focused on in-house generated linear models and the ADAMS/Chassis™ (formerly known as ADAMS/Pre™) package, a layer over the top of the standard MSC ADAMS pre- and post-processor that is strongly tailored towards productivity and consistency in vehicle analysis.

When the first edition was published there were two leading general-purpose programs, MSC ADAMS and LMS DADS. They dominated the ground vehicle and aerospace markets. LMS DADS has morphed into a product called Virtual Motion and inside MSC ADAMS there has been an explosion of so-called 'Vertical' products, which integrate technology-specific utilities (such as 'wrapping' analysis for caterpillar tracks) with a dual mode software interface geared towards the champion/standard user split described previously.

In the meantime, new products such as Simpack, Dymola and Amesim have emerged as serious commercial general-purpose products (all were somewhat nascent at the time of the first edition) along with open-source codes such as the entire Modelica endeavour (upon which Dymola is built), MBDyn from the university of Milan, SimTK, Metex and so on. What has also emerged is that software that was traditionally for other purposes, such as Matlab Simulink, has begun to extend its reach and coverage such that many useful studies can be carried out entirely within it, without recourse to a 'traditional' multibody solver.

MADYMO is a program recognised as having a multibody foundation with an embedded non-linear finite element capability. This program has been developed by TNO in the Netherlands and complements their established crash test work with dummies. Recent developments in MADYMO have included the development of biofidelic humanoid models to extend the simulation of crash test dummies to 'real-world' pedestrian impact scenarios.

Tremendous advances in computing power were readily foreseeable in 2004 and remain so. A current workstation laptop has 32 GB of RAM (or 32 million Apollo Guidance Computers) serving four separate 'cores' — really separate computers sharing the box and supporting the kind of parallel computing that was spoken of in reverent tones when using Cray-II computers remotely in the early 1990s. Models regarded as awkwardly complex only 10 years ago are now entirely manageable. Tracked vehicles, for example, can be modelled with the track represented link-by-link, including the contact interaction with deformable terrain and all the vehicle wheels; such models are currently close to the limit of what is tolerable in a normal workflow but for the next edition such models may be solving in real time on a mobile phone app.

Blundell (1999a and b, 2000a and b) published a series of four IMechE papers with the aim of summarising typical processes involved with using a general-purpose multibody program to simulate full vehicle handling manoeuvres. The first paper provided an overview of the usage of multibody systems analysis in vehicle dynamics. The second paper described suspension modelling and analysis methodologies. The third paper covered tyre modelling and provided example routines for different tyre models and data. The fourth and final paper brought the series together with a comparative study of full vehicle models, of varying complexity, simulating a double lane change manoeuvre. Results from the simulation models were compared with measured test data from the proving ground. The overall emphasis of the series of papers was to demonstrate the accuracy of simple efficient models based on parameters amenable to design sensitivity study variations rather than blindly modelling the vehicle 'as is'.

Crolla et al. (1994) also define two fundamental types of multibody systems (MBS) program, the first of which is where the equations are generated in numerical format and are solved directly using numerical integration routines embedded in the package. The second, more recent type of MBS program formulates the equations in symbolic form and often uses an independent solver. The authors also describe tool-kits as collections of routines that generate models, formulate and solve equations, and present results.



Other examples of more recently developed general-purpose codes formulate the equations algebraically and use a symbolic approach. Examples of these programs include MESA VERDE (Wittenburg and Wolz, 1985), AUTOSIM (Sayers, 1990), and RASNA Applied Motion Software (Austin and Hollars, 1992). Crolla et al. (1992) provide a summary comparison of the differences between numeric and symbolic code. As stated, MBS programs will usually automatically formulate and solve the equations of motion although in some cases, such as with the work described by Costa (1991) and Holt and Cornish (1992) and Holt (1994), a program SDFAST has been used to formulate the equations of motion in symbolic form and another program ACSL (Automatic Continuous Simulation Language) has been used to generate a solution.

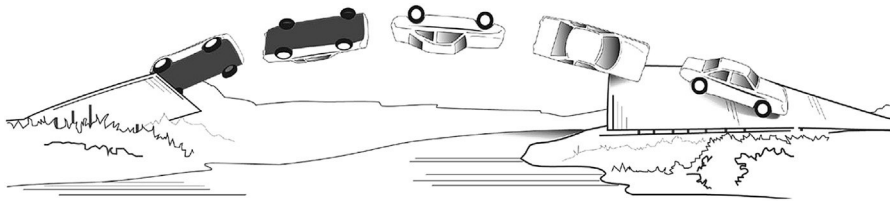
Another customised application developed by the automotive industry is described in Scapaticci et al. (1992). In this paper the authors describe how MSC ADAMS has been integrated into a system known as SARAH (Suspension Analyses Reduced ADAMS Handling). This in-house system for the automotive industry was developed by the Fiat Research Centre Handling Group and used a suspension modelling technique that ignored suspension layout but focused on the final effects of wheel centre trajectory and orientation.

This leads nicely into another category of software available, the single-purpose tool. Unlike the general-purpose tools, which can build models of anything from door latches to spacecraft, some software is written specifically for modelling vehicle dynamics.

At Leeds University a vehicle-specific system was developed under the supervision of Crolla. In this case all the commonly required vehicle dynamics studies have been embodied in their own set of programs (Crolla et al., 1994) known as VDAS (Vehicle Dynamics Analysis Software). Examples of the applications incorporated in this system included: ride/handling, suspensions, natural frequencies, mode shapes, frequency response and steady state handling diagrams. The system included a range of models and further new models could be added using a pre-processor. Single-purpose programs are described as those where the equations of motion have been developed and programmed for a specific model. Model parameters can be changed but the model is fixed unless the program is changed and recompiled. A single-purpose program for passenger cars cannot be used for motorcycles, for example.

A typical example of this type of program would be AUTOSIM described by Sayers (1990), Sharp (1997) and Mousseau et al. (1992) which is intended for vehicle handling and has been developed as a symbolic code in order to produce very fast simulations. Other examples are the Milliken Research Associates VDMS program, which is a single-purpose model for use within the Matlab environment, and IPG Carmaker. For vehicles other than cars, BikeSim and TruckSim are both available.

Generally, single-purpose programs are specifically developed for a given type of simulation but often allow flexibility as to the choice and complexity of the model. An extension of this is where the equations of motion for a fixed vehicle modelling

**FIGURE 1.8**

The Astro Spiral Jump.

approach are programmed and cannot be changed by the user such as the HVOSM (Highway-Vehicle-Object Simulation Model) developed by Raymond McHenry at Calspan in the mid 1960s. The program includes tyre and suspension models and can be used for impact studies in addition to the normal ride and handling simulations. HVOSM deserves a special mention for its contribution to the Astro Spiral Jump (Figure 1.8), a stunt used in the 1974 James Bond film 'The Man with the Golden Gun' that would have been more or less impossible to choreograph using traditional incremental stunt development methods.

The authors (Crolla et al., 1992) indicate that the University of Missouri has also developed a light vehicle dynamics simulation (LVDS) program that runs on a PC and can produce animated outputs. In the mid 1980s, Systems Technology, Inc. developed a program for vehicle dynamics analysis non-linear (VDANL) simulation. This program is based on a 13 degree of freedom, lumped parameter model (Allen et al., 1987) and has been used by researchers at Ohio State University for sensitivity analysis studies (Tandy et al., 1992).

More recently, programs such as ARAS 360 have emerged for vehicle dynamics simulation within the accident reconstruction sphere. ARAS 360 has a 15 degree-of-freedom vehicle model, non-linear tyre behaviour and a rich graphical environment including weather rendering and skid mark production.

The relative ease of computing the vehicle dynamics problem has lead to a number of novel applications for it. In widespread use in aviation, simulators are generating a lot of interest for the study of the human aspect of vehicle control. Real-time capable models that can complete their calculations in the same timescale as the world progresses, or faster, have become relatively common. Several of the commercial packages have a real-time capable implementation although this can depend greatly on the level of complexity selected. In general, single-purpose programs perform better than general-purpose programs in this arena but the relatively low complexity of the computational problem means that even general-purpose programs can be pressed into real-time use if the burden of graphical processing is off-loaded to a dedicated computing resource. The development of compelling motion cueing is still somewhat incomplete with ground vehicle simulators, unlike aircraft simulators that have a different range of sensations to reproduce. Around 2000 a project was carried out on behalf of Prodrive by Harty for Evolution Studios in Cheshire in which a 50 Hz calculation rate produced excellent behaviour including

non-linear tyre behaviour for a simulated vehicle on the Sony Playstation platform. The 'accurate' form of the model was not used for the final production release of the games since the phase delays it produced were too realistic (and hence too difficult) for many players to assimilate in a video game environment. A copy of the accurate form of the model is retained by Harty, along with a now-vintage Playstation.

For the Playstation, the tyre model was cut down to the absolute minimum complexity. This model (the 'Harty' model) is described in Chapter 5.

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## 1.9 Benchmarking exercises

A detailed comparison between the various codes is beyond the capability of most companies when selecting an MBS program. In many ways the use of multibody systems has followed on from the earlier use of finite element analysis, the latter being approximately 10 years more mature as applied commercial software. Finite element codes were subject to a rigorous and successful series of benchmarks under the auspices of NAFEMS (National Agency for Finite Elements and Standards) during the 1980s. The published results provided analysts with useful comparisons between major finite element programs such as NASTRAN and ANSYS. The tests performed compared results obtained for a range of analysis methods with various finite elements.

For the vehicle dynamics community, Kortum and Sharp (1991) recognised that with the rapid growth in available multibody systems analysis programs a similar benchmarking exercise was needed. This exercise was organised through the International Association for Vehicle System Dynamics (IAVSD). In this study the various commercially available MBS programs were used to benchmark two problems. The first was to model the Iltis military vehicle and the second a five-link suspension system. A review of the exercise is provided by Sharp (1994) where some of the difficulties involved with such a wide-ranging study are discussed. An example of the problems involved would be the comparison of results. With different investigators using the various programs at widespread locations, a simple problem occurred when the results were sent in plotted form using different size plots and inconsistent axes making direct comparisons between the codes extremely difficult. It was also very difficult to ensure that a consistent modelling approach was used by the various investigators so that the comparison was based strictly on the differences between the programs and not the models used. An example of this with the Iltis vehicle would be modelling a leaf spring for which in many programs there were at the time no standard elements within the main code. Although not entirely successful the exercise was useful in being the only known attempt to provide a comparison between all the main multibody programs at the time. It should also be recognised that in the period since the exercise most of the commercial programs have been extensively developed to add a wide range of capability.

Anderson and Hanna (1989) have carried out an interesting study where they have used two vehicles to make a comparison of three different vehicle simulation

methodologies. They have also made use of the Iltis, a vehicle of German design, which at that time was the current small utility vehicle used by the Canadian military. The Iltis was a vehicle that was considered to have performed well and had very different characteristics to the M-151 jeep that was the other vehicle in this study. The authors state that the M-151 vehicle, also used by the Canadian military, had been declared unsafe due to a propensity for rolling over.

Work has been carried out at the University of Bath (Ross-Martin et al., 1992) where the authors have compared MSC ADAMS with their own hydraulic and simulation package. The results for both programs are compared with measured vehicle test data provided in this case by Ford. The Bath model is similar to the Roll Stiffness Model described later in this book but is based on a force roll centre as described by Dixon (1987). This requires the vehicle to actually exist so that the model can use measured inputs obtained through static rig measurements, using equipment of the type described by Whitehead (1995). The roll-centre model described in this book is based on a kinematic roll centre derived using a geometric construction as described in Chapter 4, though there is little to preclude a force-based prediction by modelling the test rig on which the real vehicle is measured.

As a guide to the complexity of the models discussed in Ross-Martin et al. (1992), the Bath model required 91 pieces of information and the MSC ADAMS model although not described in detail needed 380 pieces of information. It is also stated in this paper that the MSC ADAMS model used 150 sets of non-linear data pairs that suggests detailed modelling of all the non-linear properties of individual bushes throughout the vehicle.

Limited studies continue for specific modelling challenges, such as that surrounding mobility for tracked vehicles on deformable terrain. Madsen discusses a comparison between the empirically formulated and widely used Becker soil model and a causally modelled granular soil environment in a package called Chrono::Engine (Madsen et al., 2010). He rather casually notes that a hardware-induced limit of one thousand million (a US billion) contact bodies might preclude modelling grains of sand and call for a slightly coarser approach; again we can be confident that this will probably no longer be an issue for the third edition of this book.

Since the IAVSD benchmarking exercise no new wide-ranging studies have been performed, which is interesting in its own right. It could be interpreted as meaning that most of the software is regarded as ‘correct enough’ for the applications to which it is put. It is also interesting to note a general shift in terms of the acceptance of predictive modelling such that there is no longer a widely held belief that it does not add value — many organisations have success stories to tell, and software companies carefully collate them to give confidence to new customers.

This leads to an interesting swing of the pendulum; it used to be a popular aphorism that ‘nobody believes a simulation except the man who did it, and everyone believes a measurement — except the man who did it’. Currently though, it feels to the authors like some well-presented graphics can give predictive models a credibility out of all proportion to the estimated and uncertain data on which they are based and the often incomplete studies in which they are used.

The basic principle of science — that having formed a hypothesis one should diligently search for proof that it is untrue<sup>8</sup> appears to be cast aside in the rather convenient belief that the computer models are ‘probably right’ — when nothing of the sort is true. Basic aspects of self-doubt, such as convergence checking and input sensitivity checks, are often overlooked in a misplaced desire for productivity.

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<sup>8</sup>Also known as ‘Black Swan’ search; one may produce as many white swans as one likes and while it is consistent with the hypothesis that all swans are white, it is not proof. It takes only one black swan to disprove the hypothesis completely. Black swans are alive and well in Australia.

# Kinematics and Dynamics of Rigid Bodies

# 2

*My God, it's full of stars!*

Dave Bowman, 2001: A Space Odyssey

## 2.1 Introduction

The application of a modern multibody systems (MBS) computer program requires a good understanding of the underlying theory involved in the formulation and solution of the equations of motion. Due to the three-dimensional nature of the problem the theory is best described using vector algebra. In this chapter the starting point will be the basic definition of a vector and an explanation of the notation that will be used throughout this text. The vector theory will be developed to demonstrate, using examples based on suspension systems, the calculation of new geometry and changes in body orientation, such as the steer change in a road wheel during vertical motion relative to the vehicle body. This will be extended to show how velocities and accelerations may be determined throughout a linked three-dimensional system of rigid bodies. The definition of forces and moments will lead through to the definition of the full dynamic formulations typically used in a MBS analysis code.

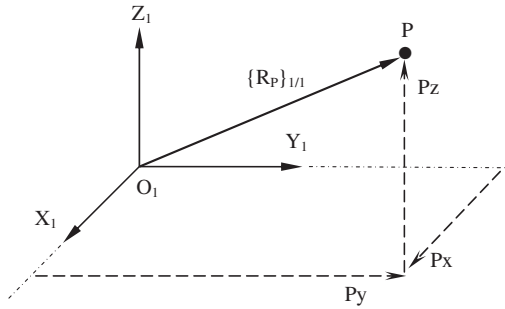
## 2.2 Theory of vectors

### 2.2.1 Position and relative position vectors

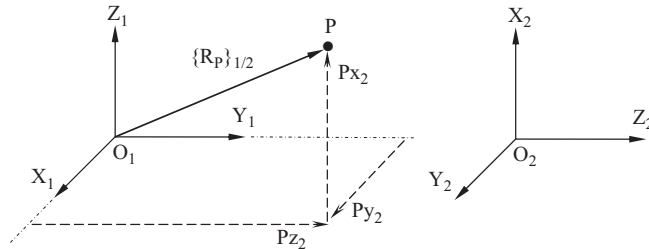
Consider the initial definition of the position vector that defines the location of point P in Figure 2.1.

In this case the vector that defines the position of P relative to the reference frame  $O_1$  may be completely described in terms of its components with magnitude  $P_x$ ,  $P_y$  and  $P_z$ . The directions of the components are defined by attaching the appropriate sign to their magnitudes.

$$\{R_P\}_{1/1} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \quad (2.1)$$

**FIGURE 2.1**

Position vector.

**FIGURE 2.2**

Resolution of position vector components.

The use of brackets  $\{ \}$  here is a shorthand representation of a column matrix and hence a vector. Note that it does not follow that any quantity that can be expressed as the terms in a column matrix is also a vector.

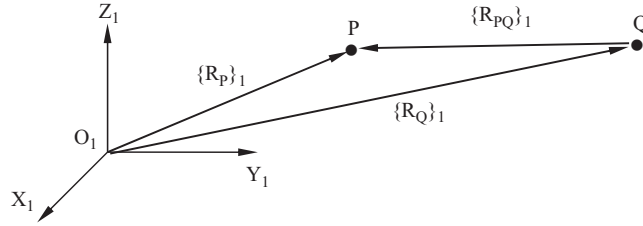
In writing the vector  $\{R_P\}_{1/1}$  the upper suffix indicates that the vector is measured relative to the axes of reference frame  $O_1$ . In order to measure a vector it is necessary to determine its magnitude and direction relative to the given axes, in this case  $O_1$ . It is then necessary to resolve it into components parallel to the axes of some reference frame that may be different from that used for measurement as shown in [Figure 2.2](#).

In this case we would write  $\{R_P\}_{1/2}$  where the lower suffix appended to  $\{R_P\}_{1/2}$  indicates the frame  $O_2$  in which the components are resolved. We can also say that in this case the vector is referred to  $O_2$ . Note that in most cases the two reference frames are the same and we would abbreviate  $\{R_P\}_{1/1}$  to  $\{R_P\}_1$ .

It is now possible in [Figure 2.3](#) to introduce the concept of a relative position vector  $\{R_{PQ}\}_1$ . The vector  $\{R_{PQ}\}_1$  is the vector from  $Q$  to  $P$ . It can also be described as the vector that describes the position of  $P$  relative to  $Q$ .

These vectors obey the triangle law for the addition and subtraction of vectors, which means that

$$\begin{aligned} \{R_{PQ}\}_1 &= \{R_P\}_1 - \{R_Q\}_1 \\ \text{or} \quad \{R_P\}_1 &= \{R_Q\}_1 + \{R_{PQ}\}_1 \end{aligned} \quad (2.2)$$



**FIGURE 2.3**

Relative position vector.

It also follows that we can write

$$\{R_{QP}\}_1 = \{R_Q\}_1 - \{R_P\}_1 \quad (2.3)$$

or

$$\{R_Q\}_1 = \{R_P\}_1 + \{R_{QP}\}_1$$

Application of Pythagoras' theorem will yield the magnitude  $|R_P|$  of the vector  $\{R_P\}_1$  as follows

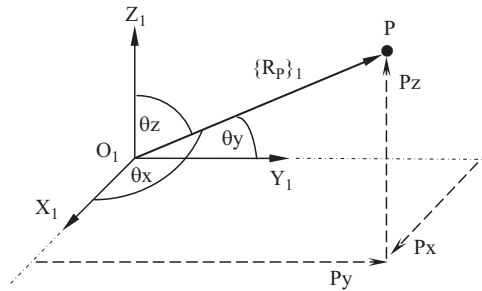
$$|R_P| = \sqrt{P_x^2 + P_y^2 + P_z^2} \quad (2.4)$$

Similarly the magnitude  $|R_{PQ}|$  of the relative position vector  $\{R_{PQ}\}_1$  can be obtained using

$$|R_{PQ}| = \sqrt{(P_x - Q_x)^2 + (P_y - Q_y)^2 + (P_z - Q_z)^2} \quad (2.5)$$

Consider now the angles  $\theta_x$ ,  $\theta_y$  and  $\theta_z$ , that the vector  $\{R_P\}_1$  makes with each of the X, Y and Z axes of frame  $O_1$  as shown in Figure 2.4. This gives the direction cosines  $l_x$ ,  $l_y$  and  $l_z$  of vector  $\{R_P\}_1$  where:

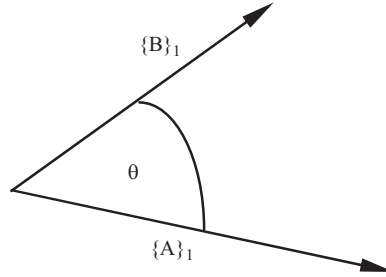
$$\begin{aligned} l_x &= \cos \theta_x = P_x / |R_P| \\ l_y &= \cos \theta_y = P_y / |R_P| \\ l_z &= \cos \theta_z = P_z / |R_P| \end{aligned} \quad (2.6)$$



**FIGURE 2.4**

Direction cosines.



**FIGURE 2.5**

Vector dot product.

These direction cosines are components of the vector  $\{l_p\}_1$  where

$$\{l_p\}_1 = \{R_p\}_1 / |R_p| \quad (2.7)$$

It can be seen that  $\{l_p\}_1$  has unit magnitude and is therefore a unit vector.

### 2.2.2 The dot (scalar) product

The dot, or scalar, product  $\{A\}_1 \bullet \{B\}_1$  of the vectors  $\{A\}_1$  and  $\{B\}_1$  yields a scalar  $C$  with magnitude equal to the product of the magnitude of each vector and the cosine of the angle between them (Figure 2.5).

Thus

$$\{A\}_1 \bullet \{B\}_1 = |C| = |A| |B| \cos \theta \quad (2.8)$$

The calculation of  $\{A\}_1 \bullet \{B\}_1$  requires the solution of

$$\{A\}_1 \bullet \{B\}_1 = \{A\}_1^T \{B\}_1 = A_x B_x + A_y B_y + A_z B_z \quad (2.9)$$

where

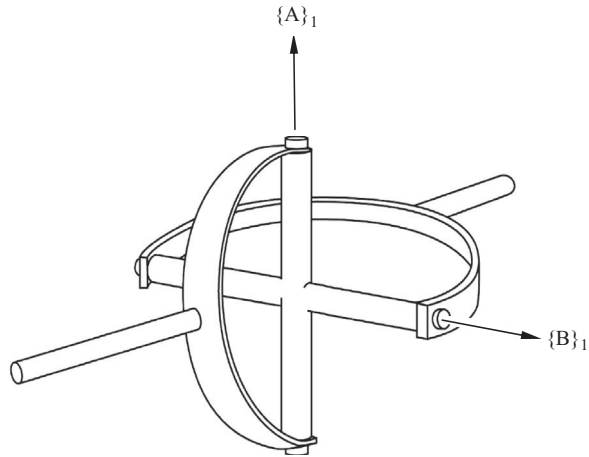
$$\{B\}_1 = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \quad \text{and} \quad \{A\}_1^T = [A_x \ A_y \ A_z] \quad (2.10)$$

The  $^T$  superscript in  $\{A\}_1^T$  indicates that the vector is transposed.

Clearly  $\{A\}_1 \bullet \{B\}_1 = \{B\}_1 \bullet \{A\}_1$  and the dot product is a commutative operation. The physical significance of the dot product will become apparent later but at this stage it can be seen that the angle  $\theta$  between two vectors  $\{A\}_1$  and  $\{B\}_1$  can be obtained from

$$\cos \theta = \{A\}_1 \bullet \{B\}_1 / |A| |B| \quad (2.11)$$

A particular case that is useful in the formulation of constraints representing joints and the like is the situation when  $\{A\}_1$  and  $\{B\}_1$  are perpendicular making  $\cos \theta = 0$ .

**FIGURE 2.6**

Application of the dot product to enforce perpendicularity.

As can be seen in [Figure 2.6](#) the equation that enforces the perpendicularity of the two spindles in the universal joint can be obtained from

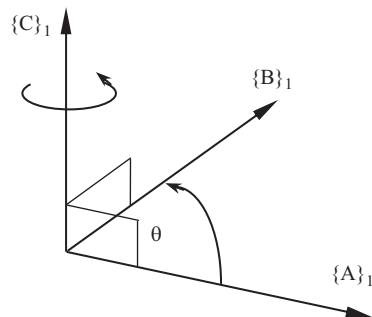
$$\{A\}_1 \bullet \{B\}_1 = 0 \quad (2.12)$$

### 2.2.3 The cross (vector) product

The cross, or vector, product of two vectors,  $\{A\}_1$  and  $\{B\}_1$  is another vector  $\{C\}_1$  given by

$$\{C\}_1 = \{A\}_1 \times \{B\}_1 \quad (2.13)$$

The vector  $\{C\}_1$  is perpendicular to the plane containing  $\{A\}_1$  and  $\{B\}_1$  as shown in [Figure 2.7](#).

**FIGURE 2.7**

Vector cross product.

The magnitude of  $\{C\}_1$  is defined as

$$|C| = |A| |B| \sin \theta \quad (2.14)$$

The direction of  $\{C\}_1$  is defined by a positive rotation about  $\{C\}_1$  rotating  $\{A\}_1$  into line with  $\{B\}_1$ . The calculation of  $\{A\}_1 \times \{B\}_1$  requires  $\{A\}_1$  to be arranged in skew-symmetric form as follows

$$\{C\}_1 = \{A\}_1 \times \{B\}_1 = [A]_1 \{B\}_1 = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \quad (2.15)$$

Multiplying this out would give the vector  $\{C\}_1$

$$\{C\}_1 = \begin{bmatrix} -A_z B_y + A_y B_z \\ A_z B_x - A_x B_z \\ -A_y B_x + A_x B_y \end{bmatrix} \quad (2.16)$$

Exchange of  $\{A\}_1$  and  $\{B\}_1$  will show that the cross product operation is not commutative and that

$$\{A\}_1 \times \{B\}_1 = -\{B\}_1 \times \{A\}_1 \quad (2.17)$$

### 2.2.4 The scalar triple product

The scalar triple product  $D$  of the vectors  $\{A\}_1$ ,  $\{B\}_1$  and  $\{C\}_1$  is defined as

$$D = \{\{A\}_1 \times \{B\}_1\} \bullet \{C\}_1 \quad (2.18)$$

### 2.2.5 The vector triple product

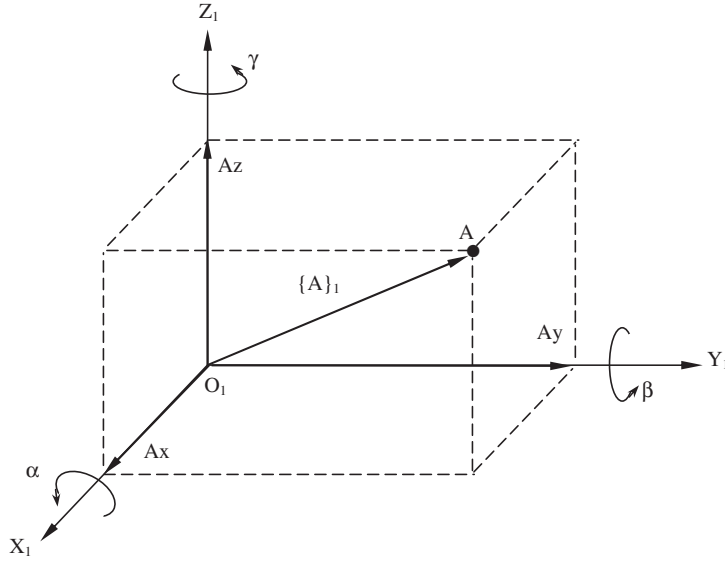
The vector triple product  $\{D\}_1$  of the vectors  $\{A\}_1$ ,  $\{B\}_1$  and  $\{C\}_1$  is defined as

$$\{D\}_1 = \{A\}_1 \times \{\{B\}_1 \times \{C\}_1\} \quad (2.19)$$

### 2.2.6 Rotation of a vector

In multibody dynamics bodies may undergo motion, which involves rotation about all three axes of a given reference frame. The new components of a vector  $\{A\}_1$ , shown in [Figure 2.8](#), may be determined as it rotates through an angle  $\alpha$  about the  $X_1$ -axis,  $\beta$  about the  $Y_1$ -axis, and  $\gamma$  about the  $Z_1$ -axis of frame  $O_1$ .

Consider first the rotation  $\alpha$  about  $O_1X_1$ . The component  $A_x$  is unchanged. The new components  $A_x'$ ,  $A_y'$  and  $A_z'$  can be found by viewing along the  $X_1$ -axis as shown in [Figure 2.9](#).

**FIGURE 2.8**

Rotation of a vector.

By inspection it is found that

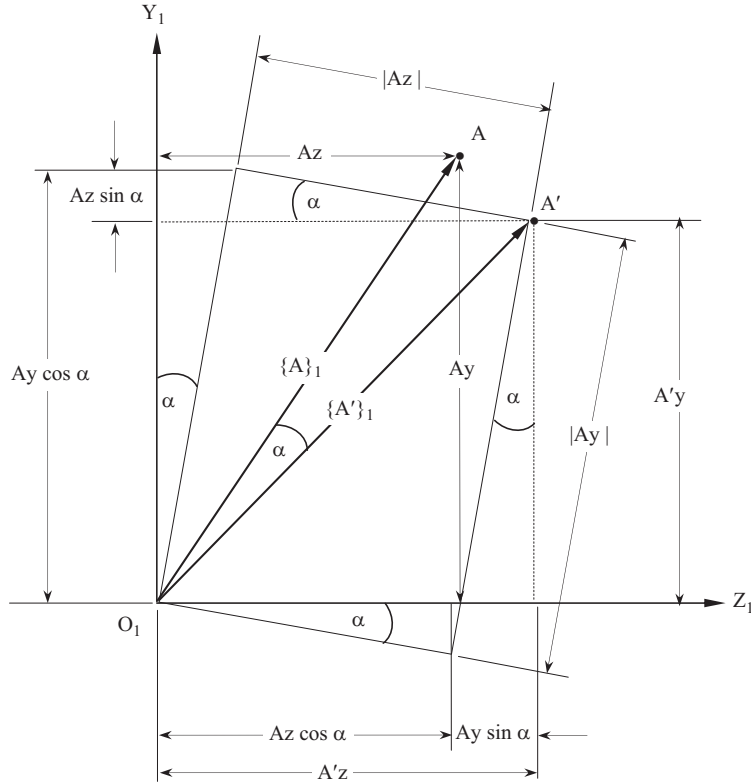
$$\begin{aligned} Ax' &= Ax \\ Ay' &= Ay \cos \alpha - Az \sin \alpha \\ Az' &= Ay \sin \alpha + Az \cos \alpha \end{aligned} \quad (2.20)$$

In matrix form this can be written

$$\begin{bmatrix} Ax' \\ Ay' \\ Az' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} \quad (2.21)$$

In a similar manner when  $\{A\}_1$  is rotated through an angle  $\beta$  about the  $Y_1$ -axis the components of a new vector  $\{A'\}_1$  can be obtained from

$$\begin{bmatrix} Ax' \\ Ay' \\ Az' \end{bmatrix} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} \quad (2.22)$$

**FIGURE 2.9**

Rotation of a vector viewed along the  $X_1$ -axis.

After a final rotation of  $\{A\}_1$  through an angle  $\gamma$  about the  $Z_1$ -axis the new components of  $\{A'\}_1$  are given by

$$\begin{bmatrix} Ax' \\ Ay' \\ Az' \end{bmatrix} = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} \quad (2.23)$$

Applying all three rotations in the sequence  $\alpha$ ,  $\beta$  and  $\gamma$  would result in the three rotation matrices being multiplied through as follows

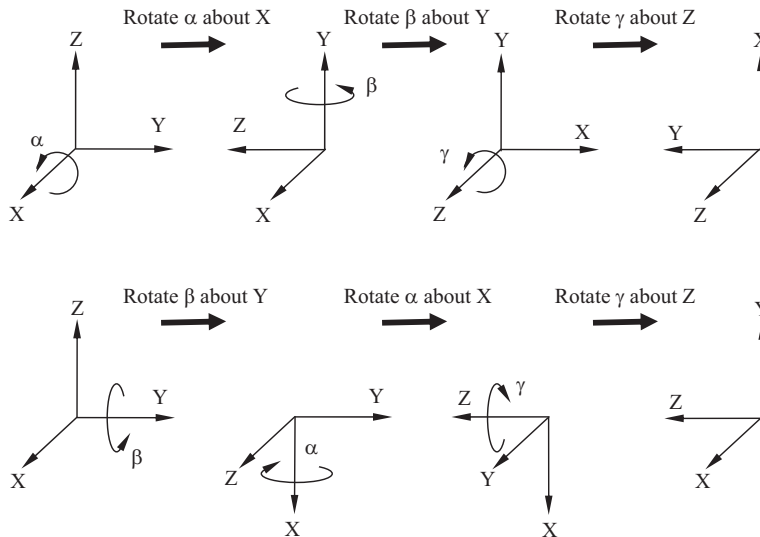
$$\begin{bmatrix} Ax' \\ Ay' \\ Az' \end{bmatrix} = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} \quad (2.24)$$

$$\begin{bmatrix} Ax' \\ Ay' \\ Az' \end{bmatrix} = \begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta & \sin \beta \sin \alpha & \sin \beta \cos \alpha \\ 0 & \cos \alpha & -\sin \alpha \\ -\sin \beta & \cos \beta \sin \alpha & \cos \beta \cos \alpha \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} \quad (2.25)$$

$$\begin{bmatrix} Ax' \\ Ay' \\ Az' \end{bmatrix} = \begin{bmatrix} \cos \gamma \cos \beta & \cos \gamma \sin \beta \sin \alpha - \sin \gamma \cos \alpha & \cos \gamma \sin \beta \cos \alpha + \sin \gamma \sin \alpha \\ \sin \gamma \cos \beta & \sin \gamma \sin \beta \sin \alpha + \cos \gamma \cos \alpha & \sin \gamma \sin \beta \cos \alpha - \cos \gamma \sin \alpha \\ -\sin \beta & \cos \beta \sin \alpha & \cos \beta \cos \alpha \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} \quad (2.26)$$

It should be noted that large rotations such as these are not commutative and therefore the angles  $\alpha$ ,  $\beta$  and  $\gamma$  cannot be considered to be the components of a vector. The order in which the rotations are applied is important. As can be seen in [Figure 2.10](#), applying equal rotations of  $90^\circ$  but in a different sequence will not result in the same final orientation of the vector.

An understanding that large rotations are not a vector is an important aspect of MBS analysis. Sets of rotations may be required as inputs to define the orientation of a rigid body or a joint. They will also form the output when the relative orientation of one body to another is requested. Note that the convention used here in [Figure 2.10](#) is based on a set of Body (X–Y–Z) rotations. It will be shown later that different conventions may be used, such as the Yaw-Pitch-Roll method based on a set of Body (Z–Y–X) rotations or the Euler angle method used in MSC.ADAMS that is based on a Body (Z–X–Z) combination.



**FIGURE 2.10**

The effect of rotation sequence on vector orientation.

### 2.2.7 Vector transformation

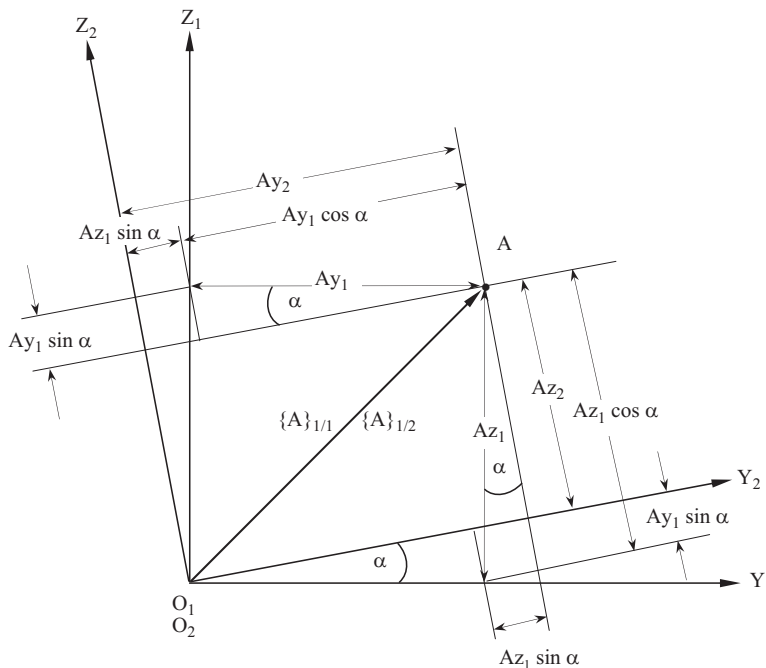
In MBS analysis it is often necessary to transform the components of a vector measured parallel to the axis of one reference frame to those measured parallel to a second reference frame. These operations should not be confused with vector rotation. In a transformation it is the magnitude and direction of the components that changes. The direction of the vector is unchanged. Consider the transformation of a vector  $\{A\}_1$ , or in full definition  $\{A\}_{1/1}$ , from reference frame  $O_1$  to reference frame  $O_2$ . Figure 2.11 represents a view back along the  $X_1$ -axis towards the origin  $O_1$ . The reference frame  $O_2$  is rotated through an angle  $\alpha$  about the  $X_1$ -axis of frame  $O_1$ .

From Figure 2.11 it can be seen that

$$\begin{aligned} Ax_2 &= Ax_1 \\ Ay_2 &= Ay_1 \cos \alpha + Az_1 \sin \alpha \\ Az_2 &= -Ay_1 \sin \alpha + Az_1 \cos \alpha \end{aligned} \quad (2.27)$$

In matrix form this can be written

$$\{A\}_{1/2} = \begin{bmatrix} Ax_2 \\ Ay_2 \\ Az_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} Ax_1 \\ Ay_1 \\ Az_1 \end{bmatrix} \quad (2.28)$$



**FIGURE 2.11**

Transformation of a vector.

This equation may be expressed as

$$\{A\}_{1/2} = [T_1]_2 \{A\}_{1/1} \quad (2.29)$$

Consider next the transformation of the vector  $\{A\}_{1/1}$  from reference frame  $O_1$  to reference frame  $O_3$ . The reference frame  $O_3$  is rotated through an angle  $\beta$  about the  $Y_1$ -axis of frame  $O_1$ . Following the same procedure as before we get

$$\{A\}_{1/3} = \begin{bmatrix} Ax_3 \\ Ay_3 \\ Az_3 \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} Ax_1 \\ Ay_1 \\ Az_1 \end{bmatrix} \quad (2.30)$$

$$\{A\}_{1/3} = [T_1]_3 \{A\}_{1/1} \quad (2.31)$$

Finally consider the transformation to frame  $O_4$  where  $O_4$  is obtained from a rotation of  $\gamma$  about the  $Z_1$ -axis of frame  $O_1$ .

$$\{A\}_{1/4} = \begin{bmatrix} Ax_4 \\ Ay_4 \\ Az_4 \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Ax_1 \\ Ay_1 \\ Az_1 \end{bmatrix} \quad (2.32)$$

$$\{A\}_{1/4} = [T_1]_4 \{A\}_{1/1} \quad (2.33)$$

The square transformation matrices  $[T_1]_2$ ,  $[T_1]_3$  and  $[T_1]_4$  are the inverses of the rotation matrices developed in [Section 2.2.6](#). This is to be expected since the method here is the reverse of that shown previously where the vector, rather than the frame, was rotated. It should be noted that a transformation matrix  $[T_m]_p$ , which transforms a vector from frame  $m$  to frame  $p$ , has a transpose  $[T_m]_p^T$  that is also its inverse  $[T_m]_p^{-1}$ .

In general terms the transformation of a vector from one frame  $m$  to another frame  $p$  may be written as

$$\{A\}_{n/p} = [T_m]_p \{A\}_{n/m} \quad (2.34)$$

### 2.2.8 Differentiation of a vector

The differentiation of a vectors  $\{A\}_1$  with respect to a scalar variable, such as time  $t$ , results in another vector given by

$$\frac{d}{dt} \{A\}_1 = \begin{bmatrix} \frac{dAx}{dt} \\ \frac{dAy}{dt} \\ \frac{dAz}{dt} \end{bmatrix} \quad (2.35)$$



Differentiation with respect to time is often denoted by the Newtonian dot giving

$$\{\dot{A}\}_1 = \begin{bmatrix} \dot{A}_x \\ \dot{A}_y \\ \dot{A}_z \end{bmatrix} \quad (2.36)$$

If the frames used for measurement and reference differ it is necessary to distinguish between, for example,  $\frac{d}{dt} \{A\}_{m/n}$  and  $\{\dot{A}\}_{m/n}$  since

$$\frac{d}{dt} \{A\}_{m/n} \neq \{\dot{A}\}_{m/n} \quad (2.37)$$

In evaluating  $\frac{d}{dt} \{A\}_{m/n}$ , we measure  $\{A\}$  in frame m, transform to frame n and then differentiate  $\{A\}_{m/n}$  with respect to time. The notation  $\{\dot{A}\}_{m/n}$ , however, implies that  $\{\dot{A}\}_m$  is determined first and that this vector is then transformed to frame n.

Consider the vector  $\{A\}_{1/1}$  shown in Figure 2.12. The vector lies in the  $X_1 Y_1$  plane of frame  $O_1$  and rotates at a constant speed of  $\omega$  rad/s about the  $Z_1$ -axis. Frame 2 has its  $Z$ -axis coincident with  $Oz_1$  and its  $X$ -axis is coincident with and rotates with  $\{A\}_{1/1}$ .

If  $OX_1$  and  $OX_2$  were coincident at time  $t = 0$ , then where  $A$  is the magnitude of  $\{A\}_{1/1}$

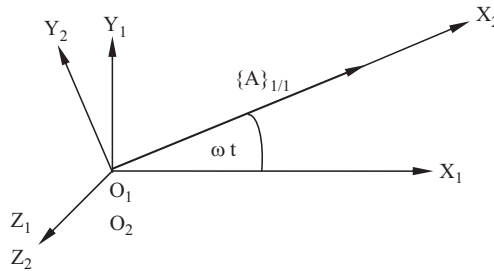
$$\{A\}_{1/1}^T = [A \cos \omega t \quad A \sin \omega t \quad 0] \quad (2.38)$$

Transforming to frame  $O_2$  gives

$$\{A\}_{1/2}^T = [A \quad 0 \quad 0] \quad (2.39)$$

Differentiating this with respect to time gives the following since the magnitude  $A$  does not vary with time.

$$\frac{d}{dt} \{A\}_{1/2}^T = [0 \quad 0 \quad 0] \quad (2.40)$$



**FIGURE 2.12**

Vector rotating with constant velocity.

Now

$$\{\dot{A}\}_{1/1}^T = [-\omega A \sin \omega t \quad \omega A \cos \omega t \quad 0] \quad (2.41)$$

and

$$\{\dot{A}\}_{1/2}^T = [0 \quad \omega A \quad 0] \quad (2.42)$$

proving that

$$\frac{d}{dt} \{A\}_{1/2} \neq \{\dot{A}\}_{1/2} \quad (2.43)$$

### 2.2.9 Integration of a vector

Integration of the vector  $\{A\}_1$  with respect to the scalar variable  $t$  is given by

$$\int \{A\}_1 dt = \begin{bmatrix} \int A_x dt \\ \int A_y dt \\ \int A_z dt \end{bmatrix} \quad (2.44)$$

### 2.2.10 Differentiation of the dot product

The dot product of two vectors  $\{A\}_1$  and  $\{B\}_1$  is given by

$$\{A\}_1 \bullet \{B\}_1 = \{A\}_1^T \{B\}_1 = [A_x \ B_x \ + \ A_y \ B_y \ + \ A_z \ B_z] \quad (2.45)$$

Differentiation of  $\{A\}_1 \bullet \{B\}_1$  with respect to time  $t$  gives

$$\frac{d}{dt} (\{A\}_1 \bullet \{B\}_1) = [A_x \dot{B}_x + B_x \dot{A}_x + A_y \dot{B}_y + B_y \dot{A}_y + A_z \dot{B}_z + B_z \dot{A}_z] \quad (2.46)$$

or

$$\frac{d}{dt} (\{A\}_1 \bullet \{B\}_1) = \{A\}_1 \bullet \{\dot{B}\}_1 + \{\dot{A}\}_1 \bullet \{B\}_1 \quad (2.47)$$

The result of this is a scalar. The rule for differentiation of the dot product is similar to the differentiation of the product of two scalars  $u$  and  $v$ .

$$\frac{d}{dt} (uv) = u \frac{dv}{dt} + v \frac{du}{dt} \quad (2.48)$$

### 2.2.11 Differentiation of the cross product

The cross product of two vectors  $\{A\}_1$  and  $\{B\}_1$  is given by

$$\{A\}_1 \times \{B\}_1 = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} -A_z B_y + A_y B_z \\ A_z B_x - A_y B_z \\ -A_y B_x + A_x B_y \end{bmatrix} \quad (2.49)$$

Differentiation of this vector with respect to time  $t$  gives

$$\frac{d}{dt} (\{A\}_1 \times \{B\}_1) = \begin{bmatrix} -A_z \dot{B}_y - \dot{A}_z B_y + A_y \dot{B}_z + \dot{A}_y B_z \\ A_z \dot{B}_x + \dot{A}_z B_x - A_x \dot{B}_z - \dot{A}_y B_z \\ -A_y \dot{B}_x - \dot{A}_y B_x + A_x \dot{B}_y + \dot{A}_x B_y \end{bmatrix} \quad (2.50)$$

or

$$\frac{d}{dt} (\{A\}_1 \times \{B\}_1) = \{A\}_1 \times \{\dot{B}\}_1 + \{\dot{A}\}_1 \times \{B\}_1 \quad (2.51)$$

The result of this operation is a vector. Note again that the rule for the differentiation of the cross product is similar to the rule for the differentiation of the product of two scalars.

### 2.2.12 Summary

1. A vector is expressed in terms of the magnitudes of its three orthogonal components listed, in natural order, as the elements of a column matrix. The brackets  $\{ \}$  are used as a shorthand representation of a vector. Thus the vector  $\{A\}_m$  with components of magnitude  $A_x$ ,  $A_y$  and  $A_z$  is represented by

$$\{A\}_m = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (2.52)$$

2. The suffices  $m$  and  $n$  appended to the vector  $\{A\}_{m/n}$  indicate that its components were measured relative to the axes of frame  $m$  but that the vector was then resolved into components parallel to the axes of frame  $n$ . We say that the vector is referred or transformed to frame  $n$ . If  $m$  is equal to  $n$  then the vector may be written as  $\{A\}_m$ .
3. The magnitude of the vector  $\{A\}_m$  is represented by  $|A|$ , or simply by  $A$  when the meaning is unambiguous. This magnitude is given by

$$|A| = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (2.53)$$

4. The cosines of the angles that  $\{A\}_m$  makes with the X, Y and Z axes respectively of frame m are known as its direction cosines  $\{l\}_m$ . This vector is derived from  $\{A\}_m$  as follows

$$\{l\}_m = \{A\}_m / |A| \quad (2.54)$$

Since the magnitude of  $\{l\}_m$  is unity, it is called a unit vector.

5. If  $\{A\}_m$ ,  $\{B\}_m$  and  $\{C\}_m$  are vectors of the same dimensions and  $\{C\}_m$  is the resultant of  $\{A\}_m$  and  $\{B\}_m$  then the equation

$$\{C\}_m = \{A\}_m + \{B\}_m \quad (2.55)$$

expresses the triangle law for the addition or subtraction of vectors. This equation is only valid if all vectors are referred to the same frame.

6. The dot product  $\{A\}_m \cdot \{B\}_m$  of the vectors  $\{A\}_m$  and  $\{B\}_m$  is defined as a scalar whose magnitude is  $|A| |B| \cos \theta$ ,  $\theta$  being the angle between the vectors. The dot product is evaluated in terms of a matrix product as follows

$$\{A\}_m \cdot \{B\}_m = \{A\}_m^T \{B\}_m \quad (2.56)$$

7. The cross product  $\{A\}_m \times \{B\}_m$  of the vectors  $\{A\}_m$  and  $\{B\}_m$  is defined as a vector  $\{C\}_m$  whose magnitude is  $|A| |B| \sin \theta$ ,  $\theta$  being the angle between  $\{A\}_m$  and  $\{B\}_m$ . The vector  $\{C\}_m$  is perpendicular to the plane containing the other two and its direction is the direction of advance of a right-handed screw, lying parallel to  $\{C\}_m$ , when subjected to a rotation which would bring  $\{A\}_m$  into alignment with  $\{B\}_m$  by the shortest path. The cross product is evaluated in terms of a matrix product as follows

$$\{C\}_m = \{A\}_m \times \{B\}_m = [A]_m \{B\}_m \quad (2.57)$$

where

$$[A]_m = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix} \text{ and } \{B\}_m = \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} \quad (2.58)$$

$[A]_m$  is known as the skew-symmetric form of the vector  $\{A\}_m$ .

8. The angle  $\theta$  between the vectors  $\{A\}_m$  and  $\{B\}_m$  may be determined from

$$\cos \theta = \{A\}_m^T \{B\}_m / |A| |B| \quad (2.59)$$

9. If vectors  $\{A\}_m$  and  $\{B\}_m$  are parallel then  $\{A\}_m$  can be represented using a scalar f as follows

$$\{A\}_m = f \{B\}_m \quad (2.60)$$

10. If vectors  $\{A\}_m$  and  $\{B\}_m$  are perpendicular then

$$\{A\}_m \bullet \{B\}_m = \{A\}_m^T \{B\}_m = 0 \quad (2.61)$$

11. The scalar triple product  $D$  of the vectors  $\{A\}_m$ ,  $\{B\}_m$  and  $\{C\}_m$  is a scalar defined by

$$D = \{\{A\}_m \times \{B\}_m\} \bullet \{C\}_m \quad (2.62)$$

12. The vector triple product  $\{D\}_m$  of the vectors  $\{A\}_m$ ,  $\{B\}_m$  and  $\{C\}_m$  is a vector  $\{D\}_m$  defined by

$$\{D\}_m = \{A\}_m \times \{\{B\}_m \times \{C\}_m\} \quad (2.63)$$

13. If the vector  $\{A\}_m^T = [A_x \ A_y \ A_z]$  is rotated through angle  $+\alpha$  about the  $x$ -axis of frame  $m$  then the new vector  $\{A'\}_m$  is given by

$$\{A'\}_m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (2.64)$$

For rotation about  $O_m Y_m$  and  $O_m Z_m$ , the square matrix above is replaced by those given in [Eqns \(2.22\)](#) and [\(2.23\)](#)

14. If the  $X$ -axes of frames  $m$  and  $p$  coincide and the  $Y$ -axis of frame  $p$  is rotated by  $+\alpha$ , relative to the corresponding axis of frame  $m$ , then the vector  $\{A\}_{n/m}$  is transformed from frame  $m$  to frame  $p$  according to the relationship

$$\{A\}_{n/p} = [T_m]_p \{A\}_{n/m} \quad (2.65)$$

where

$$\{T\}_{m/p} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \quad (2.66)$$

The other two transformations are given by [Eqns \(2.30\)](#) and [\(2.32\)](#).

15. The transformations of 14 above may be combined as indicated.

$$[T_m]_q = [T_p]_q [T_n]_p [T_m]_n \quad (2.67)$$

This chain of matrices is to be read from right to left.

16. Differentiation of the vector  $\{A\}_m^T = [A_x \ A_y \ A_z]$  with respect to the scalar variable  $t$  is defined by

$$\frac{d}{dt} \{A\}_m = \begin{bmatrix} \frac{dA_x}{dt} \\ \frac{dA_y}{dt} \\ \frac{dA_z}{dt} \end{bmatrix} \quad (2.68)$$

The result of this operation is a vector.

17. In general,

$$\frac{d}{dt} \{A\}_{m/n} \neq \{\dot{A}\}_{m/n} \quad (2.69)$$

where the dot denotes differentiation with respect to time.

18. Integration of the vector  $\{A\}_m^T = [A_x \ A_y \ A_z]$  with respect to the scalar variable  $t$  produces another vector defined by

$$\int \{A\}_m dt = \begin{bmatrix} \int A_x dt \\ \int A_y dt \\ \int A_z dt \end{bmatrix} \quad (2.70)$$

19. Differentiation of the dot product  $\{A\}_m \cdot \{B\}_m$  with respect to time  $t$  is defined by

$$\frac{d}{dt} (\{A\}_m \cdot \{B\}_m) = \{A\}_m \cdot \{\dot{B}\}_m + \{\dot{A}\}_m \cdot \{B\}_m \quad (2.71)$$

where

$$\{\dot{A}\}_m = \frac{d}{dt} \{A\}_m \text{ and } \{\dot{B}\}_m = \frac{d}{dt} \{B\}_m$$

20. Differentiation of the cross product  $\{A\}_m \times \{B\}_m$  with respect to time  $t$  follows the same rule as that for the dot product. Hence,

$$\frac{d}{dt} (\{A\}_m \times \{B\}_m) = \{A\}_m \times \{\dot{B}\}_m + \{\dot{A}\}_m \times \{B\}_m \quad (2.72)$$

## 2.3 Geometry analysis

### 2.3.1 Three point method

In order to establish the position of any point in space, vector theory can be used to work from three points for which the coordinates are already established. Consider the following example shown in [Figure 2.13](#).

In this example the positions of A, B and C are taken to be known as are the lengths AD, BD and CD. The position of D is unknown and must be solved. In terms of vectors this can be expressed using the following known inputs:

$$\{\mathbf{R}_A\}_1^T = [A_x \ A_y \ A_z]$$

$$\{\mathbf{R}_B\}_1^T = [B_x \ B_y \ B_z]$$

$$\{\mathbf{R}_C\}_1^T = [C_x \ C_y \ C_z]$$

$$|\mathbf{R}_{DA}|$$

$$|\mathbf{R}_{DB}|$$

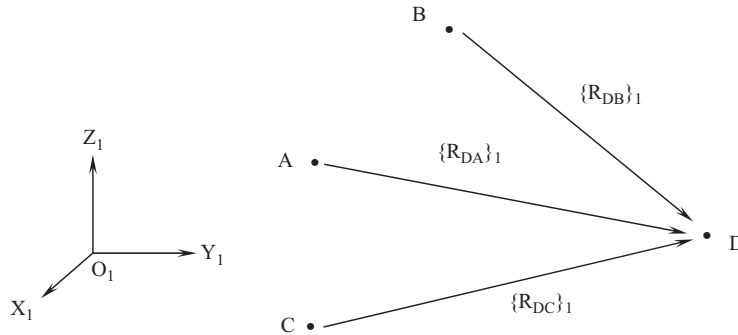
$$|\mathbf{R}_{DC}|$$

In order to solve the three unknowns  $D_x$ ,  $D_y$  and  $D_z$ , which are the components of the position vector  $\{\mathbf{R}_D\}_1$ , it is necessary to set up three equations as follows:

$$|\mathbf{R}_{DA}|^2 = (D_x - A_x)^2 + (D_y - A_y)^2 + (D_z - A_z)^2 \quad (2.73)$$

$$|\mathbf{R}_{DB}|^2 = (D_x - B_x)^2 + (D_y - B_y)^2 + (D_z - B_z)^2 \quad (2.74)$$

$$|\mathbf{R}_{DC}|^2 = (D_x - C_x)^2 + (D_y - C_y)^2 + (D_z - C_z)^2 \quad (2.75)$$



**FIGURE 2.13**

Use of position vectors for geometry analysis.

Multiplying out these three equations leads to:

$$|R_{DA}|^2 = (Dx^2 - 2Dx Ax + Ax^2) + (Dy^2 - 2Dy Ay + Ay^2) + (Dz^2 - 2Dz Az + Az^2) \quad (2.76)$$

$$|R_{DB}|^2 = (Dx^2 - 2Dx Bx + Bx^2) + (Dy^2 - 2Dy By + By^2) + (Dz^2 - 2Dz Bz + Bz^2) \quad (2.77)$$

$$|R_{DC}|^2 = (Dx^2 - 2Dx Cx + Cx^2) + (Dy^2 - 2Dy Cy + Cy^2) + (Dz^2 - 2Dz Cz + Cz^2) \quad (2.78)$$

At this stage we need to introduce some numerical data to demonstrate how a solution can be obtained. This will also demonstrate how cumbersome the algebra will become and the need to utilise computer software to solve these problems. As an example we can take the following coordinates for points A, B and C:

$$\{R_A\}_1 = [103 \ 350 \ 142] \text{ mm}$$

$$\{R_B\}_1 = [-127 \ 350 \ 128] \text{ mm}$$

$$\{R_C\}_1 = [-15 \ 500 \ 540] \text{ mm}$$

The lengths of the three rigid links can be taken as:

$$|R_{DA}| = 172.064 \text{ mm}$$

$$|R_{DB}| = 183.527 \text{ mm}$$

$$|R_{DC}| = 401.103 \text{ mm}$$

Substituting these numerical values into the equations above we get

$$172.064^2 = (Dx - 103)^2 + (Dy - 350)^2 + (Dz - 142)^2$$

$$183.527^2 = (Dx + 127)^2 + (Dy - 350)^2 + (Dz - 128)^2$$

$$401.103^2 = (Dx + 15)^2 + (Dy - 500)^2 + (Dz - 540)^2$$

Multiplying out gives

$$\begin{aligned} 29606.02 &= (Dx^2 - 206 Dx + 10609) + (Dy^2 - 700 Dy + 122500) \\ &\quad + (Dz^2 - 284 Dz + 20164) \end{aligned}$$

$$\begin{aligned} 33682.16 &= (Dx^2 + 254 Dx + 16129) + (Dy^2 - 700 Dy + 122500) \\ &\quad + (Dz^2 - 256 Dz + 16384) \end{aligned}$$

$$\begin{aligned} 160883.617 &= (Dx^2 + 30 Dx + 225) + (Dy^2 - 1000 Dy + 250000) \\ &\quad + (Dz^2 - 1080 Dz + 291600) \end{aligned}$$



Combining the terms in these three equations leads to

$$-123667 = (Dx^2 - 206 Dx) + (Dy^2 - 700 Dy) + (Dz^2 - 284 Dz) \quad (2.79)$$

$$-121331 = (Dx^2 + 254 Dx) + (Dy^2 - 700 Dy) + (Dz^2 - 256 Dz) \quad (2.80)$$

$$-380942 = (Dx^2 + 30 Dx) + (Dy^2 - 1000 Dy) + (Dz^2 - 1080 Dz) \quad (2.81)$$

Subtracting Eqn (2.79) from Eqn (2.80) gives

$$2336 = 460 Dx + 28 Dz \quad (2.82)$$

$$Dz = 83.43 - 16.43 Dx$$

Subtracting Eqn (2.81) from Eqn (2.80) gives

$$259611 = 224 Dx + 300 Dy + 824 Dz \quad (2.83)$$

Substituting the expression for Dz given in Eqn (2.82) into Eqn (2.83) gives:

$$259611 = 224 Dx + 300 Dy + 68746.32 - 13538.32 Dx$$

$$190864.68 = -13314.32 Dx + 300 Dy$$

This gives

$$Dy = 636.22 + 44.38 Dx$$

Substituting the Dy and Dz back into Eqn (2.79) gives

$$\begin{aligned} -123667 = & (Dx^2 - 206 Dx) + (636.22^2 + 1969.58 Dx^2 - 445354 - 31066 Dx) + \\ & (83.43^2 + 269.9 Dx^2 - 23694.12 + 4666.12 Dx) - 66355.33 = 2240.52 Dx^2 \\ & + 27123.5 Dx \end{aligned} \quad (2.84)$$

Rearranging Eqn (2.84) gives

$$2240.48 Dx^2 + 27123.5 Dx + 66355.33 = 0 \quad (2.85)$$

This is now in the familiar form of a quadratic equation  $ax^2 + bx + c = 0$  for which the solution is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore the solution to Eqn (2.85) is obtained from

$$Dx = \frac{26605.88 \pm \sqrt{26605.88^2 - 4 \times 2240.48 \times 66355.33}}{2 \times 2240.48}$$

This gives two solutions

$$Dx = -3.403\text{mm or } -8.703\text{mm}$$

The fact that there are two possible solutions for  $Dx$  illustrates the nonlinearity of this geometric analysis. In this case inspection of the two solutions does not immediately identify which one should be eliminated. Trying each value in the equation for  $Dy$  gives

$$Dy = 636.22 + 44.38 \times (-3.403) = 485.19\text{mm}$$

or

$$Dy = 636.22 + 44.38 \times (-8.703) = 249.98\text{mm}$$

Similarly trying each value of  $Dx$  in the equation for  $Dz$  gives

$$Dz = 83.43 - 16.43 \times (-3.403) = 139.34\text{mm}$$

or

$$Dz = 83.43 - 16.43 \times (-8.703) = 226.42\text{mm}$$

Using a computer program, written in BASIC, to check these answers gives the following two solutions:

$$Dx = -3.403\text{mm or } -8.703\text{mm}$$

$$Dy = 485.194\text{mm or } 249.986\text{mm}$$

$$Dz = 139.338\text{mm or } 226.413\text{mm}$$

In this case the first solution can be identified as the correct one, by inspection only, on the evidence of the  $Dy$  coordinate.

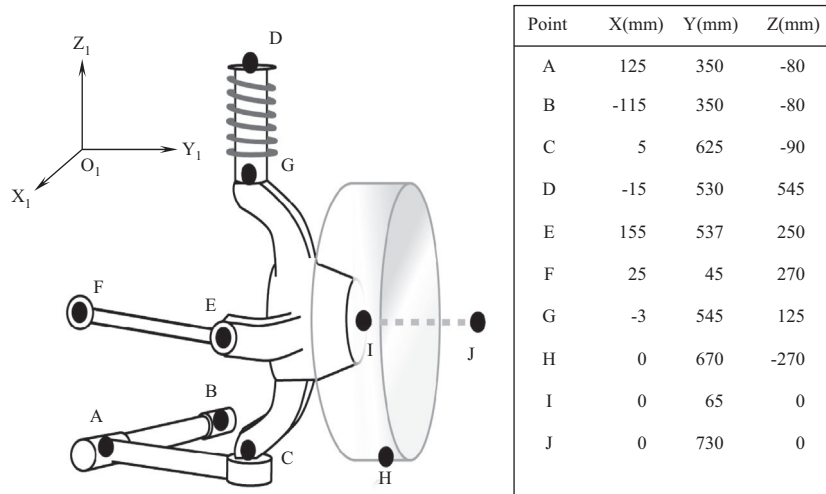
### 2.3.2 Vehicle suspension geometry analysis

The following example, based on a McPherson Strut type of suspension, illustrates the steps that could be followed to determine the geometry for the suspension system shown in [Figure 2.14](#).

The purpose of the analysis is to devise a sequence of calculations that would allow the calculation of the positions of the movable points in the suspension,  $\{R_C\}_1$ ,  $\{R_E\}_1$ ,  $\{R_G\}_1$ ,  $\{R_I\}_1$ ,  $\{R_J\}_1$  and  $\{R_H\}_1$ , once the suspension is displaced from the coordinates given in [Figure 2.14](#). In this example, the two points I and J are used to define the axis of rotation of the wheel.

The following sequence, shown in [Figure 2.15](#), can be used to locate the moveable points.

In order to impart motion to the suspension, resulting from for example hitting a bump, we can shorten the strut DG by a given amount. This will result in links DE and DC also shortening. Points C, E and G will move to new positions that can be

**FIGURE 2.14**

Suspension geometry data.

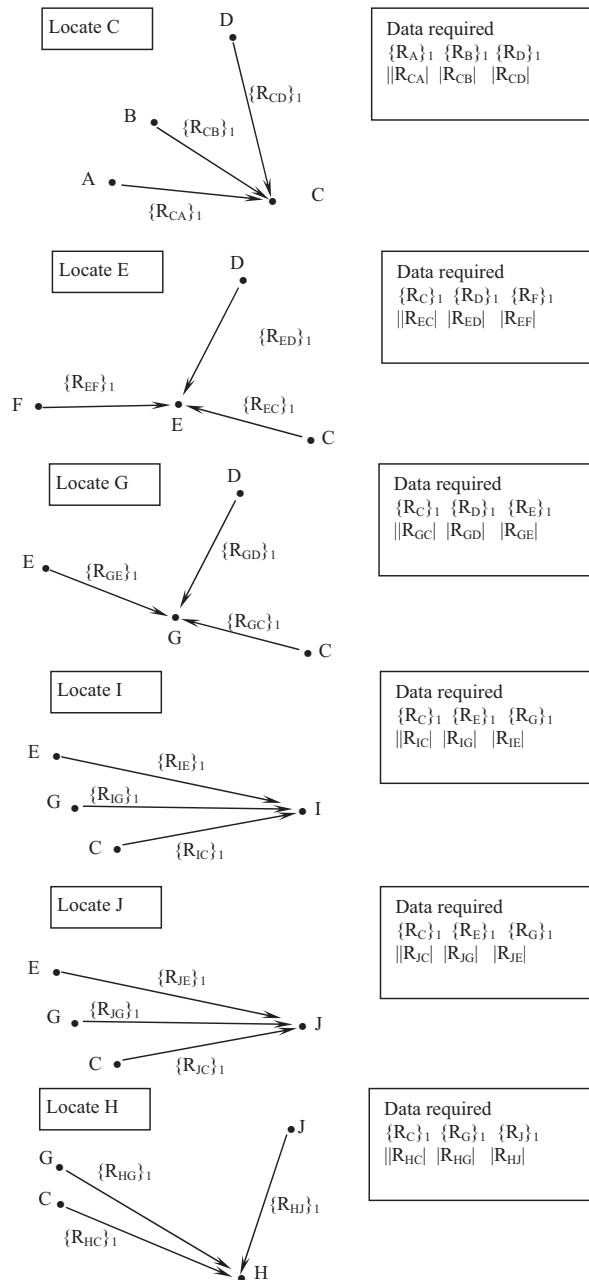
designated  $C'$ ,  $E'$  and  $G'$ . It should be noted that for the analysis using vectors here  $C'$  and  $E'$  are not the actual final positions but are only used to find the magnitudes of  $|R_{DC'}|$  and  $|R_{DE'}|$  so that the analysis may progress using the sequence shown in Figure 2.15. Having calculated the new positions of all the movable nodes the movement of the tyre contact patch, in this case taken to be point H, could be used to establish, for example, the lateral movement or half track change. The change in orientation of the wheel will also be of interest. The new positions  $I'$  and  $J'$  can be compared with the undisplaced positions I and J to determine the change in steer angle as shown in Figure 2.16.

The bump steer can be determined by finding the angle  $\delta$  between the projection of  $IJ$  and  $I'J'$  onto the global  $X_1Y_1$  plane. The projection is achieved by setting the  $z$  coordinates of all four vectors to zero and then rearranging the vector dot product as shown in Eqn (2.86).

$$\cos \delta = \{R_{IJ}\}_1 \cdot \{R_{I'J'}\} / |R_{IJ}| |R_{I'J'}| \quad (2.86)$$

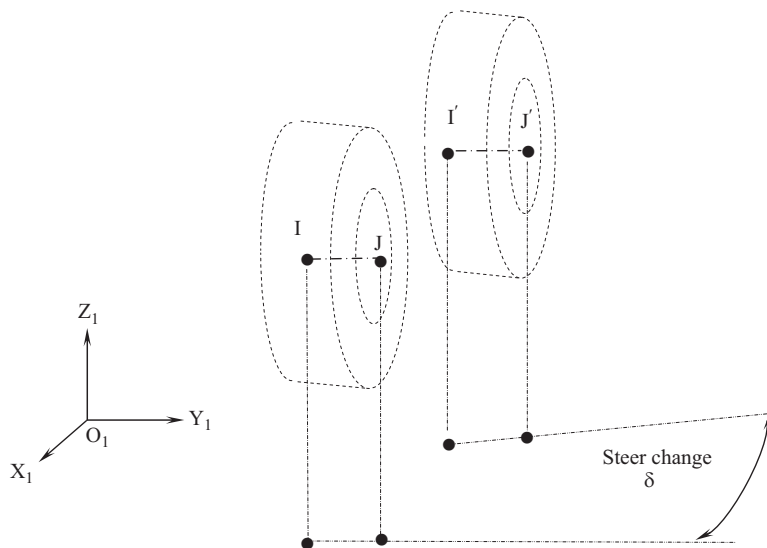
## 2.4 Velocity analysis

Consider the rigid body, Body 2, shown in Figure 2.17. In this case we are initially only interested in motion in the  $X_1Y_1$  plane. The body moves and rotates through an angle  $\delta\gamma$ , measured in radians, about the  $Z_1$ -axis.



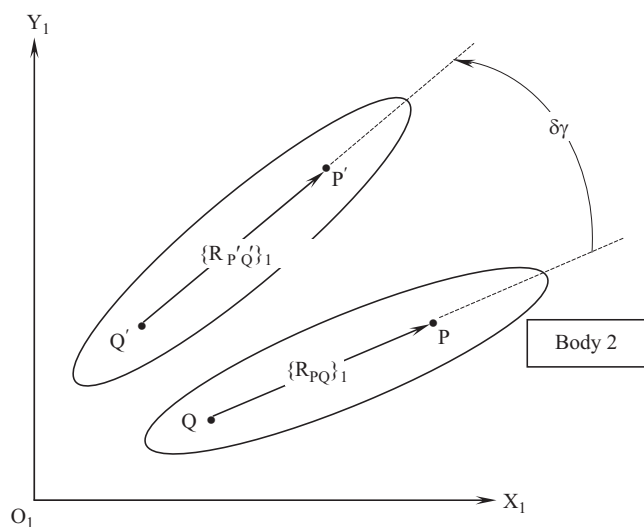
**FIGURE 2.15**

Calculation sequence to solve suspension geometry.



**FIGURE 2.16**

Using vectors to determine bump steer.



**FIGURE 2.17**

Motion of a vector attached to a rigid body.

The vector  $\{R_{PQ}\}_1$  moves with the body to a new position  $\{R_{P'Q'}\}_1$ . The new vector  $\{R_{P'Q'}\}_1$  is defined by the transformation

$$\{R_{P'Q'}\}_1 = [A] \{R_{PQ}\}_1 \quad (2.87)$$

where  $[A]$  is the rotation matrix that rotates  $\{R_{PQ}\}_1$  onto  $\{R_{P'Q'}\}_1$ . Expanding this gives

$$\begin{bmatrix} P'Q'_x \\ P'Q'_y \\ P'Q'_z \end{bmatrix} = \begin{bmatrix} \cos \delta\gamma & -\sin \delta\gamma & 0 \\ \sin \delta\gamma & \cos \delta\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} \quad (2.88)$$

Assuming that  $\delta\gamma$  is small so that we can take  $\cos \delta\gamma = 1$  and  $\sin \delta\gamma = \delta\gamma$  (rads) leads to

$$\begin{bmatrix} P'Q'_x \\ P'Q'_y \\ P'Q'_z \end{bmatrix} = \begin{bmatrix} 1 & -\delta\gamma & 0 \\ \delta\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} \quad (2.89)$$

The change in relative position  $\delta \{R_{PQ}\}_1$  is given by

$$\delta \{R_{PQ}\}_1 = \{R_{P'Q'}\}_1 - \{R_{PQ}\}_1 \quad (2.90)$$

or

$$\begin{bmatrix} \delta PQ_x \\ \delta PQ_y \\ \delta PQ_z \end{bmatrix} = \begin{bmatrix} 1 & -\delta\gamma & 0 \\ \delta\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} \quad (2.91)$$

This gives

$$\begin{bmatrix} \delta PQ_x \\ \delta PQ_y \\ \delta PQ_z \end{bmatrix} = \begin{bmatrix} 0 & -\delta\gamma & 0 \\ \delta\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} \quad (2.92)$$

If this change takes place in time  $\delta t$  then

$$\frac{\delta}{\delta t} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\delta\gamma}{\delta t} & 0 \\ \frac{\delta\gamma}{\delta t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} PQ_x \\ PQ_y \\ PQ_z \end{bmatrix} \quad (2.93)$$

In the limit  $\delta t$  approaches zero and we can write

$$\frac{d}{dt} \begin{bmatrix} PQx \\ PQy \\ PQz \end{bmatrix} = \begin{bmatrix} 0 & -\frac{d\gamma}{dt} & 0 \\ \frac{d\gamma}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} PQx \\ PQy \\ PQz \end{bmatrix} \quad (2.94)$$

which can be written

$$\begin{bmatrix} V_{PQx} \\ V_{PQy} \\ V_{PQz} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & 0 \\ \omega_z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} PQx \\ PQy \\ PQz \end{bmatrix} \quad (2.95)$$

Note that generally rotations cannot be represented as vector quantities unless they are very small, as in finite element programs. Hence angular velocities obtained by differencing rotations over very small time intervals are in fact vector quantities.

If the rigid link also undergoes small rotations  $\delta\alpha$  about the X-axis and  $\delta\beta$  about the Y-axis then the full expression is

$$\begin{bmatrix} V_{PQx} \\ V_{PQy} \\ V_{PQz} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} PQx \\ PQy \\ PQz \end{bmatrix} \quad (2.96)$$

Note that this matrix is the skew-symmetric form of the angular velocity vector  $[\omega_x \ \omega_y \ \omega_z]^T$ . In general terms we write

$$\{V_{PQ}\}_1 = \{\omega_2\}_1 \times \{R_{PQ}\}_1 \quad (2.97)$$

The direction of the relative velocity vector  $\{V_{PQ}\}_1$  is perpendicular to the line of the relative position vector  $\{R_{PQ}\}_1$  as shown in [Figure 2.18](#). The two points P and Q are fixed in the same rigid body, Body 2. As such there can be no component of motion along the line PQ and any relative motion must therefore be perpendicular to this line.

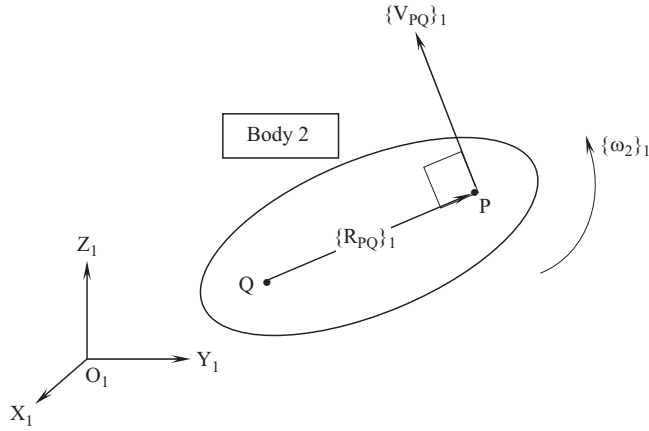
This relationship can be expressed mathematically in two ways. The first of these uses the vector dot product to enforce perpendicularity as shown in [Eqn \(2.98\)](#).

$$\{V_{PQ}\}_1 \bullet \{R_{PQ}\}_1 = 0 \quad (2.98)$$

The second method uses the vector cross product as shown in [Eqn \(2.99\)](#).

$$\{V_{PQ}\}_1 = \{\omega_2\}_1 \times \{R_{PQ}\}_1 \quad (2.99)$$

More often than not we use the cross product as this will yield the angular velocity vector  $\{\omega_2\}_1$ . This may be required for a later acceleration analysis. In

**FIGURE 2.18**

Relative velocity vector.

developing the equations to solve the velocities in a system of interconnected rigid bodies the triangle law of vector addition can also be used, as shown in Eqn (2.100).

$$\{V_P\}_1 = \{V_Q\}_1 + \{V_{PQ}\}_1 \quad (2.100)$$

## 2.5 Acceleration analysis

Given that acceleration is the time rate of change of velocity we can develop the equations required for an acceleration analysis using the vectors shown in Figure 2.19.

It is possible to develop equations that would yield the relative acceleration vector  $\{A_{PQ}\}_1$  using the angular velocity vector  $\{\omega_2\}_1$  and the angular acceleration vector  $\{\alpha_2\}_1$  as follows

$$\{A_{PQ}\}_1 = \frac{d}{dt} \{V_{PQ}\}_1 \quad (2.101)$$

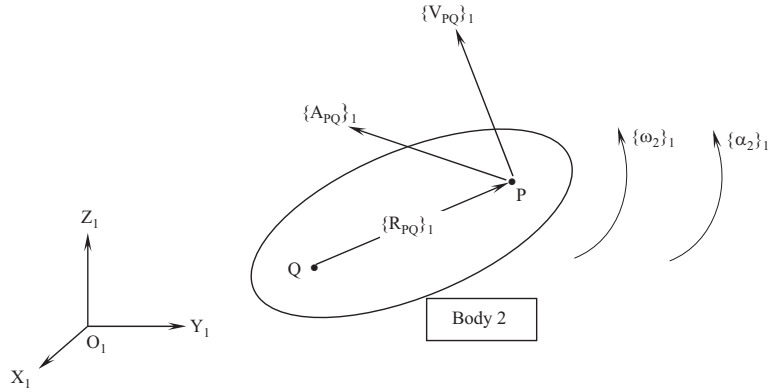
$$\{A_{PQ}\}_1 = \frac{d}{dt} \{ \{\omega_2\}_1 \times \{R_{PQ}\}_1 \} \quad (2.102)$$

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \frac{d}{dt} \{R_{PQ}\}_1 + \frac{d}{dt} \{ \{\omega_2\}_1 \times \{R_{PQ}\}_1 \} \quad (2.103)$$

Since it is known that

$$\frac{d}{dt} \{R_{PQ}\}_1 = \{V_{PQ}\}_1 \quad (2.104)$$



**FIGURE 2.19**

Relative acceleration vector.

$$\frac{d}{dt} \{\omega_2\}_1 = \{\alpha_2\}_1 \quad (2.105)$$

We can therefore write

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \{V_{PQ}\}_1 + \{\alpha_2\}_1 \times \{R_{PQ}\}_1 \quad (2.106)$$

Since it is also known that

$$\{V_{PQ}\}_1 = \{\omega_2\}_1 \times \{R_{PQ}\}_1 \quad (2.107)$$

This leads to the expression

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \{\{\omega_2\}_1 \times \{R_{PQ}\}_1\} + \{\alpha_2\}_1 \times \{R_{PQ}\}_1 \quad (2.108)$$

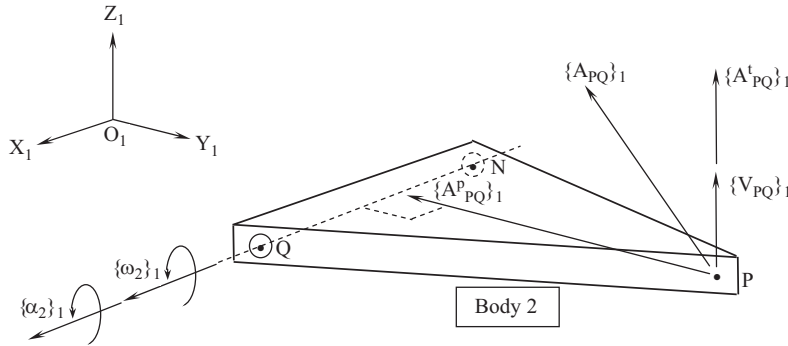
The acceleration vector  $\{A_{PQ}\}_1$  can be considered to have a centripetal component  $\{A^p_{PQ}\}_1$  and a transverse component  $\{A^t_{PQ}\}_1$ . This is illustrated in [Figure 2.20](#) where one of the arms from a double wishbone suspension system is shown.

In this case the centripetal component of acceleration is given by

$$\{A^p_{PQ}\}_1 = \{\omega_2\}_1 \times \{\{\omega_2\}_1 \times \{R_{PQ}\}_1\} \quad (2.109)$$

Note that as the suspension arm is constrained to rotate about the axis NQ, ignoring at this stage any possible deflection due to compliance in the suspension bushes, the vectors  $\{\omega_2\}_1$  for the angular velocity of Body 2 and  $\{\alpha_2\}_1$  for the angular acceleration would act along the axis of rotation through NQ. The components of these vectors would adopt signs consistent with producing a positive rotation about this axis as shown in [Figure 2.20](#).

When setting up the equations to solve a velocity or acceleration analysis it may be desirable to reduce the number of unknowns based on the knowledge that a particular body is constrained to rotate about a known axis as shown here. The velocity vector  $\{\omega_2\}_1$  could, for example, be represented as follows

**FIGURE 2.20**

Centripetal and transverse components of acceleration vectors.

$$\{\omega_2\}_1 = f\omega_2 \{R_{QN}\}_1 \quad (2.110)$$

In this case, since  $\{\omega_2\}_1$  is parallel to the relative position vector  $\{R_{QN}\}_1$  a scale factor  $f\omega_2$  can be introduced. This would reduce the problem from the three unknown components,  $\omega_{x2}$ ,  $\omega_{y2}$  and  $\omega_{z2}$  of the vector  $\{\omega_2\}_1$  to a single unknown  $f\omega_2$ . A similar approach could be used for an acceleration analysis with, for example

$$\{\alpha_2\}_1 = f\alpha_2 \{R_{QN}\}_1 \quad (2.111)$$

It can also be seen from [Figure 2.20](#) that the centripetal acceleration acts towards, and is perpendicular to, the axis of rotation of the body. This relationship can be proved having found the centripetal acceleration by use of the dot product with

$$\{A^P_{PQ}\}_1 \bullet \{R_{QN}\}_1 = 0 \quad (2.112)$$

The transverse component of acceleration is given by:

$$\{A^t_{PQ}\}_1 = \{\alpha_2\}_1 \times \{R_{PQ}\}_1 \quad (2.113)$$

Note that the transverse component of acceleration is also perpendicular to in this case the vector  $\{R_{PQ}\}_1$  as defined by the dot product with

$$\{A^t_{PQ}\}_1 \bullet \{R_{PQ}\}_1 = 0 \quad (2.114)$$

Note that although the vector  $\{A^t_{PQ}\}_1$  is shown to be acting in the same direction as the vector  $\{V_{PQ}\}_1$  in [Figure 2.20](#), this may not necessarily be the case. A reversal of  $\{A^t_{PQ}\}_1$  would correspond to a reversal of  $\{\alpha_2\}_1$ . This would indicate that point P is moving in a certain direction but in fact decelerating.

The resultant acceleration vector  $\{A_{PQ}\}_1$  is found to give the expression shown in [Eqn \(2.115\)](#) using the triangle law to add the centripetal and transverse components as follows

$$\{A_{PQ}\}_1 = \{A^P_{PQ}\}_1 + \{A^t_{PQ}\}_1 \quad (2.115)$$

For the example shown here in Figure 2.20 the analysis may often focus on suspension movement only and assume the vehicle body to be fixed and not moving. This would mean that the velocity or acceleration at point Q,  $\{A_Q\}_1$  would be zero. Since we can say, based on the triangle law, that

$$\{A_{PQ}\}_1 = \{A_P\}_1 - \{A_Q\}_1 \quad (2.116)$$

it therefore follows in this case that since Q is fixed

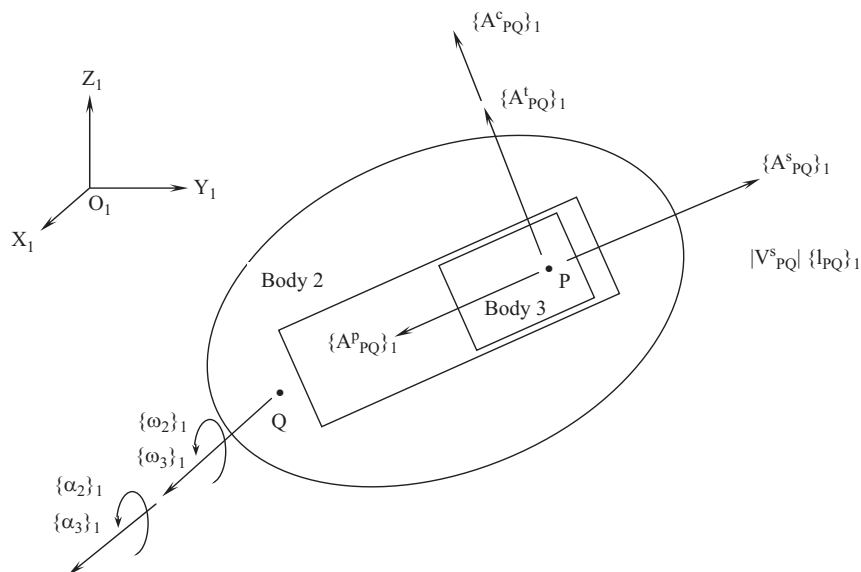
$$\{A_P\}_1 = \{A_{PQ}\}_1 \quad (2.117)$$

Note that the same principle could be used when solving the velocities for this problem and that we could write

$$\{V_P\}_1 = \{V_{PQ}\}_1 \quad (2.118)$$

Finally, it should be noted for the particular example, shown here in Figure 2.20, we could work from either point Q or point N when solving for the velocities or accelerations and obtain the same answers for the velocity or acceleration at point P.

Combining the relative acceleration already obtained for points on a rigid body translating and rotating in space with sliding motion will introduce two more components of relative acceleration. This is best explained by considering the situation shown in Figure 2.21 where point P is located on Body 3 and point Q is located on Body 2. In this case Body 3 is constrained to move and rotate with Body 2 but has an additional relative sliding degree of freedom that allows it to move, relative to Body 2



**FIGURE 2.21**

Relative acceleration with sliding motion.

2, along a slot with an axis aligned with the two points P and Q. To simplify the understanding, the two bodies are assumed to rotate, as shown, about an axis passing through point Q. Hence the relative acceleration vectors can be assumed to be acting in the directions shown, as point P moves away from point Q. The angular velocity and acceleration vectors for Body 2 and Body 3 will be the same and either may be used in the subsequent formulations.

The first of the two new components is easy to comprehend and is associated with the additional sliding motion. Since the direction of motion is known to be constrained to act along the line PQ it is possible to define the sliding acceleration,  $\{A_{PQ}^s\}_1$ , using the magnitude of the sliding acceleration  $|A_{PQ}^s|$  factored with the unit vector  $\{l_{PQ}\}_1$ , acting along the line from Q to P, as follows

$$\{A_{PQ}^s\}_1 = |A_{PQ}^s| \{l_{PQ}\}_1 \quad (2.119)$$

The second of the two new components,  $\{A_{PQ}^c\}_1$ , is known as the Coriolis acceleration and requires more detailed explanation. As a starting point we can assume that both bodies are rotating as shown in [Figure 2.21](#).

In deriving the Coriolis term, consider first the formulation for the velocity vector  $\{V_{PQ}\}_1$ . In addition to the component of velocity associated with rigid body motion that would act in a perpendicular direction to the line PQ there is an additional component of sliding velocity. This sliding component can be defined using the magnitude of the sliding velocity  $|V_{PQ}^s|$  factored with the unit vector  $\{l_{PQ}\}_1$ , acting along the line from Q to P. The formulation of  $\{V_{PQ}\}_1$  now becomes

$$\{V_{PQ}\}_1 = \{\omega_2\}_1 \times \{R_{PQ}\}_1 + |V_{PQ}^s| \{l_{PQ}\}_1 \quad (2.120)$$

Differentiating this with respect to time yields

$$\{A_{PQ}\}_1 = \frac{d}{dt} \{V_{PQ}\}_1 = \frac{d}{dt} \{ \{\omega_2\}_1 \times \{R_{PQ}\}_1 + |V_{PQ}^s| \{l_{PQ}\}_1 \} \quad (2.121)$$

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \{V_{PQ}\}_1 + \{\alpha_2\}_1 \times \{R_{PQ}\}_1 + |V_{PQ}^s| \frac{d}{dt} \{l_{PQ}\}_1 + |A_{PQ}^s| \{l_{PQ}\}_1 \quad (2.122)$$

Since

$$\frac{d}{dt} \{R_{PQ}\}_1 = \{V_{PQ}\}_1 = \{\omega_2\}_1 \times \{R_{PQ}\}_1 \quad (2.123)$$

it therefore follows that

$$\frac{d}{dt} \{l_{PQ}\}_1 = \{\omega_2\}_1 \times \{l_{PQ}\}_1 \quad (2.124)$$

and

$$|V_{PQ}^s| \{l_{PQ}\}_1 = \{V_{PQ}^s\}_1 \quad (2.125)$$

and

$$|A_{PQ}^s| \{l_{PQ}\}_1 = \{A_{PQ}^s\}_1 \quad (2.126)$$

Combining these we therefore obtain

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \{V_{PQ}\}_1 + \{\alpha_2\}_1 \times \{R_{PQ}\}_1 + \{\omega_2\}_1 \times \{V_{PQ}^s\}_1 + \{A_{PQ}^s\}_1 \quad (2.127)$$

Substituting

$$\{V_{PQ}\}_1 = \{\omega_2\}_1 \times \{R_{PQ}\}_1 + \{V_{PQ}^s\}_1 \quad (2.128)$$

gives the following

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \{\{\omega_2\}_1 \times \{R_{PQ}\}_1 + \{V_{PQ}^s\}_1\} + \{\alpha_2\}_1 \times \{R_{PQ}\}_1 + \{\omega_2\}_1 \times \{V_{PQ}^s\}_1 + \{A_{PQ}^s\}_1 \quad (2.129)$$

It therefore follows that

$$\{A_{PQ}\}_1 = \{\omega_2\}_1 \times \{\{\omega_2\}_1 \times \{R_{PQ}\}_1\} + \{\alpha_2\}_1 \times \{R_{PQ}\}_1 + 2\{\omega_2\}_1 \times \{V_{PQ}^s\}_1 + \{A_{PQ}^s\}_1 \quad (2.130)$$

In summary, we can now identify all four components of acceleration associated with the combined rotation and sliding motion as the centripetal acceleration  $\{A_{PQ}^p\}_1$ , the transverse acceleration  $\{A_{PQ}^t\}_1$ , the Coriolis acceleration  $\{A_{PQ}^c\}_1$  and the sliding acceleration  $\{A_{PQ}^s\}_1$ , where:

$$\{A_{PQ}^p\}_1 = \{\omega_2\}_1 \times \{\{\omega_2\}_1 \times \{R_{PQ}\}_1\} \quad (2.131)$$

$$\{A_{PQ}^t\}_1 = \{\alpha_2\}_1 \times \{R_{PQ}\}_1 \quad (2.132)$$

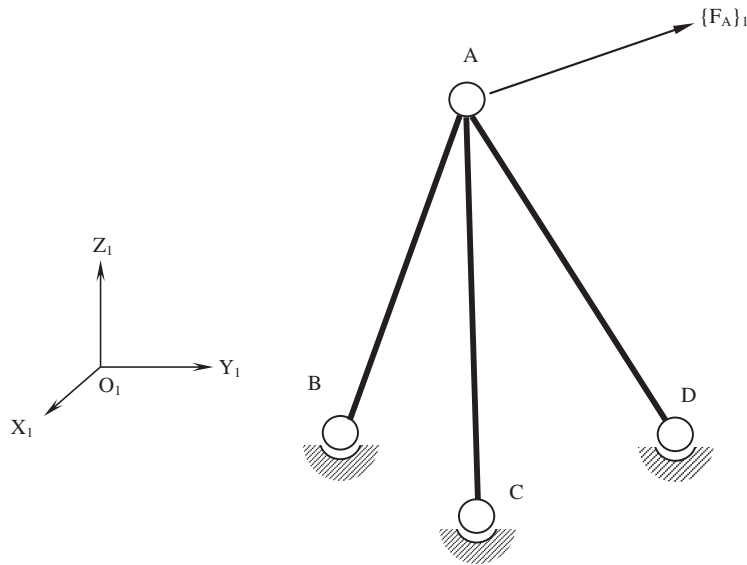
$$\{A_{PQ}^c\}_1 = 2\{\omega_2\}_1 \times \{V_{PQ}^s\}_1 \quad (2.133)$$

$$\{A_{PQ}^s\}_1 = |A_{PQ}^s| \{l_{PQ}\}_1 \quad (2.134)$$

---

## 2.6 Static force and moment definition

Before progressing to the development of the equations of motion associated with large displacement rigid body dynamic motion it is necessary to examine the use of vectors for static analysis in MBS. In this case we define static analysis as the study of forces acting on a body or series of bodies where motion takes place in the absence of accelerations. That is, a system that is either at rest or moving in a straight line with constant velocity.

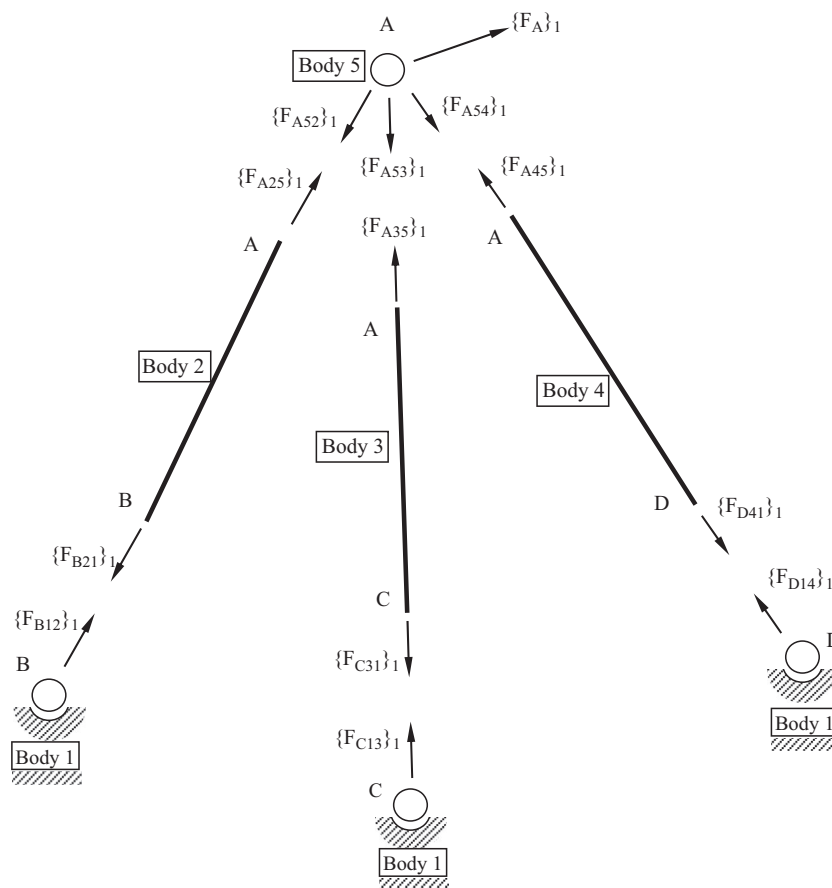
**FIGURE 2.22**

Tripod structure.

In order to demonstrate the use of vectors for representation of forces consider the following example, shown in Figure 2.22, which involves a tripod structure comprising three links with ball or spherical joints at each end.

Before attempting any analysis to determine the distribution of forces it is necessary to prepare a free body diagram and label the bodies and forces in an appropriate manner as shown in Figure 2.23.

The notation used here to describe the forces depends on whether the force is an applied external force (action-only force) or an internal force resulting from the interconnection of bodies (action–reaction force). In this example we have an applied force at point  $A$ . In order to fully define a force we must be able to specify the point of application, the line of sight and the sense of the force. In this case we can use the notation  $\{F_A\}_1$  to define the force where the subscript  $A$  defines the point of application and the components of the vector  $F_{Ax}$ ,  $F_{Ay}$  and  $F_{Az}$  would define both the line of sight and the sense of the force. Where the force is the result of an interaction we can use, for example, the following  $\{F_{A52}\}_1$  that specifies that the force is acting at  $A$  on Body 5 due to its interaction with Body 2. Note that for this example we are assuming that  $A$  is a point that could be, for example, taken to be at the centre of a bush or spherical joint. It should also be noted that in assigning identification numbers to the bodies we are taking Body 1 to be a fixed and unmoving body that may also be referred to as a ground body. In this example the ground body is not in one place as such but can be considered to be located at the positions  $B$ ,  $C$  and  $D$  where fixed anchorages are provided.

**FIGURE 2.23**

Free body diagram of tripod structure.

For the action–reaction forces shown here, Newton’s Third Law would apply so that for the interaction of Body 5 and Body 2 we can say that  $\{F_{A52}\}_1$  and  $\{F_{A25}\}_1$  are equal and opposite or

$$\{F_{A52}\}_1 = -\{F_{A25}\}_1 \quad (2.135)$$

In this example we are looking at linkages that are pin jointed, or have spherical joints at each end. An example of a similar linkage would be a tie rod in a suspension system. As both ends of the linkage are pin jointed, the force by definition must act along the linkage. In this case we could use this information to reduce the number of unknowns by using a scale factor as shown in Eqns (2.136)–(2.138).

$$\{F_{A25}\}_1 = f_2 \{R_{AB}\}_1 \quad (2.136)$$

$$\{F_{A35}\}_1 = f_3 \{R_{AC}\}_1 \quad (2.137)$$

$$\{F_{A45}\}_1 = f_4 \{R_{AD}\}_1 \quad (2.138)$$

In this case the solution would be assisted as the three unknowns  $F_{A25x}$ ,  $F_{A25y}$  and  $F_{A25z}$ , associated with Eqn (2.136), for example, would be reduced to a single unknown  $f_2$ . To maintain rigour the choice of position vector, for example,  $\{R_{AB}\}_1$  rather than  $\{R_{BA}\}_1$ , has also been used so that comparing Eqn (2.136) with the forces as drawn in the free body diagram shown in Figure 2.23 would be consistent with the scale factors having positive values. Should the solution yield negative scale factors this would simply involve a reversal of the sense of the force from that initially assumed in the free body diagram.

In this case, since the forces act along the line of the linkages, we equate forces throughout the structure as follows

$$\{F_{A52}\}_1 = -\{F_{A25}\}_1 = \{F_{B21}\}_1 = -\{F_{B12}\}_1 \quad (2.139)$$

$$\{F_{A53}\}_1 = -\{F_{A35}\}_1 = \{F_{C31}\}_1 = -\{F_{C13}\}_1 \quad (2.140)$$

$$\{F_{A54}\}_1 = -\{F_{A45}\}_1 = \{F_{D41}\}_1 = -\{F_{D14}\}_1 \quad (2.141)$$

Clearly by solving the unknown forces acting on the pin, Body 5, which connects to all three linkages, the complete force distribution in this system will be known. Setting up the equation of equilibrium for Body 5 we get

$$\sum \{F_5\}_1 = \{0\}_1 \quad (2.142)$$

$$\{F_A\}_1 + \{F_{A52}\}_1 + \{F_{A53}\}_1 + \{F_{A54}\}_1 = \{0\}_1 \quad (2.143)$$

Using the information developed in Eqns (2.136)–(2.141) we can now write

$$\{F_A\}_1 + f_2 \{R_{BA}\}_1 + f_3 \{R_{CA}\}_1 + f_4 \{R_{DA}\}_1 = \{0\}_1 \quad (2.144)$$

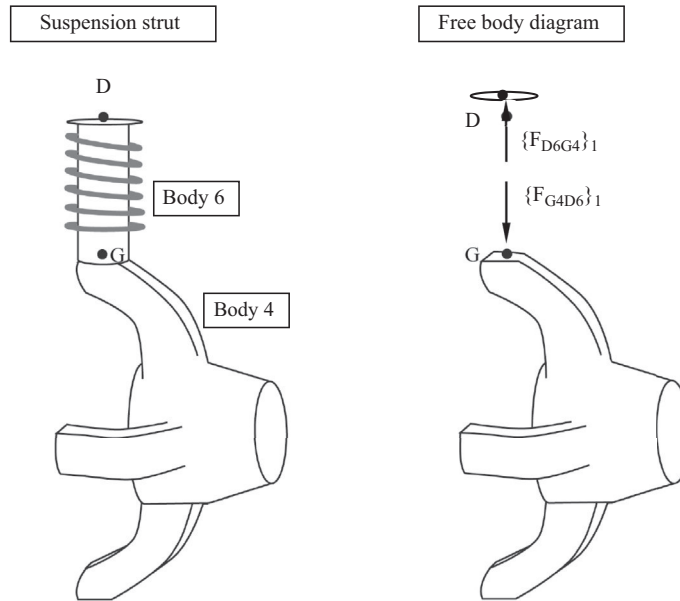
The direction of the position vectors as defined in Eqn (2.144) should be carefully noted. These have been selected to maintain the correct positive sign convention throughout the equation. Expanding Eqn (2.144) would lead to

$$\begin{bmatrix} F_{Ax} \\ F_{Ay} \\ F_{Az} \end{bmatrix} + f_2 \begin{bmatrix} B_{Ax} \\ B_{Ay} \\ B_{Az} \end{bmatrix} + f_3 \begin{bmatrix} C_{Ax} \\ C_{Ay} \\ C_{Az} \end{bmatrix} + f_4 \begin{bmatrix} D_{Ax} \\ D_{Ay} \\ D_{Az} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.145)$$

For a given applied force  $\{F_A\}_1$ , and taking the geometry of the points A, B, C and D to be known, Eqn (2.145) yields the three equations required to solve the three unknowns  $f_2$ ,  $f_3$  and  $f_4$  and hence solve the force distribution in this example.

The notation that has been used to define forces in the example shown can be used where the forces represent reactions generated at a point in a multibody system.



**FIGURE 2.24**

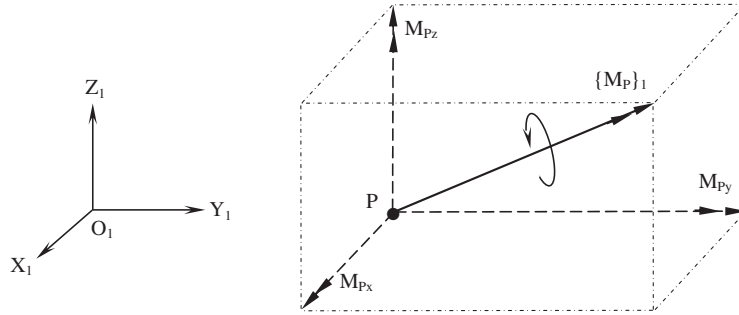
Vector notation for line of sight forces.

The notation needs to be altered for forces acting along the line of sight of a force element, such as a spring or damper, connecting two bodies as shown in Figure 2.24.

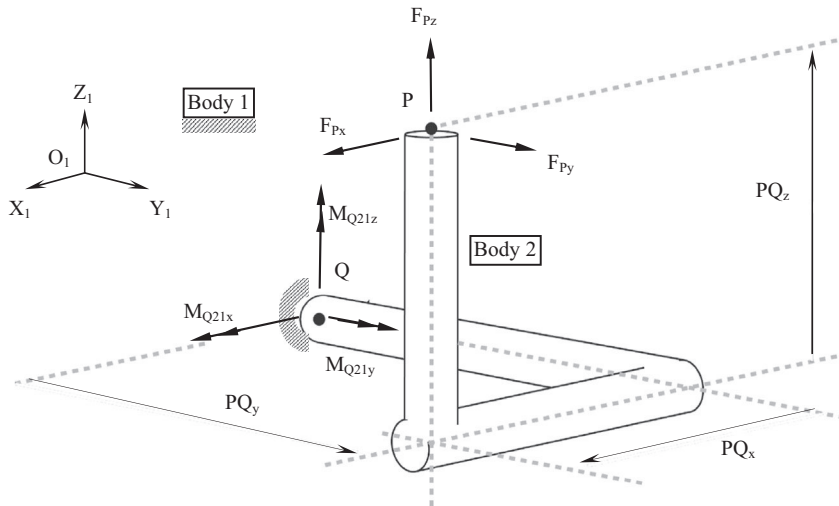
The vector representation of a moment is not as straightforward to interpret as the vector representation of a force. The moment  $\{M_P\}_1$  shown acting about point P in Figure 2.25 is represented by a vector that is orientated along an axis about which the moment acts. The length of the vector represents the magnitude of the moment and the direction of the vector is that which is consistent with a positive rotation about the axis as shown. The components of the vector,  $M_{Px}$ ,  $M_{Py}$  and  $M_{Pz}$  are resolved parallel to a reference frame, in this case  $O_1$ . The double-headed arrows used in Figure 2.25 are intended to distinguish the moment vector from that of a force.

Considering next the equilibrium of the body shown in Figure 2.26 it can be seen that a force acting at point P will produce a reaction moment at point Q. In order to simplify the diagram only the components of the applied force and the resulting moment are shown here. Since we are looking only at the derivation of the moment at Q the reaction force at Q is also omitted.

The moment generated at point Q, which is the point where Body 2 is fixed to the non-moving ground part Body 1, is designated as  $\{M_{Q21}\}_1$ . Applying the same principle as used for a force we would read this as the moment acting at point Q, on Body 2 due to its connection to Body 1, with components resolved parallel to the axes of reference frame  $O_1$ . For convenience and to assist with the interpretation of the derivation that follows the components of the force  $\{F_P\}_1$ , the moment  $\{M_{Q21}\}_1$  and the

**FIGURE 2.25**

Vector representation of a moment.

**FIGURE 2.26**

Reaction moment between two bodies.

relative position vector  $\{R_{PQ}\}_1$  have all been set up in Figure 2.26 to have components that are positive.

Using the traditional approach the three equations of moment equilibrium,  $\sum M_{Qx} = 0$ ,  $\sum M_{Qy} = 0$  and  $\sum M_{Qz} = 0$  could be transformed to produce the following three equations:

$$M_{Q21x} = -PQ_z \times F_{Py} + PQ_y \times F_{Pz} \quad (2.146)$$

$$M_{Q21y} = PQ_z \times F_{Px} - PQ_x \times F_{Pz} \quad (2.147)$$

$$M_{Q21z} = PQ_x \times F_{Py} - PQ_y \times F_{Px} \quad (2.148)$$

Taking the above Eqns (2.146)–(2.148) and arranging in matrix form gives

$$\begin{bmatrix} M_{Qx} \\ M_{Qy} \\ M_{Qz} \end{bmatrix} = \begin{bmatrix} 0 & -PQ_z & PQ_y \\ PQ_z & 0 & -PQ_x \\ -PQ_y & PQ_x & 0 \end{bmatrix} \begin{bmatrix} F_{Px} \\ F_{Py} \\ F_{Pz} \end{bmatrix} \quad (2.149)$$

From Eqn (2.149) it can be seen that the moment  $\{M_Q\}_1$  can be found from the cross product of the relative position vector  $\{R_{PQ}\}_1$  and the force  $\{F_P\}_1$ .

$$\{M_Q\}_1 = \{R_{PQ}\}_1 \times \{F_P\}_1 \quad (2.150)$$

It should be noted that in using the vector cross product to compute the moment of a force about a point that the order of the operation is critical. The relative position vector is crossed with the force so that it is the relative position vector that is arranged in skew-symmetric form and not the force vector. The relative position vector must also be the vector from the point about which the moment is taken to the point of application of the force.

## 2.7 Dynamics of a particle

In the remaining sections of this chapter the authors have broadly followed the approach given by D'Souza and Garg (1984) to derive the equations of motion for a rigid body. The starting point is to consider the dynamics of a particle, a body for which the motion is restricted to translation without rotation. The resultant moment acting on the body is therefore zero. In the absence of rotation, the velocity and acceleration will be the same at all points on the body and hence a particle may be treated as a point mass.

From Newton's Second Law it can be seen that the time rate of change of linear momentum  $\{L\}_1$  for a particle is equal to the resultant force acting on it.

$$\sum \{F\}_1 = \frac{d}{dt} \{L\}_1 = \frac{d}{dt} (m\{V\}_1) \quad (2.151)$$

The resultant force is represented by the vector  $\sum \{F\}_1$ ,  $m$  is the mass and  $\{V\}_1$  is the velocity vector measured relative to an inertial reference frame  $O_1$ . The linear momentum of the body is  $m\{V\}_1$ . The components of the vectors in Eqn (2.151) are all resolved parallel to the axes of  $O_1$ . Taking the mass to be constant, Eqn (2.151) may be written as

$$\sum \{F\}_1 = m \frac{d}{dt} \{V\}_1 = m\{A\}_1 \quad (2.152)$$

The accelerations in Eqn (2.152) are also measured relative to the inertial reference frame  $O_1$ . Since the velocities and accelerations used here are measured relative

to a nonmoving frame we refer to them as absolute. Expanding the vector equation given in (2.152) gives

$$\sum \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (2.153)$$

## 2.8 Linear momentum of a rigid body

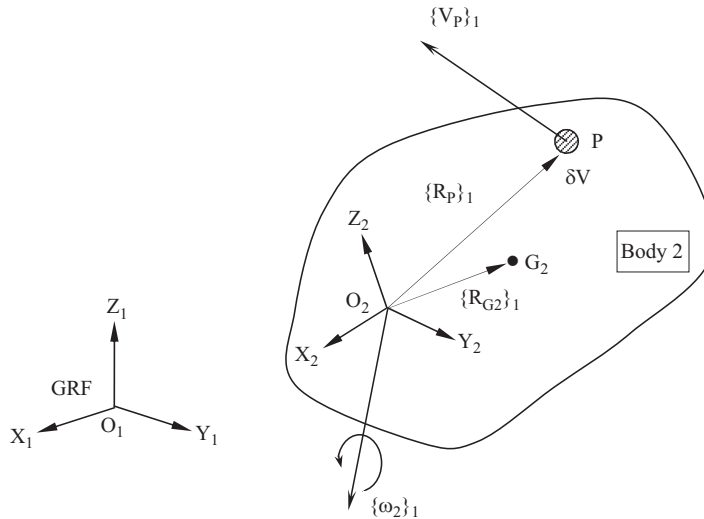
As a body translates and rotates in space it will have linear momentum  $\{L\}_1$  associated with translation and angular momentum  $\{H\}_1$  associated with rotation. For the rigid body, Body 2, shown in Figure 2.27 the mass centre is located at  $G_2$  by the vector  $\{R_{G2}\}_1$  relative to the reference frame  $O_2$ .

A small element of material with a volume  $\delta V$  is located at  $P$  relative to  $O_2$  by the position vector  $\{R_P\}_1$ . Assuming Body 2 to be of uniform density  $\rho$ , we can say that the element of material has a mass  $\delta m$  given by

$$\delta m = \rho \delta V \quad (2.154)$$

and that as the element becomes infinitesimal the mass of the body  $m_2$  is given by

$$m_2 = \rho \int_{\text{vol}} dV = \int_{\text{vol}} dm \quad (2.155)$$



**FIGURE 2.27**

Linear momentum of a rigid body. GRF, ground reference frame.

The position of the centre of mass  $\{\mathbf{R}_{G2}\}_1$  is then found by integrating the elemental first moments of mass about frame  $O_2$  and dividing by the total mass  $m_2$ .

$$\{\mathbf{R}_{G2}\}_1 = \frac{1}{m_2} \int_{\text{vol}} \{\mathbf{R}_P\}_1 dm \quad (2.156)$$

The linear momentum  $\{\mathbf{L}_2\}_1$  of the body is the linear momenta of the elements of mass that comprise the body. If the position of an element of mass is given by  $\{\mathbf{R}_P\}_1$  then the velocity  $\{\mathbf{V}_P\}_1$  of  $dm$  is given using the triangle law of vector addition by

$$\{\mathbf{V}_P\}_1 = \{\mathbf{V}_{O2}\}_1 + \{\mathbf{V}_{PO2}\}_1 \quad (2.157)$$

$$\{\mathbf{V}_P\}_1 = \{\mathbf{V}_{O2}\}_1 + \{\boldsymbol{\omega}_2\}_1 \times \{\mathbf{R}_P\}_1$$

The linear momentum  $\{\mathbf{L}\}_1$  of the body is therefore found by integrating the mass particles factored by their velocity vectors  $\{\mathbf{V}_P\}_1$  over the volume of the body

$$\{\mathbf{L}_2\}_1 = \int_{\text{vol}} \{\mathbf{V}_P\}_1 dm \quad (2.158)$$

Using the expression for  $\{\mathbf{V}_P\}_1$  given in Eqn (2.157) leads to

$$\{\mathbf{L}_2\}_1 = \{\mathbf{V}_{O2}\}_1 \int_{\text{vol}} dm + \{\boldsymbol{\omega}_2\}_1 \times \int_{\text{vol}} \{\mathbf{R}_P\}_1 dm \quad (2.159)$$

Using the expressions given in Eqns (2.155) and (2.156) we get an expression for the linear momentum of Body 2 in terms of the overall mass  $m_2$  and the velocity at the mass centre  $\{\mathbf{V}_{G2}\}_1$ .

$$\{\mathbf{L}_2\}_1 = m_2 \{ \{\mathbf{V}_{O2}\}_1 + \{\boldsymbol{\omega}_2\}_1 \times \{\mathbf{R}_{G2}\}_1 \} = m_2 \{\mathbf{V}_{G2}\}_1 \quad (2.160)$$

It can be noted that problems are often set up in multibody dynamics so that the body frame coincides with the mass centre. In this case for Body 2,  $O_2$  and  $G_2$  would be coincident so that  $\{\mathbf{R}_{G2}\}_1$  would be zero and the linear momentum would be obtained directly from

$$\{\mathbf{L}_2\}_1 = m_2 \{\mathbf{V}_{G2}\}_1 = m_2 \{\mathbf{V}_{O2}\}_1 \quad (2.161)$$

---

## 2.9 Angular momentum

The angular momentum  $\{\mathbf{H}_P\}_1$  of the particle of material with mass  $dm$  in Figure 2.27 can be found by taking the moment of the linear momentum of the particle about the frame  $O_2$  as follows

$$\{\mathbf{H}_P\}_1 = \{\mathbf{R}_P\}_1 \times \{ \{\mathbf{V}_{O2}\}_1 + \{\boldsymbol{\omega}_2\}_1 \times \{\mathbf{R}_P\}_1 \} dm \quad (2.162)$$

Integrating this over the volume of the body leads to the angular momentum  $\{H_2\}_1$  of the rigid body about the frame  $O_2$  as follows

$$\begin{aligned}\{H_2\}_1 &= \int_{\text{vol}} \{R_p\}_1 \times \left\{ \{V\}_{O_2} + \{\omega_2\}_1 \times \{R_p\}_1 \right\} dm \\ &= -\{V\}_{O_2} \times \int_{\text{vol}} \{R_p\}_1 dm + \int_{\text{vol}} \{R_p\}_1 \times \left\{ \{\omega_2\}_1 \times \{R_p\}_1 \right\} dm\end{aligned}\quad (2.163)$$

Note that in the second line of Eqn (2.163), the vector  $\{V_{O_2}\}_1$  comes out of the integral and hence the order of the cross product is reversed necessitating the reverse of sign for  $\{V_{O_2}\}_1$ . If for Body 2, frame  $O_2$  is positioned either at the mass centre, so that  $\int_{\text{vol}} \{R_p\}_1 dm = \{0\}_1$ , or at a point of attachment to the nonmoving ground, where  $\{V_{O_2}\}_1 = \{0\}_1$ , then Eqn (2.163) reduces to the more convenient form, which we will assume from now on.

$$\{H_2\}_1 = \int_{\text{vol}} \{R_p\}_1 \times \left\{ \{\omega_2\}_1 \times \{R_p\}_1 \right\} dm \quad (2.164)$$

If we now take the general case for any body (ignore body subscripts) and expand the vectors into their full form we get

$$\{H\}_1 = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}, \quad \{R\}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \{\omega\}_1 = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Applying the vector cross product, making use of the skew-symmetric form of a vector in the normal manner, leads to

$$\begin{aligned}\{R\}_1 \times \left\{ \{\omega\}_1 \times \{R\}_1 \right\} &= \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} -\omega_z y + \omega_y z \\ -\omega_z x - \omega_x z \\ -\omega_y x + \omega_x y \end{bmatrix} \\ &= \begin{bmatrix} y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z) \\ z(\omega_y z - \omega_z y) - x(\omega_x y - \omega_y x) \\ x(\omega_z x - \omega_x z) - y(\omega_y z - \omega_z y) \end{bmatrix}\end{aligned}\quad (2.165)$$

Substituting Eqn (2.165) into Eqn (2.164) gives the general expression for the angular momentum  $\{H\}_1$  of a body, where for simplicity now the integral sign is taken to indicate integration over the volume of the body.

$$\{H\}_1 = \int \{R\}_1 \times \left\{ \{\omega\}_1 \times \{R\}_1 \right\} dm \quad (2.166)$$

$$H_x = \omega_x \int (y^2 + z^2) dm - \omega_y \int xy dm - \omega_z \int xz dm \quad (2.167)$$

$$H_y = -\omega_x \int xy dm + \omega_y \int (x^2 + z^2) dm - \omega_z \int yz dm \quad (2.168)$$

$$H_z = -\omega_x \int xz dm - \omega_y \int yz dm + \omega_z \int (x^2 + y^2) dm \quad (2.169)$$

It is now possible to substitute into [Eqns \(2.167\)–\(2.169\)](#) the following general terms for the moments of inertia  $I_{xx}$ ,  $I_{yy}$  and  $I_{zz}$  of the rigid body.

$$I_{xx} = \int (y^2 + z^2) dm \quad (2.170)$$

$$I_{yy} = \int (x^2 + z^2) dm \quad (2.171)$$

$$I_{zz} = \int (x^2 + y^2) dm \quad (2.172)$$

In addition we can introduce the products of inertia  $I_{xy}$ ,  $I_{yz}$  and  $I_{xz}$ .

$$I_{xy} = I_{yx} = - \int xy dm \quad (2.173)$$

$$I_{yz} = I_{zy} = - \int yz dm \quad (2.174)$$

$$I_{xz} = I_{zx} = - \int xz dm \quad (2.175)$$

This allows [Eqns \(2.167\)–\(2.169\)](#) to be arranged in matrix form as follows

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad (2.176)$$

If we return to our earlier consideration of Body 2 shown in [Figure 2.27](#), the matrix equation in [Eqn \(2.176\)](#) would lead to

$$\{H_2\}_{1/1} = [I_2]_{2/1} \{\omega_2\}_{1/1} \quad (2.177)$$

In writing the vectors  $\{H_2\}_{1/1}$   $\{\omega_2\}_{1/1}$  we revert to the full definition of a vector used here where the upper suffix indicates that the vector is measured relative to the axes of reference frame  $O_1$  and the lower suffix indicates that the components of the vector are resolved parallel to the axes of frame  $O_1$ . The matrix  $[I_2]_{2/1}$  is the moment of inertia matrix for Body 2 about its mass centre  $G_2$  located at frame  $O_2$ . The use of

the upper and lower suffixes here indicates that the moments of inertia have been measured relative to frame  $O_2$  but transformed to frame  $O_1$ . This is necessary so that the vector operation in Eqn (2.171) is consistent. This is only possible if the vectors and matrix are referred to the same frame, which in this case is  $O_1$ .

Note that in this form Eqn (2.177) is not practical since the orientation of frame  $O_2$  relative to frame  $O_1$  will change as the body rotates requiring the recomputation of  $[I_2]_{2/1}$  at each time step. The matrix  $[I_2]_{2/2}$ , or in simpler form  $[I_2]_2$ , is constant since it is measured relative to and referred to a frame that is fixed in Body 2 and hence only needs to be determined once for an undeformable body. When considering the equation for the angular momentum of a body it is preferable therefore to consider all quantities to be referred to a frame fixed in the body, in this case frame  $O_2$ .

$$\{H_2\}_{1/2} = [I_2]_{2/2} \{\omega_2\}_{1/2} \quad (2.178)$$

Before progressing to develop the equations used to describe the dynamics of rigid bodies translating and rotating in three-dimensional space, the definition of the moments of inertia introduced here requires further consideration.

## 2.10 Moments of inertia

From our previous consideration of the angular momentum of a rigid body, we see that there are three moments of inertia and three products of inertia the values of which must be specified to analyse the rotational motion of the body. Before considering the three-dimensional situation it is useful to start with the two-dimensional inertial properties associated with plane motion.

For the Body 2 shown in Figure 2.28 we assume that a constraint has been applied that allows the body to move only in the  $X_1Y_1$  plane of frame  $O_1$ .

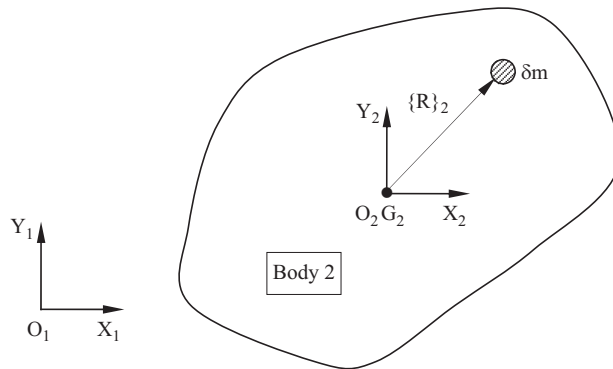


FIGURE 2.28

Moment of inertia for plane motion.



As such the body has three degrees of freedom, which are translation in the  $X_1$  direction, translation in the  $Y_1$  direction and rotation about the  $Z_1$ -axis. From the expressions given earlier for the moments of inertia it can be seen that these are in fact the second moments of the mass distribution about the chosen frame fixed in the body.

For the body shown in [Figure 2.28](#), we are only interested in rotation about the  $Z_1$ -axis and as such only require the  $I_{zz}$  moment of inertia. To indicate that this is for Body 2 we will refer to this as  $I_{2zz}$ . Considering the particle of mass  $\delta m$  located by a general position vector  $\{R\}_2$  at point P, we see that we are not only measuring the vector with respect to frame  $O_2$  but we are also referring the vector to frame  $O_2$ .

Ignoring the  $z$  coordinate as this is for plane motion in  $X_1Y_1$  we can take the general case and say that the position of the element is given by  $\{R\}_2^T = [x \ y \ 0]$ . The moment of inertia  $I_{2zz}$  is found by summing the second moments of the elements of mass over the volume of the body.

$$\begin{aligned} I_{2zz} &= \sum |R|_2^2 \delta m \\ &= \sum (x^2 + y^2) \delta m \end{aligned} \quad (2.179)$$

In the limit as the volume of the element becomes infinitesimal, we can obtain an expression for the moment of inertia  $I_{2zz}$  as follows

$$I_{2zz} = \int (x^2 + y^2) dm \quad (2.180)$$

The moment of inertia  $I_{2zz}$  in [Eqn \(2.180\)](#) is therefore the integral of the mass elements, each multiplied by the square of the radial distance from the  $z$ -axis. This distance is referred to as the radius of gyration, in this case  $k_{2zz}$  where this is given by

$$k_{2zz} = \sqrt{\frac{I_{2zz}}{m_2}} \quad (2.181)$$

From this we can use the radius of gyration to express the moment of inertia in more general terms as

$$I_{2zz} = m_2 k_{2zz}^2 \quad (2.182)$$

It is now possible to demonstrate how the moment of inertia may be derived for a standard shape. This is demonstrated in the following example where the moment of inertia is derived for the rectangle shown in [Figure 2.29](#). This again considers the two-dimensional case for plane motion where the rectangular body, Body 2, is constrained to only move in the  $X_1Y_1$  plane of frame  $O_1$ .

Taking this body to have a thickness of  $t$  and a density of  $\rho$ , we can say that the mass  $\delta m$  of the small element of mass with dimensions  $\delta x$  and  $\delta y$  is given by

$$\delta m = \rho t \delta x \delta y \quad (2.183)$$

The moment of inertia  $I_{2zz}$  is again found by summing the second moments of the elements of mass over the volume of the body.

$$\begin{aligned} I_{2zz} &= \sum (x^2 + y^2) \delta m \\ &= \rho t \sum (x^2 + y^2) \delta x \delta y \end{aligned} \quad (2.184)$$

This leads to the following equation as the volume of the element becomes infinitesimal.

$$I_{2zz} = \rho t \int_{-d/2}^{d/2} \int_{-b/2}^{b/2} (x^2 + y^2) dx dy \quad (2.185)$$

Solving this double integral gives

$$\begin{aligned} I_{2zz} &= \rho t \int_{-d/2}^{d/2} \left[ \frac{x^3}{3} + y^2 x \right]_{-b/2}^{b/2} dy \\ I_{2zz} &= \rho t \int_{-d/2}^{d/2} \left( \frac{b^3}{24} + \frac{y^2 b}{2} \right) - \left( -\frac{b^3}{24} - \frac{y^2 b}{2} \right) dy \\ I_{2zz} &= \rho t \int_{-d/2}^{d/2} \left( \frac{b^3}{12} + y^2 b \right) dy \\ I_{2zz} &= \rho t \left[ \frac{b^3 y}{12} + \frac{y^3 b}{3} \right]_{-d/2}^{d/2} \\ I_{2zz} &= \rho t \left( \frac{b^3 d}{24} + \frac{d^3 b}{24} \right) - \left( -\frac{b^3 d}{24} + \frac{d^3 b}{24} \right) \\ I_{2zz} &= \rho t \left( \frac{b^3 d + d^3 b}{12} \right) \end{aligned} \quad (2.186)$$

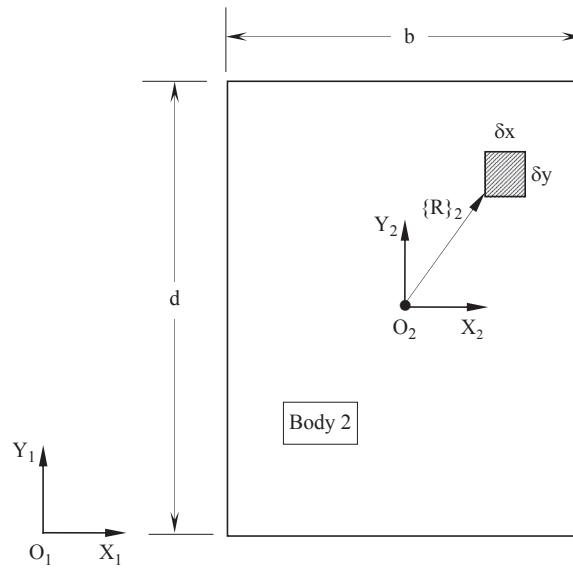
Since the mass of the rectangle body  $m_2$  is given by  $m_2 = \rho t b d$ , we can write

$$I_{2zz} = m_2 \left( \frac{b^2 + d^2}{12} \right) \quad (2.187)$$

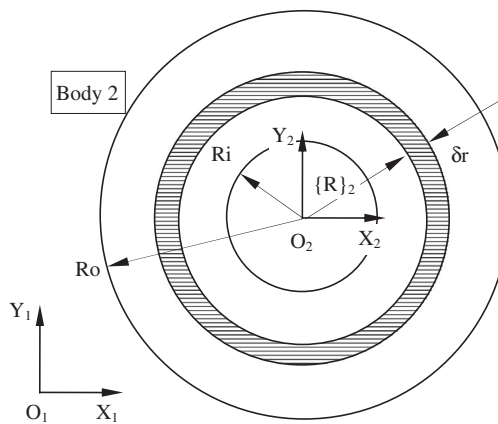
Another example of a standard shape is the ring shown in [Figure 2.30](#). This again considers the two-dimensional case for plane motion, Body 2 is constrained to only move in the  $X_1 Y_1$  plane of frame  $O_1$ .

Taking this body to have a thickness of  $t$  and a density of  $\rho$ , we can say that the mass  $\delta m$  of the small elemental ring of mass, at a radius  $R = |R|_2$ , with radial width  $\delta r$  is given by

$$\delta m = \rho t 2 \pi R \delta r \quad (2.188)$$


**FIGURE 2.29**

Moment of inertia for a rectangle.


**FIGURE 2.30**

Moment of inertia for a ring.

The moment of inertia  $I_{2zz}$  is again found by summing the second moments of the elements of mass over the volume of the body.

$$\begin{aligned} I_{2zz} &= \sum R^2 \delta m \\ &= 2 \pi \rho t \sum R \cdot R^2 \delta r \end{aligned} \quad (2.189)$$

This leads to the following equation as the volume of the element becomes infinitesimal.

$$I_{2zz} = 2 \pi \rho t \int_{R_i}^{R_o} R^3 dr \quad (2.190)$$

Solving this integral gives

$$I_{2zz} = 2 \pi \rho t \left[ \frac{R^4}{4} \right]_{R_i}^{R_o}$$

$$I_{2zz} = \frac{\pi t \rho}{2} (R_o^4 - R_i^4)$$

Since the mass of the ring  $m_2$  is given by  $m_2 = \rho t \pi (R_o^2 - R_i^2)$ , we can write

$$I_{2zz} = \frac{m_2}{2} \frac{(R_o^4 - R_i^4)}{(R_o^2 - R_i^2)} \quad (2.191)$$

$$I_{2zz} = \frac{m_2}{2} (R_o^2 + R_i^2)$$

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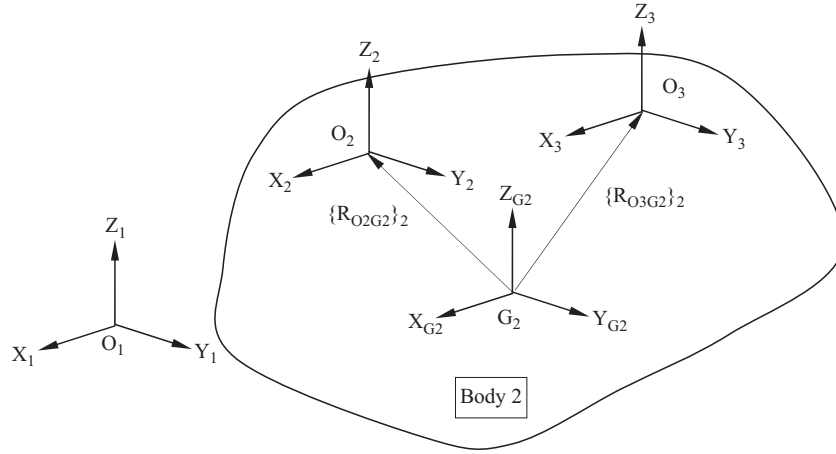
## 2.11 Parallel axes theorem

If a rigid body comprises rigidly attached combinations of regular shapes, such as those just described, the overall inertial properties of the body may be found using the parallel axes theorem. Returning to the three-dimensional situation we can consider the two parallel axes systems  $O_2$  and  $O_3$  both fixed in Body 2 as shown in [Figure 2.31](#).

It is now possible to show that there is an inertia matrix for Body 2 associated with frame  $O_2$ , which would be written  $[I_2]_{2/2}$ . In a similar manner it is possible to determine the terms in a moment of inertia matrix  $[I_2]_{3/2}$ , where the use of the upper suffix here indicates that the moments of inertia have been measured relative to the origin of frame  $O_3$  and the lower suffix indicates that the terms in the matrix are transformed to frame  $O_2$ . Since  $O_2$  and  $O_3$  are parallel the matrix  $[I_2]_{3/3}$  would be identical to  $[I_2]_{3/2}$ .

The positions of the frames  $O_2$  and  $O_3$  relative to  $G_2$  the mass centre of Body 2 can be given by

$$\{R_{O_2 G_2}\}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}, \quad \{R_{O_3 G_2}\}_2 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \quad (2.192)$$

**FIGURE 2.31**

Parallel axes theorem.

where according to the triangle law of vector addition if  $a$ ,  $b$  and  $c$  are the components of the relative position vector  $\{R_{O3O2}\}_2$  we can write

$$\{R_{O3O2}\}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \{R_{O3G2}\}_2 = \begin{bmatrix} x_2 + a \\ y_2 + b \\ z_2 + c \end{bmatrix} \quad (2.193)$$

On this basis it is possible to relate a moment of inertia, for example  $I_{2x_3x_3}$  for frame  $O_3$  to  $I_{2x_2x_2}$  for frame  $O_2$ .

$$\begin{aligned} I_{2x_3x_3} &= \int (y_3^2 + z_3^2) dm \\ &= \int [(y_2 + b)^2 + (z_2 + c)^2] dm \\ &= \int [(y_2^2 + 2y_2 b + b^2) + (z_2^2 + 2z_2 c + c^2)] dm \\ &= I_{2x_2x_2} + 2b \int y_2 dm + 2c \int z_2 dm + (b^2 + c^2) m_2 \\ &= I_{2x_2x_2} + 2m_2(by_2 + cz_2) + m_2(b^2 + c^2) \end{aligned} \quad (2.194)$$

If we take the situation where  $O_2$  is coincident with  $G_2$  the mass centre of Body 2, such that  $x_2$ ,  $y_2$  and  $z_2$  are zero, then [Eqn \(2.194\)](#) can be simplified to:

$$I_{2x_3x_3} = I_{2x_2x_2} + m_2(b^2 + c^2) \quad (2.195)$$

In a similar manner it is possible to relate a product of inertia, for example  $I_2 y_3 z_3$  for frame  $O_3$  to  $I_2 y_2 z_2$  for frame  $O_2$ .

$$\begin{aligned}
 I_2 y_3 z_3 &= - \int y_3 z_3 \, dm \\
 &= - \int (y_2 + b) (z_2 + c) \, dm \\
 &= - \int y_2 z_2 + c y_2 + b z_2 + bc \, dm \\
 &= I_2 y_2 z_2 - m_2 (c y_2 + b z_2) - m_2 bc
 \end{aligned} \tag{2.196}$$

Taking again  $O_2$  to lie at the mass centre  $G_2$  we can simplify Eqn (2.196) to

$$I_2 y_3 z_3 = I_2 y_2 z_2 - m_2 b c \tag{2.197}$$

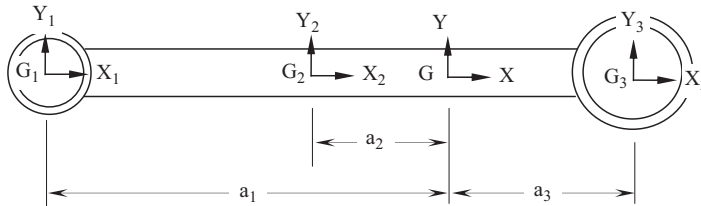
On the basis of the derivation of the relationships in Eqns (2.195) and (2.197), we can find in a similar manner the full relationship between  $[I_2]_{3/2}$  and  $[I_2]_{2/2}$  to be

$$[I_2]_{3/2} = [I_2]_{2/2} + m_2 \begin{bmatrix} b^2 + c^2 & -ab & -ac \\ -ab & c^2 + a^2 & -bc \\ -ac & -bc & a^2 + b^2 \end{bmatrix} \tag{2.198}$$

A practical application of the parallel axes theorem given in Eqn (2.198) is provided using the simplified representation of a tie rod as shown in Figure 2.32. The body can be considered as an assembly of three components with centres of mass at  $G_1$ ,  $G_2$  and  $G_3$ . The mass centre of the entire body is located at  $G$ . The components have masses  $m_1$ ,  $m_2$  and  $m_3$  and moments of inertia about the local  $z$ -axis at each mass centre  $I_{G1zz}$ ,  $I_{G2zz}$  and  $I_{G3zz}$ .

Applying the parallel axes theorem would, in this case, give a moment of inertia  $I_{Gzz}$  for the body using

$$I_{Gzz} = I_{G1zz} + m_1 a_1^2 + I_{G2zz} + m_2 a_2^2 + I_{G3zz} + m_3 a_3^2 \tag{2.199}$$



**FIGURE 2.32**

Application of parallel axes theorem.

## 2.12 Principal axes

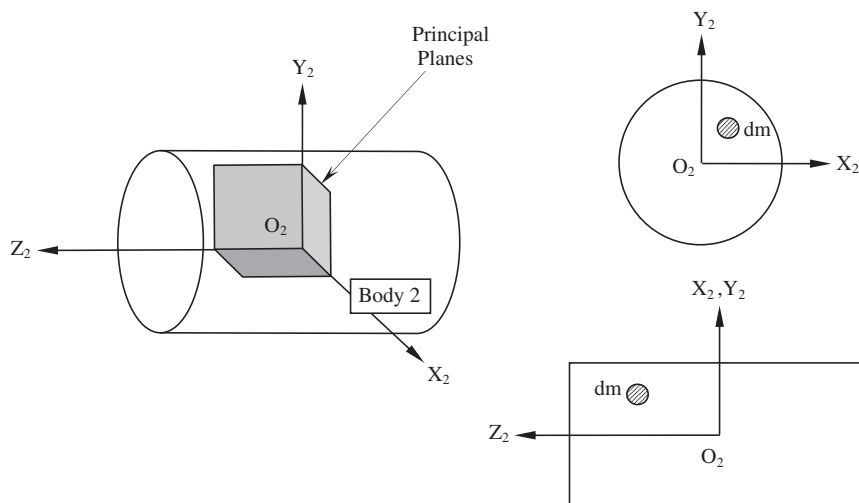
The principal axes of any rigid body are those for which the products of inertia are all zero resulting in an inertia matrix of the form

$$[I] = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \quad (2.200)$$

where in this case  $I_1$ ,  $I_2$  and  $I_3$  are the principal moments of inertia. The three planes formed by the principal axes are referred to as the principal planes, as shown in [Figure 2.33](#). In this example the geometry chosen is a solid cylinder to demonstrate the concept.

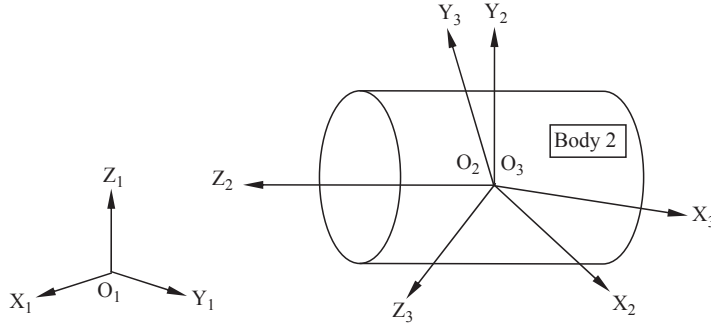
For the cylinder the principal axes are represented by the frame  $O_2$  positioned at the mass centre of the body. In this case each of the principal planes is a plane of symmetry for the body. As can be seen for each element of mass with positive coordinates there are other elements of mass, reflected in each of the principal planes, with negative coordinates. The result of this is that the products of inertia are all zero.

Returning to the consideration of the angular momentum of a body given in [Eqn \(2.178\)](#), this can now be written as shown in [Eqn \(2.201\)](#) where  $I_1$ ,  $I_2$  and  $I_3$  are the principal moments of inertia for Body 2 taken about the origin of frame  $O_2$ . The moment of inertia matrix is referred to the principal axes, again frame  $O_2$  and the products of inertia are zero.



**FIGURE 2.33**

Principal axes for a body.



**FIGURE 2.34**

Transformation to principal axes.

$$\{H_2\}_{1/2} = [I_2]_{2/2} \{\omega_2\}_{1/2}$$

$$\begin{bmatrix} H_{2x} \\ H_{2y} \\ H_{2z} \end{bmatrix} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \begin{bmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{bmatrix} = \begin{bmatrix} I_1 & \omega_{2x} \\ I_2 & \omega_{2y} \\ I_3 & \omega_{2z} \end{bmatrix} \quad (2.201)$$

The principal axes and the principal moments of inertia may be obtained by considering the two frames  $O_3$  and  $O_2$  both located at the mass centre in Body 2, as shown in Figure 2.34. The axes associated with frame  $O_2$  are again taken to be the principal axes of the body. The angular momentum  $\{H_2\}_{1/2}$  referred to  $O_2$  can be transformed from the angular momentum  $\{H_2\}_{1/3}$  referred to  $O_3$  using the rotational transformation matrix  $[T_3]_2$  as described in Section 2.2.7.

$$\{H_2\}_{1/2} = [T_3]_2 \{H_2\}_{1/3} \quad (2.202)$$

Since  $\{H_2\}_{1/2} = [I_2]_{2/2} \{\omega_2\}_{1/2}$  and  $\{H_2\}_{1/3} = [I_2]_{2/3} \{\omega_2\}_{1/3}$  it follows from Eqn (2.202) that

$$[I_2]_{2/2} \{\omega_2\}_{1/2} = [T_3]_2 [I_2]_{2/3} \{\omega_2\}_{1/3} \quad (2.203)$$

If we premultiply a transformation matrix  $[T]$  by its transpose  $[T]^T$  a property of these matrices is that we produce an identity matrix.

$$[T]^T [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.204)$$

This allows us to further develop Eqn (2.203) to give

$$[I_2]_{2/2} \{\omega_2\}_{1/2} = [T_3]_2 [I_2]_{2/3} [T_3]_2^T [T_3]_2 \{\omega_2\}_{1/3} \quad (2.205)$$



Since we also know that  $\{\omega_2\}_{1/2} = [T_3]_2 \{\omega_2\}_{1/3}$  we can therefore write

$$[I_2]_{2/2} = [T_3]_2 [I_2]_{2/3} [T_3]_2^T \quad (2.206)$$

In this case we require a transformation matrix  $[T_3]_2$ , which is orthogonal and will lead to a diagonal matrix  $[I_2]_{2/2}$ . If we now take [Eqn \(2.206\)](#) and premultiply by  $[T_3]_2^T$  we get

$$[T_3]_2^T [I_2]_{2/2} = [I_2]_{2/3} [T_3]_2^T$$

$$\begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix}$$

$$\begin{bmatrix} T_{11}I_1 & T_{21}I_2 & T_{31}I_3 \\ T_{12}I_1 & T_{22}I_2 & T_{32}I_3 \\ T_{13}I_1 & T_{23}I_2 & T_{33}I_3 \end{bmatrix} = \begin{bmatrix} I_{xx}T_{11} + I_{xy}T_{12} + I_{xz}T_{13} & I_{xx}T_{21} + I_{xy}T_{22} + I_{xz}T_{23} & I_{xx}T_{31} + I_{xy}T_{32} + I_{xz}T_{33} \\ I_{xy}T_{11} + I_{yy}T_{12} + I_{yz}T_{13} & I_{xy}T_{21} + I_{yy}T_{22} + I_{yz}T_{23} & I_{xy}T_{31} + I_{yy}T_{32} + I_{yz}T_{33} \\ I_{xz}T_{11} + I_{yz}T_{12} + I_{zz}T_{13} & I_{xz}T_{21} + I_{yz}T_{22} + I_{zz}T_{23} & I_{xz}T_{31} + I_{yz}T_{32} + I_{zz}T_{33} \end{bmatrix} \quad (2.207)$$

Equating the first columns from the matrices on either side of [Eqn \(2.207\)](#) leads to the eigenvalue equation in [\(2.208\)](#)

$$I_1 \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} \quad (2.208)$$

This equation can be rearranged to give

$$\begin{bmatrix} I_{xx} - I_1 & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I_1 & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I_1 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2.209)$$

For a nontrivial solution to [Eqn \(2.209\)](#), we require the determinant of the square matrix containing  $I_1$  to be zero leading to the characteristic equation

$$\begin{vmatrix} I_{xx} - I_1 & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I_1 & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I_1 \end{vmatrix} = 0 \quad (2.210)$$

The process used to find the determinant is documented in standard texts dealing with the mathematical manipulation of matrices, but in general a three by three square matrix may be summarised as follows

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} = A(EI - HF) - B(DI - GF) + C(DH - GE) \quad (2.211)$$

The solution of Eqn (2.210) leads to a cubic equation in  $I_1$  with three positive real roots, these being the three principal moments of inertia  $I_1$ ,  $I_2$  and  $I_3$ . If each of these is substituted in turn into equations that equate all three columns on either side of Eqn (2.207) we get

$$I_1 \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} \quad (2.212)$$

$$I_2 \begin{bmatrix} T_{21} \\ T_{22} \\ T_{23} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} T_{21} \\ T_{22} \\ T_{23} \end{bmatrix} \quad (2.213)$$

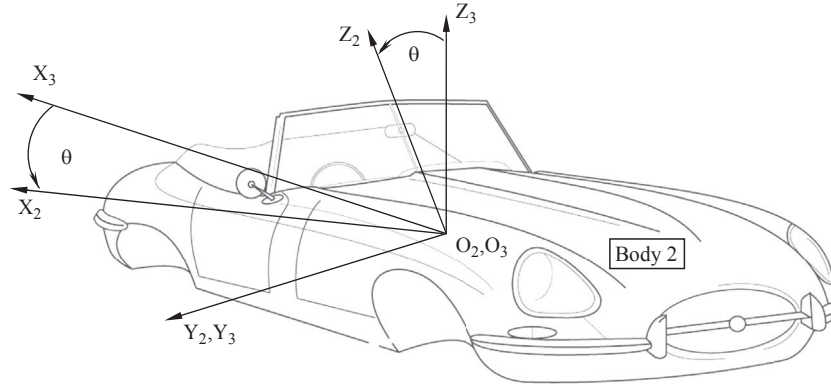
$$I_3 \begin{bmatrix} T_{31} \\ T_{32} \\ T_{33} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} T_{31} \\ T_{32} \\ T_{33} \end{bmatrix} \quad (2.214)$$

The solution of Eqns (2.212)–(2.214) thus yields all the terms in  $[T_3]_2$ , the transformation matrix from frame  $O_3$  to  $O_2$ . In summary,  $I_1$ ,  $I_2$  and  $I_3$  are the eigenvalues of the inertia matrix  $[I_2]_{2/3}$  and are also the principal moments of inertia for Body 2 these being the diagonal terms in the matrix  $[I_2]_{2/2}$ . The three column matrices in Eqns (2.212)–(2.214)

$$\begin{bmatrix} T_{11} \\ T_{12} \\ T_{13} \end{bmatrix} \quad \begin{bmatrix} T_{21} \\ T_{22} \\ T_{23} \end{bmatrix} \quad \begin{bmatrix} T_{31} \\ T_{32} \\ T_{23} \end{bmatrix}$$

are the eigenvectors of  $[I_2]_{2/3}$ . If each vector is now normalised so that the length of the vector is unity, we get the direction cosines between each of the axes of  $O_2$ , the principal axes of Body 2, and  $O_3$ .

We can now consider a practical application of this with regard to vehicle dynamics where the body of a vehicle will generally be the largest and most significant mass in the model. For the vehicle body, Body 2, shown in Figure 2.35 we can take frame  $O_3$  to be positioned at the mass centre and orientated so that the x-axis is along the centre line and pointing to the rear of the vehicle and the z-axis is vertical. The  $X_3Z_3$  plane is thus a plane of symmetry. It should be noted that in reality this assumption involves some approximation due to the asymmetry of the masses that

**FIGURE 2.35**

Vehicle body coordinate system.

may be lumped with the vehicle body, such as the engine, battery, exhaust system and fuel tank. The frame  $O_2$ , shown in Figure 2.35 is also positioned at the mass centre and has its  $Y_2$ -axis coincident with the  $Y_3$ -axis of frame  $O_3$ . The frame  $O_2$  represents the principal axes of the vehicle body and is obtained by a transformation from  $O_3$  represented by the rotation through an angle  $\theta$  about the  $Y_2$ - and  $Y_3$ -axes.

In determining the products of inertia for this body it can be seen that, for every element of mass with a positive  $y$  coordinate there exists an equivalent element with a negative  $y$  coordinate. As a result we get

$$I_{xy} = I_{yx} = - \int xy \, dm = 0$$

$$I_{yz} = I_{zy} = - \int yz \, dm = 0$$

The inertia matrix for Body 2  $[I_2]_{2/3}$  measured from frame  $O_2$  and referred to  $O_3$  is therefore

$$[I_2]_{2/3} = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_2 & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \quad (2.215)$$

From this it can be seen that the  $y$ -axis is a principal axis and is normal to the plane of symmetry. The principal moment of inertia  $I_2$  is therefore equal to  $I_{yy}$ . From Section 2.2.7 we can see that the matrix  $[T_3]_2$  that transforms from frame  $O_3$  to  $O_2$  is given by

$$[T_3]_2 = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \quad (2.216)$$

From Eqn (2.206), we can see that using the transformation matrix  $[T_3]_2$  given for this particular case will lead to

$$[I_2]_{2/2} = [T_3]_2 [I_2]_{2/3} [T_3]_2^T$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_2 & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} I_{xx} \cos \theta - I_{xz} \sin \theta & 0 & I_{xx} \sin \theta + I_{xz} \cos \theta \\ 0 & I_2 & 0 \\ I_{xz} \cos \theta - I_{zz} \sin \theta & 0 & I_{xz} \sin \theta + I_{zz} \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} =$$

$$\begin{bmatrix} I_{xx} \cos^2 \theta - 2I_{xz} \sin \theta \cos \theta + I_{zz} \sin^2 \theta & 0 & I_{xx} \sin \theta \cos \theta + I_{xz} \cos^2 \theta - I_{xz} \sin^2 \theta - I_{zz} \sin \theta \cos \theta \\ 0 & I_2 & 0 \\ I_{xx} \sin \theta \cos \theta + I_{xz} \cos^2 \theta - I_{xz} \sin^2 \theta - I_{zz} \sin \theta \cos \theta & 0 & I_{xx} \sin^2 \theta + 2I_{xz} \sin \theta \cos \theta + I_{zz} \cos^2 \theta \end{bmatrix} \quad (2.217)$$

Multiplying out the matrix equation in (2.217) leads to the following expressions for the principal moments of inertia  $I_1$  and  $I_3$ .

$$I_1 = I_{xx} \cos^2 \theta - 2 I_{xz} \sin \theta \cos \theta + I_{zz} \sin^2 \theta \quad (2.218)$$

$$I_3 = I_{xx} \sin^2 \theta + I_{xz} \cos^2 \theta + I_{xz} \sin^2 \theta - I_{zz} \sin \theta \cos \theta \quad (2.219)$$

Equating now the zero elements on the left-hand side of Eqn (2.217) with the terms in either row1column3 or column1row3 gives

$$0 = I_{xx} \sin \theta \cos \theta - I_{xz} \sin^2 \theta + I_{xz} \cos^2 \theta - I_{zz} \sin \theta \cos \theta \quad (2.220)$$

This can be rearranged to give

$$0 = I_{xz} (\cos^2 \theta - \sin^2 \theta) + (I_{xx} - I_{zz}) \sin \theta \cos \theta \quad (2.221)$$

From trigonometric addition formulae we can make use of

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \quad \text{and} \quad \sin 2\theta = 2 \sin \theta \cos \theta$$

which leads to

$$0 = I_{xz} \cos 2\theta + \frac{1}{2} (I_{xx} - I_{zz}) \sin 2\theta \quad (2.222)$$

Rearranging Eqn (2.222) leads to an expression from which we can determine a value for  $\theta$  using the known values for  $I_{xx}$ ,  $I_{zz}$  and  $I_{xz}$ .

$$\tan 2\theta = \frac{I_{xz}}{\frac{1}{2} (I_{zz} - I_{xx})} \quad (2.223)$$

Using the value obtained for  $\theta$  in Eqn (2.223) it is now possible to substitute this back into Eqns (2.218) and (2.219) and obtain values for the two unknown principal moments of inertia  $I_1$  and  $I_3$ .

### 2.13 Equations of motion

If we consider the rigid body, Body 2, shown in Figure 2.36 we can formulate six equations of motion corresponding with the six degrees of freedom resulting from unconstrained motion.

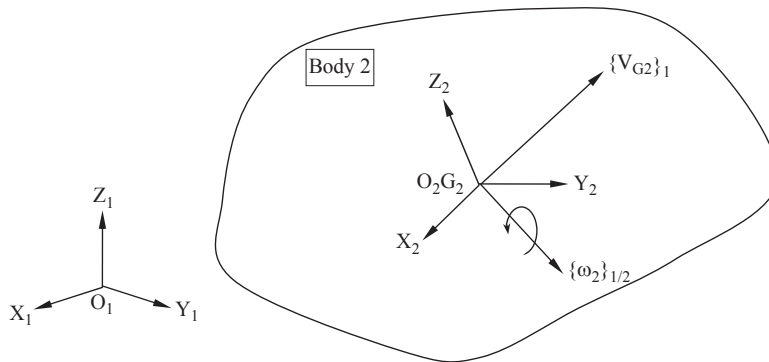
From our earlier consideration of linear momentum given in Eqn (2.161), we can write

$$\sum \{F_2\}_1 = \frac{d}{dt} \{L_2\}_1 = m_2 \frac{d}{dt} \{V_{G_2}\}_1 \quad (2.224)$$

Expressing this in the familiar form of Newton's Second Law we get

$$\sum \{F_2\}_1 = m_2 \{A_{G_2}\}_1 \quad (2.225)$$

The vector equation given in Eqn (2.225) will thus yield the three equations associated with the translational motion of the body. It may be noted that for these



**FIGURE 2.36**

Rigid body motion.

equations the vectors in Eqn (2.225) may be conveniently referred to the fixed reference frame  $O_1$ .

In the same way that a resultant force acting on the body produces a change in linear momentum, a resultant moment will produce a change in angular momentum. If we consider the expression for angular momentum given in Eqn (2.178) we can obtain the equations of motion associated with rotational motion. For the rotational equations it is convenient to refer the vectors to the reference frame  $O_2$  fixed in and rotating with Body 2.

$$\sum \{M_{G2}\}_{\frac{1}{2}} = \frac{d}{dt} \{H_2\}_{\frac{1}{2}} = \frac{d}{dt} [I_2] \{\omega_2\}_{\frac{1}{2}} \quad (2.226)$$

It can also be shown that

$$\frac{d}{dt} \{H_2\}_{\frac{1}{2}} = \frac{d}{dt} \{H_2\}_{\frac{1}{2}} + \{\omega_2\}_{\frac{1}{2}} \times \{H_2\}_{\frac{1}{2}} \quad (2.227)$$

$$\{M_{G2}\}_{\frac{1}{2}} = [I_2]_{\frac{1}{2}} \{\alpha_2\}_{\frac{1}{2}} + \{\omega_2\}_{\frac{1}{2}} \times \{H_2\}_{\frac{1}{2}} \quad (2.228)$$

Expanding Eqn (2.228) gives

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_{2x} \\ \alpha_{2y} \\ \alpha_{2z} \end{bmatrix} + \begin{bmatrix} 0 & -\omega_{2z} & \omega_{2y} \\ \omega_{2z} & 0 & -\omega_{2x} \\ -\omega_{2y} & \omega_{2x} & 0 \end{bmatrix} \begin{bmatrix} H_{2x} \\ H_{2y} \\ H_{2z} \end{bmatrix} \quad (2.229)$$

Substituting in now terms for  $H_{2x}$ ,  $H_{2y}$  and  $H_{2z}$  leads to the equations given in Eqns (2.230)–(2.232). For convenience we can drop the subscript for Body 2.

$$M_x = I_{xx} \alpha_x + I_{xy}(\alpha_y - \omega_x \omega_z) + I_{xz}(\alpha_z + \omega_x \omega_y) + (I_{zz} - I_{yy})\omega_y \omega_z + I_{yz}(\omega_y^2 - \omega_z^2) \quad (2.230)$$

$$M_y = I_{xy}(\alpha_x - \omega_y \omega_z) + I_{yy} \alpha_y + I_{yz}(\alpha_z - \omega_x \omega_y) + (I_{xx} - I_{zz})\omega_x \omega_z + I_{xz}(\omega_z^2 - \omega_x^2) \quad (2.231)$$

$$M_z = I_{xz}(\alpha_x - \omega_y \omega_z) + I_{yz}(\alpha_y + \omega_x \omega_z) + I_{zz} \alpha_z + (I_{yy} - I_{xx})\omega_x \omega_y + I_{xy}(\omega_x^2 - \omega_y^2) \quad (2.232)$$

In summary, the rotational equations of motion for Body 2 may be written in vector form as

$$\sum \{M_{G2}\}_{\frac{1}{2}} = [I_2]_{\frac{1}{2}} \{\alpha_2\}_{\frac{1}{2}} + [\omega_2]_{\frac{1}{2}} [I_2]_{\frac{1}{2}} \{\omega_2\}_{\frac{1}{2}} \quad (2.233)$$

Hence we can see that in setting up the equations of motion for any rigid body, the translational equations for all bodies in a system may conveniently be referred to a single-fixed inertial frame  $O_1$ . The rotational equations, however, are better referred to a body-centred frame, in this case  $O_2$ . A considerable simplification in these equations will result if frame  $O_2$  is selected such that its axes are the principal axes of the body ( $I_1 = I_{xx}$ ,  $I_2 = I_{yy}$ ,  $I_3 = I_{zz}$ ) and the products of inertia are zero. The equations that result are known as Euler's equations of motion:

$$M_x = I_1 \alpha_x + (I_3 - I_2) \omega_y \omega_z \quad (2.234)$$

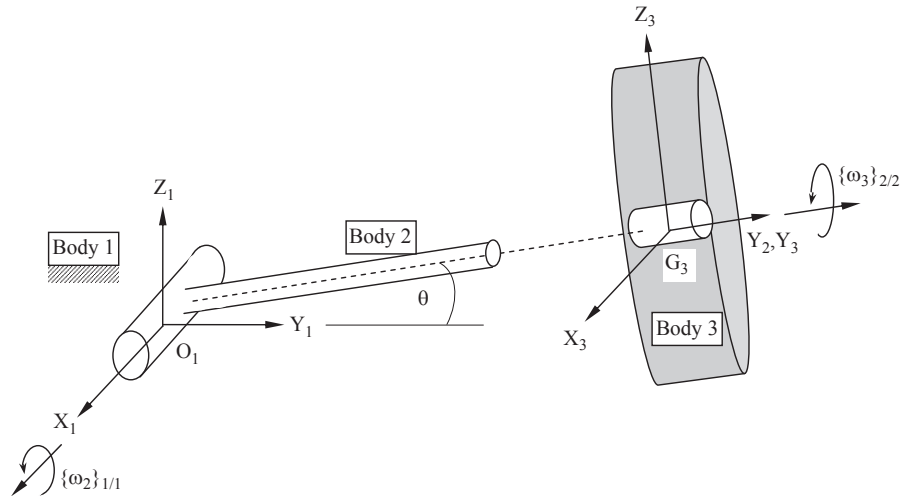
$$M_y = I_2 \alpha_y + (I_1 - I_3) \omega_x \omega_z \quad (2.235)$$

$$M_z = I_3 \alpha_z + (I_2 - I_1) \omega_x \omega_y \quad (2.236)$$

The equations given in Eqns (2.234)–(2.236) become even simpler when the motion of a body is constrained so that rotation takes place in one plane only. If, for example, rotation about the x- and y-axes are prevented then Eqn (2.236) reduces to the more familiar form associated with two-dimensional motion.

$$M_z = I_3 \alpha_z \quad (2.237)$$

The following example also demonstrates how gyroscopic effects associated with three-dimensional motion may be identified. If we consider the swing arm suspension system shown in Figure 2.37 we can take the suspension arm Body 2 to be constrained by a revolute joint to rotate with a constant angular velocity of 10 rad/s about the axis of the joint as shown. Whilst this motion is in progress the road wheel Body 3 is also rotating with a constant angular velocity of 100 rad/s about the axis of the revolute joint representing the wheel bearing.



**FIGURE 2.37**

Swing arm suspension dynamics.

The following data may be used to represent the mass properties of the road wheel

$$m_3 = 16 \text{ kg}$$

$$I_{31} = I_{3xx} = 0.5 \text{ kgm}^2$$

$$I_{32} = I_{3yy} = 1.0 \text{ kgm}^2$$

$$I_{33} = I_{3zz} = 0.5 \text{ kgm}^2$$

In order to determine the reaction torque on the wheel bearing when the axle is still in a horizontal position,  $\theta = 0$ , we can use the following to give us the required angular velocity vector  $\{\omega_3\}_{1/3}$ .

$$\{\omega_2\}_{1/2} = \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \text{ rad/s} \quad \{\omega_3\}_{2/3} = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix} \text{ rad/s} \quad \{\omega_3\}_{1/3} = \begin{bmatrix} 10 \\ 100 \\ 0 \end{bmatrix} \text{ rad/s}$$

In the absence of angular acceleration, [Eqn \(2.233\)](#) can be adapted to give for this problem

$$\sum \{M_3\}_{1/3} = [\omega_3]_{1/3} [I_3]_{1/3} \{\omega_3\}_{1/3} \quad (2.238)$$

Expanding this gives

$$\sum \begin{bmatrix} M_{3x} \\ M_{3y} \\ M_{3z} \end{bmatrix}_{1/3} = \begin{bmatrix} 0 & -\omega_{3z} & \omega_{3y} \\ \omega_{3z} & 0 & -\omega_{3x} \\ -\omega_{3y} & \omega_{3x} & 0 \end{bmatrix}_{1/3} \begin{bmatrix} I_{3xx} & 0 & 0 \\ 0 & I_{3yy} & 0 \\ 0 & 0 & I_{3zz} \end{bmatrix}_{1/3} \begin{bmatrix} \omega_{3x} \\ \omega_{3y} \\ \omega_{3z} \end{bmatrix}_{1/3} \quad (2.239)$$

Substituting in the numerical data for this problem gives

$$\sum \begin{bmatrix} M_{3x} \\ M_{3y} \\ M_{3z} \end{bmatrix}_{1/3} = \begin{bmatrix} 0 & 0 & 100 \\ 0 & 0 & -10 \\ -100 & 10 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 10 \\ 100 \\ 0 \end{bmatrix} \text{ Nm}$$

$$\sum \begin{bmatrix} M_{3x} \\ M_{3y} \\ M_{3z} \end{bmatrix}_{1/3} = \begin{bmatrix} 0 & 0 & 100 \\ 0 & 0 & -10 \\ -100 & 10 & 0 \end{bmatrix} \begin{bmatrix} 5.0 \\ 100 \\ 0 \end{bmatrix} \text{ Nm} \quad (2.240)$$

$$\sum \begin{bmatrix} M_{3x} \\ M_{3y} \\ M_{3z} \end{bmatrix}_{1/3} = \begin{bmatrix} 0 \\ 0 \\ 500 \end{bmatrix} \text{ Nm}$$



As can be seen from the result in Eqn (2.240), the reaction torque on the wheel bearing is about the z-axis and is due to gyroscopic effects as the wheel spins about the y-axis and rotates about the x-axis.

It should be noted that in addition to the derivation of the equations of motion based on the direct application of Newton's Laws, variational methods, including for example Lagrange's equations, provide an elegant alternative and are often employed in MBS formulations. Many texts on classical dynamics, such as (D'Souza and Garg, 1984), include a thorough treatment of these methods.

Variational methods are attractive for a number of reasons. Equations are formulated using kinetic energy and work resulting in scalar rather than vector terms. Solutions can also be more efficient since constraint forces that do not perform work can be omitted. Variational methods also make use of generalised rather than physical coordinates reducing the number of equations required.

The theory and methods described in this chapter form a basis for the MBS formulations covered in the next chapter. The vector notation used here will be used to describe the part equations and the constraint equations required to represent joints constraining relative motion between interconnected bodies. In Chapter 4 the vector-based methods described here will be used to carry out a range of analyses from first principles on a double-wishbone suspension system and to compare the calculated results with those found using MSC.ADAMS.

# Multibody Systems Simulation Software

# 3

*We can only see a short distance ahead, but we can see plenty there that needs to be done.*

Alan Turing

## 3.1 Overview

There exists a range of commercial computer packages that can be used to solve problems in multibody systems analysis. In addition to the commercial packages that may be licensed there are also programs developed by academic institutions that may be available, albeit without the level of development and support that would be expected when buying the software from an established program developer. The first version of this book centred on MSC ADAMS but the progression of other software providers and their diversity of approach has broadened the coverage in this chapter handsomely. The purpose of this chapter is in principle to equip the reader to understand the capabilities of any multibody systems (MBS) analysis programs used in vehicle dynamics.

General-purpose MBS programs are able to address a large set of problems across a wide range of engineering industries and are not restricted to the applications in vehicle dynamics discussed here. MBS software within the automotive industry is used to simulate the performance of anything that moves.

From door latches to gearbox synchromesh, models are increasingly being used for large amplitude nonlinear vibration problems, even in areas that were unthinkable 20 years ago, such as timing chain dynamics. Many of the general-purpose programs have developed toolkits that allow a system to be exercised and validated in isolation before being included in a larger system. Previously, a separate subsystem test rig and full vehicle model would have been prepared, leading to transcription errors when moving between the two and requiring the upkeep of two models when design changes needed to be tracked. The analyst will often wish to validate the performance of a suspension model over a range of displacements and loads before the assembly of a full vehicle model that may be used for ride, handling and durability studies. A detailed model may include representations of the body, sub-frames, suspension arms, struts, anti-roll bars, steering system, engine, drivetrain and tyres. Some of the elements may be structurally compliant and others

```

UNITS/FORCE = NEWTON, MASS = KILOGRAM, LENGTH = MILLIMETER, TIME = SECOND
PART/1, GROUND
MARKER/1, PART = 1, QP = 0, 0, -100
MARKER/2, PART = 1, FLOATING
PART/2, MASS = 11.64, CM = 22, IP = 5.099E+005, 3.851E+005, 6.002E+005
, VX = -40230
MARKER/3, PART = 2, QP = 681.85, 0, 802.28, REULER = 90D, 30.3D, 270D
GRAPHICS/57, CYLINDER, CM = 6, LENGTH = -150, RADIUS = 18
VARIABLE/26, FUNCTION = IF((VARVAL(25)+15.0D):
SFORCE/18, TRANSLATIONAL, I = 38, J = 26, FUNCTION = VARVAL(26) * 8.96
DIFF/4, IC = 0, IMPLICIT, FUNCTION = DIF(4)-varval(29)
TFSISO/100, X = 100, U = 101, Y = 102, NUMERATOR = 986.96, DENOMINATOR = 986.96
, 44.429, 1

```

**FIGURE 3.1**

A fragment of a model in the language expected by the solver, which is technically human readable but somewhat difficult to work with.

may have elaborate behaviour captured in empirical models embedded within dedicated subroutines of the solver. Examples of the latter include elastomer elements, dampers, tyres and human beings.

The main analysis code consists of a number of integrated programs that perform three-dimensional kinematic, static, quasi-static or dynamic analysis of mechanical systems. These programs may be thought of as the core solver. In addition there are a number of auxiliary programs, which can be supplied to link with the core solver. These programs typically capture the embedded empirical models and are usually kept separate to the core solver to allow the software vendor to sell a modular system and reduce costs for customers who have no need of certain items. Frequently the architecture of the solver is open enough that users may from time to time develop their own embodiment of empirical models and link them to the core solver; the Harty tyre model is one such example of this.

Prior to submitting an analysis the model needs to be described to the solver in a way that it understands and most solvers have some form of proprietary information format, as shown in [Figure 3.1](#), in which they expect to read the description. It is typically true that this format can be a little difficult to prepare and also to read once it has been prepared, and so for several decades now additional programs have been available to allow users to describe a mechanism or system in a broadly ‘human-friendly’ way<sup>1</sup> before allowing a machine translation to occur into the solver language.

These programs can also generally handle libraries of commonly used blocks and allow their reuse and recombination to analyse hitherto unexamined systems. In this manner it saves a large amount of labour in model preparation. Such a program used to be known as a *preprocessor* and is still often a completely separate executable from the main solver. When working at the edge of the available computational budget, it can be advantageous to work with the preprocessor to define the model

<sup>1</sup>In the last 40 years tremendous strides have been made in human–computer interfacing and there is no reason to believe it will stop where we are now.

and then shut it down before starting the solver. Modern preprocessors are typically graphically rich and extremely interactive, which has the effect of making them surprisingly resource-hungry.

Once a model has been defined the core solver will assemble the equations of motion and solve them automatically. It is also possible to include differential equations, transfer functions and other items directly in the solution, which allows the modelling of a variety of control systems, thermal effects and many others.

The preprocessor can link or interface with CAD systems, other analysis software (e.g. finite element, control system or hydraulic modelling), software used for advanced visualisation or software for downstream processing as part of a larger experiment in progress. The combined use of these systems can lead to the development of what may be referred to as ‘virtual prototypes’: numerical models that can simulate the tests and conditions to which a real prototype would be subject during the development of a new engineering product. The extent of these interfaces is diagrammatically represented in [Figure 3.2](#).

Note that although the software will be continually developed after publication, there is good reason to suspect<sup>2</sup> that this architecture will continue for some time, since it is both computationally and organisationally convenient.

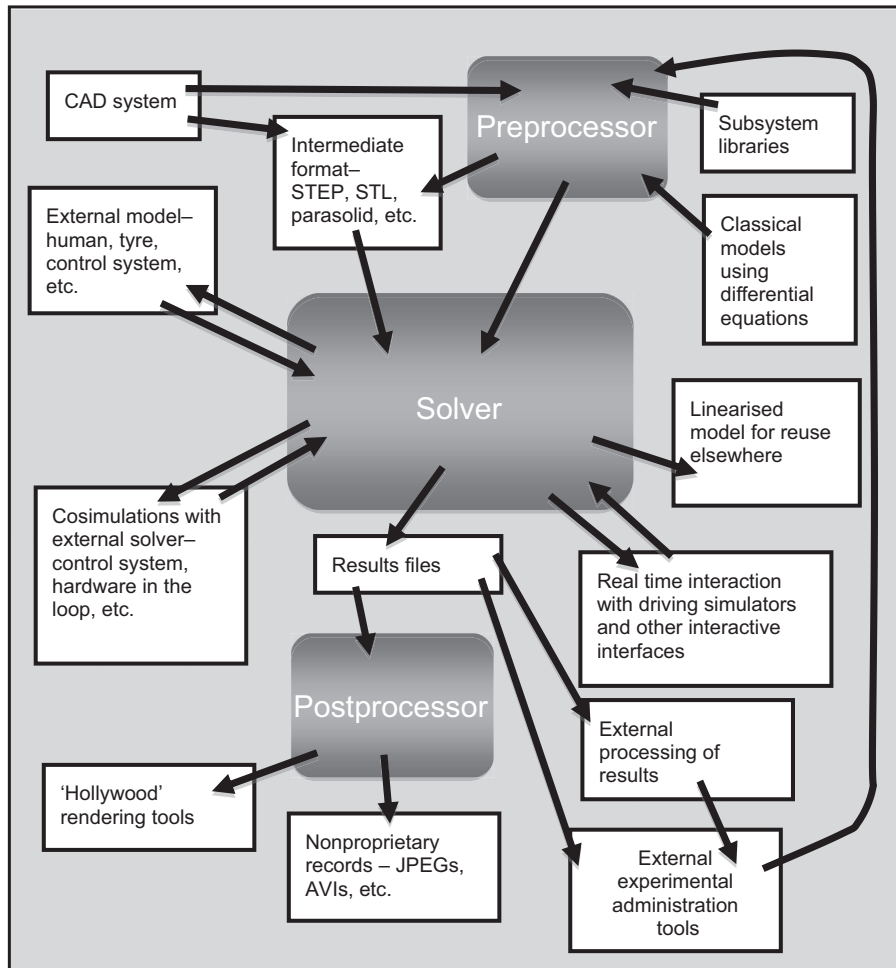
The first step in any simulation is to prepare a data set that will define the system being modelled. This will include a description of the rigid parts, connecting joints, motion generators, forces and compliances. With many modern packages the user is spared from actually looking at anything other than a graphical representation of the data set. Nevertheless in most codes the data set is user friendly in that the data statements are easily understood with few restrictions on format ([Figure 3.3](#)); this is a great step compared to the fixed-length formatting often used in early programs, where a 2 (integer 2) would cause a failure if a 2. (real 2.0) was expected and vice versa.

For some applications that may not be commercially worth supporting or that are more advanced, users can prepare their own user-written subroutines in languages, such as Fortran, ANSI ‘C’, Python, .NET or any of the other myriad of supported languages available. They can be linked with the main code to be invoked at solution time, often in the form of a dynamically linked library. It is worth mentioning that Fortran persists despite its age simply because of the well integrated and very complete function library and the existence of legacy code; however, it could be wise for new practitioners to learn their art in C.

For each rigid body in the system it is necessary to define the mass, centre of mass location, and mass moments of inertia. Each body will possess a set of

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<sup>2</sup>In his rather brilliant book ‘Thinking, Fast and Slow’ (2012), Daniel Kahneman observes that if something has already persisted for a period of time, it is reasonable to think that it will continue to persist. The printed page has been around for about five centuries but recording media such as the IBM printed card, magnetic data tape, floppy disks and compact flash cards have come and gone very quickly. So ‘3 Box’ analysis packages have another couple of decades in them, probably.

**FIGURE 3.2**

Interaction of a general-purpose program with other information.

```

marker create marker_name=m_fr_shadow_to_ground &
  location = &
    (fr_wheel_rad), &
    0.0, &
    (ground_height) &
  orientation=0, 0, 0

constraint create joint planar &
  joint_name=j_ground_to_fr_shadow &
  i_marker=m_ground_to_fr_shadow &
  j_marker=m_fr_shadow_to_ground
  
```

**FIGURE 3.3**

A more readable format for text files has emerged.

coordinates that can be defined in global or local coordinate systems and are considered to move with the part during the simulation. These points are used to define centre of mass locations, joint locations and orientations, force locations and directions. The relative motion between different parts in the system can usually be constrained using joints, joint primitives, couplers, gears and user-defined constraints.

There is an important conceptual difference between some of the packages, which is worth describing at this point.

Some packages, such as MSC ADAMS, bring each body into being with six degrees of freedom (DOF) — three translational and three rotational. Once the DOF have been created, they are connected to other DOF using mathematical relationships — such as the relative displacement of two points, each on a different body, being equal to zero (a spherical joint). This is set up as a constraint equation to restrict the DOF, and has to be explicitly solved at each step of the solution.

Other packages, such as Simpack, bring each body into being with only the DOF required to generate the mechanism. In this way there is no iterative solution of constrained DOF to arrive at a solution satisfying the constraint equations — instead those equations are never generated. This is much more like the classical approach, where we might consider a single DOF system and work with only one equation; nobody solves six equations to solutions of zero if they are considering a single DOF model with pencil and paper. The reduced DOF approach has clear benefits at run-time but does carry some overhead in terms of the software assembling the equations of motion.

The two styles can be somewhat confusing if one is unaware of these differences. Unfortunately, language being a rather limited and imprecise thing, the two software packages described typically use the same words but mean different things.

In Simpack, a ‘joint’ is declared with a body to bring about the DOF of interest. Thus for a falling stone — always the multibody problem of choice in much the same way as ‘Hello World’ is the task of choice when learning a new coding language — a body is declared with a prismatic joint. This results in only 1 DOF and thus the minimum amount of computation.

To make a pendulum — step two on any journey into new software — Simpack again has only 1 DOF by declaring a ‘joint’ with only rotation about the pivot point.

In MSC ADAMS, a falling stone can be modelled simply as a single body with 6 DOF and no joints, which eases the model preparation significantly. However, for the pendulum model MSC ADAMS requires five equations to constrain out the five unwanted motions in the solution, which are solved for zero deflection and thus have forces available as part of the solution.

In Simpack, to make a double pendulum the two bodies are ‘chained’ together, each with a single DOF, to make a 2 DOF model. Each model has a ‘joint’ which is really a declaration of the desired DOF. For MSC ADAMS there are now 10 equations of motion being solved for a solution of zero, and two we care about. By the time a four-bar linkage is modelled — rather confusingly, with three links and the ground — there are 17 equations being solved to zero.

Using the Simpack-style method of defining bodies chained to one another, it is obvious this approach fails when the chain makes a loop. To close the loop, Simpack makes use of ADAMS-style constraint equations, computing the forces required to solve for a relative motion of zero in the constrained DOF.

Thus in ADAMS, all ‘joints’ are mathematical constraints but in Simpack, ‘joints’ are actually DOF and ‘constraints’ are mathematical constraints.

The next step in building the model would typically be the definition of external forces and internal force elements. External forces can be constant, time histories or functionally dependent on any state variable. These forces can also be defined to be translational or rotational. They can act in the global system or can act in the local system of the body so that they effectively ‘follow’ the part during the simulation.

Users can also set up internal force elements acting between two parts to represent springs, dampers, cables or rubber mounts. Internal force elements can act along the line of sight between the points the force element connects on the two parts, or can act in some arbitrarily defined direction. In the latter case, some additional effects can be introduced, that the user may not have had in mind; it is up to the user to be in control of what they want to model. These force elements are often referred to as action–reaction forces as they always produce equal and opposite forces on the two parts connected by the force element. The elements can also be defined to act in only tension or compression and may be linear or nonlinear. Additional forces can be defined as action-only forces, in which the universe exerts a force on the model with no internal reaction. Aerodynamic forces might be modelled using an action-only force.

It is also useful if the MBS analysis program allows the definition of elaborate mathematical equations within the data set. This enables the user to formulate an expression involving user-defined constants, system constants, system variables, arithmetic IF’s, FORTRAN 90 or ANSI ‘C’ library functions, standard mathematical functions or ‘off-the-shelf’ functions supplied with the main code to represent events, such as impacts. The access to system variables can be a powerful modelling tool. The user can effectively access any displacement, velocity, acceleration or other force in the system when defining the force equation. Forces can also be defined as a function of time to vary or switch on and off as the simulation progresses. Caution is needed to ensure formulations are continuous in the time domain to avoid problems during the numerical solution of the resulting equations. Recent versions of many software packages also include a general contact force model between geometries associated with the rigid bodies.

Enforced displacement input can be defined at certain joints to be either constant or time dependent. When a motion is defined at a joint it may be translational or rotational. The motion effectively provides another constraint so that the DOF at that joint is lost to the motion. Motion expressions can be defined using all the functions available as for force definitions, except that the only system variable that should be accessed is time. While it is possible to connect imposed motions to variables other than time, such use is normally deprecated within the documentation for the software since it can lead to very poor numerical conditioning that

can be difficult for a novice user to recognise. An example of this might be the steering inputs to a vehicle model, for which the preferred closed loop embodiment would calculate a demanded position in response to vehicle states and apply forces to attempt to deliver the demanded position. Both an open loop embodiment (playing in a recorded steer history) and a closed loop embodiment are discussed in Chapter 6.

An MBS analysis program will often provide a number of elements with the capability to model flexibility of bodies and elastic connections between parts. These may include features for modelling beam elements, rubber bushings or mounts, plus a general stiffness and damping field element. At various positions in a model rigid parts can be elastically connected together in preference to using a rigid constraint element, such as a joint or joint primitive. Vehicle suspension bushes can be represented by a set of six action—reaction forces, which will hold the two parts together. In the simplest form, the equations of force are linear and uncoupled. The user is only required to provide the six diagonal coefficients of stiffness and damping. For more complicated cases a general-purpose statement can be used to provide a linear representation of a flexible body or connection. Further elaboration is difficult with the standard items in the program; simple nonlinearity is usually easy to achieve by making forces reference data stored as splines, but frequency-dependent characteristics, such as displayed by many elastomers, need a more specialised representation that may only be possible using dedicated subroutines, such as those described in Chavan et al. (2010).

Using recent advances in software techniques to allow the combination of component mode synthesis representations for stiff, small amplitude linear dynamic behaviour of structural elements, one or more of the major structural parts of the system may be represented in modal form to study the influence of its flexibility on the behaviour of the system as a whole. A disadvantage of this method of working is the opportunity to consume large amounts of computing resources solving these models, if care is not taken to ensure the flexibility is germane to the task at hand. Where a full representation of the flexibility of the structure is unnecessary, a simpler representation is possible using joints as ‘hinges’ and an associated stiffness at keypoints in the structure. The preference for one approach or the other is largely governed by the time and data extant. This level of abstraction requires a high degree of understanding of the structural behaviour of elements of the system and can easily lead to poorly conditioned numerical problems if carelessly performed, raising solution times drastically. Worse still, it can lead to ‘plausible-but-wrong’ answers, particularly if mass properties are poorly distributed.

Using component mode synthesis, a complete set of modal components can be used with a full vehicle comprehensive model. This approach confuses accuracy with usefulness in a manner that is becoming increasingly common. The use of such models works against volatility of design, and such models cannot be effectively used with an emerging design but belong to a new generation of mathematical prototypes for use in a later vehicle programme. The notion that too much complexity is a bad thing is discussed in later chapters of this book.



For full vehicle applications it is important to obtain a usefully accurate model for the tyres and the associated forces generated at the tyre—road surface contact patch. For each tyre on the vehicle model, the program will calculate the three orthogonal forces and three orthogonal torques acting at the wheel centre as a result of the conditions at the tyre—road surface contact patch. In order to perform these calculations it is necessary to continuously update the tyre model regarding the position, velocity and orientation of the wheel centre marker and any changes in the topography of the road surface. Once this information has been received the tyre model must then calculate the set of forces acting at the contact patch. Once these forces have been calculated they can be resolved back to the wheel centre. The MBS analysis program will then integrate through time to find the new position and orientation of the vehicle and then repeat the process.

It should be noted that commercial software is undergoing continual development and as such the description provided here is limited to the software features required to carry out the simulations described in this text. Elements, such as springs, dampers, bushes and bump stops are described in this chapter as these are considered fundamental components of an MBS modelling system.

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## 3.2 Modelling features

### 3.2.1 Planning the model

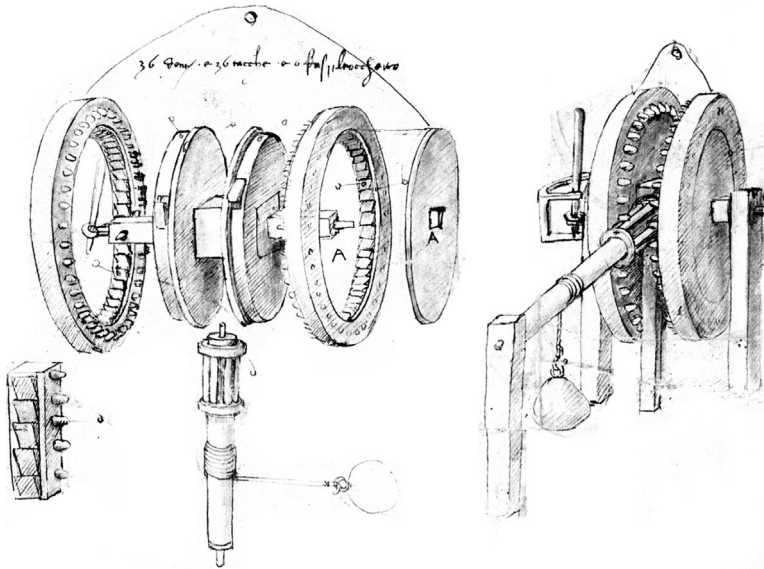
Before progressing to the methods used to describe the typical elements of a multibody system model, it is necessary to outline some of the planning that goes into the development of the model. The first step should be to sketch out a system schematic, which would typically illustrate items, such as the parts, joints, imparted motions and applied forces. The model may well include other elements that include, for example, springs, dampers and beams. The drawing of a schematic is an important first step as it will help the user to not only plan the data that will need to be collated but more importantly to estimate the DOF in the system and develop an understanding of how the mechanism will work.

The use of a modern graphical user interface (GUI) should not discourage this process, which has been used to help understand mechanisms for centuries (Figure 3.4).

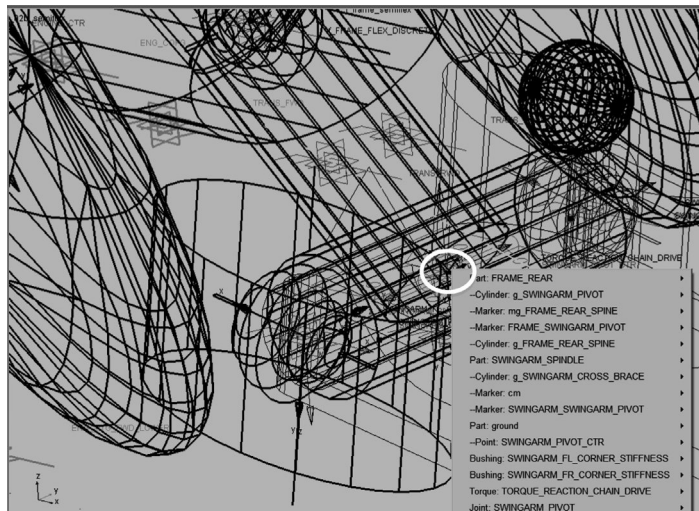
Many users of modern GUIs are tempted to believe that this step is no longer necessary. The authors suggest this is not the case and will continue not to be the case until a convenient graphical engine for exploring models is incorporated into the interface.<sup>3</sup> The primary issue is that many entities in a multibody model are collocated — i.e. at the same place (Figure 3.5).

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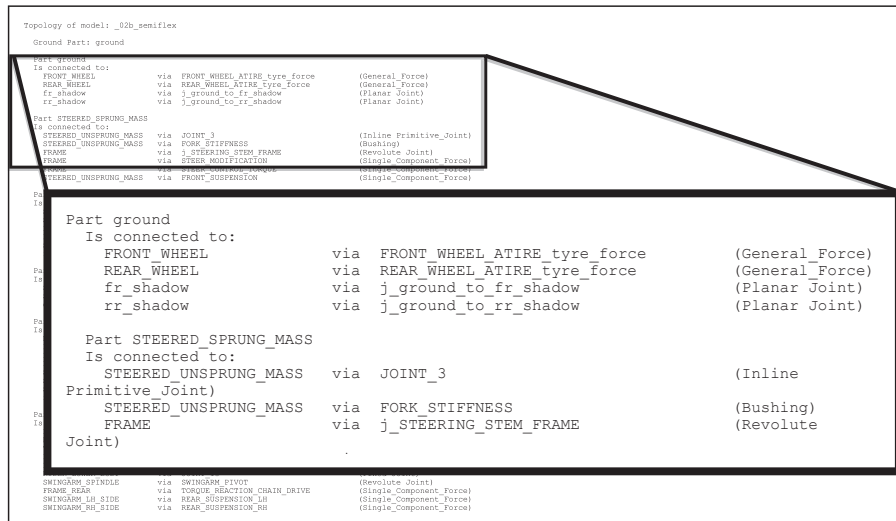
<sup>3</sup>See, for example, Maneesh Agrawala's work on interactive exploded diagrams, first published in 2004 but not as yet incorporated in any multibody software known to the authors (Wilmot et al., 2004). A 2009 YouTube clip reveals an easy and instinctive way to explore system models were it to be incorporated ([http://www.youtube.com/watch?v=NL2QFLiM\\_mY](http://www.youtube.com/watch?v=NL2QFLiM_mY)).

**FIGURE 3.4**

Good enough for Leonardo da Vinci, a mechanism sketch can help clarify the form the model must have.

**FIGURE 3.5**

Illustrating the Graphical User Interface Problem with a Typical Multibody Model. The top left corner of the selection box (circled) often offers the choice of more than 10 items at that point in space.

**FIGURE 3.6**

A text-based topology map.

Many software packages offer some form of topology map for the model, some in a text-based form (Figure 3.6) and some graphical. Some of these are more helpful than others. The MSC textual topology map, for example, is quite useful if the model is simple but rapidly becomes incomprehensible for a complex model.

Both MSC and Simpack provide graphical topology maps (Figure 3.7) but again for complex models this has varying degrees of utility.

Figure 3.8 provides an example of a system schematic for a double wishbone suspension system. Users may develop their own style when drawing a schematic, but the symbols shown in Figure 3.9 are provided as a suggested starting point for the sketching of the various elements of a model.

### 3.2.2 Coordinate systems

The three-dimensional description of a multibody system requires the use of coordinate systems, not only to set up the configuration and physical properties of the model, but also to describe the calculated outputs, such as the displacements, velocities and accelerations. Although in general coordinate systems can be Cartesian, cylindrical or spherical, general-purpose programs tend to employ Cartesian systems for reasons of algorithmic convenience. Program documentation refers to coordinate systems variously as frames of reference (or just frames), triads, ‘coords’ and markers; other terms will probably emerge from time to time. ‘Frame’ is used here to avoid confusion between *multibody system* and *coordinate system*.

The concepts are common to every attempt to describe a multibody system in software and not restricted to any one package. There are three types of right-handed Cartesian frames that will be used in this text:

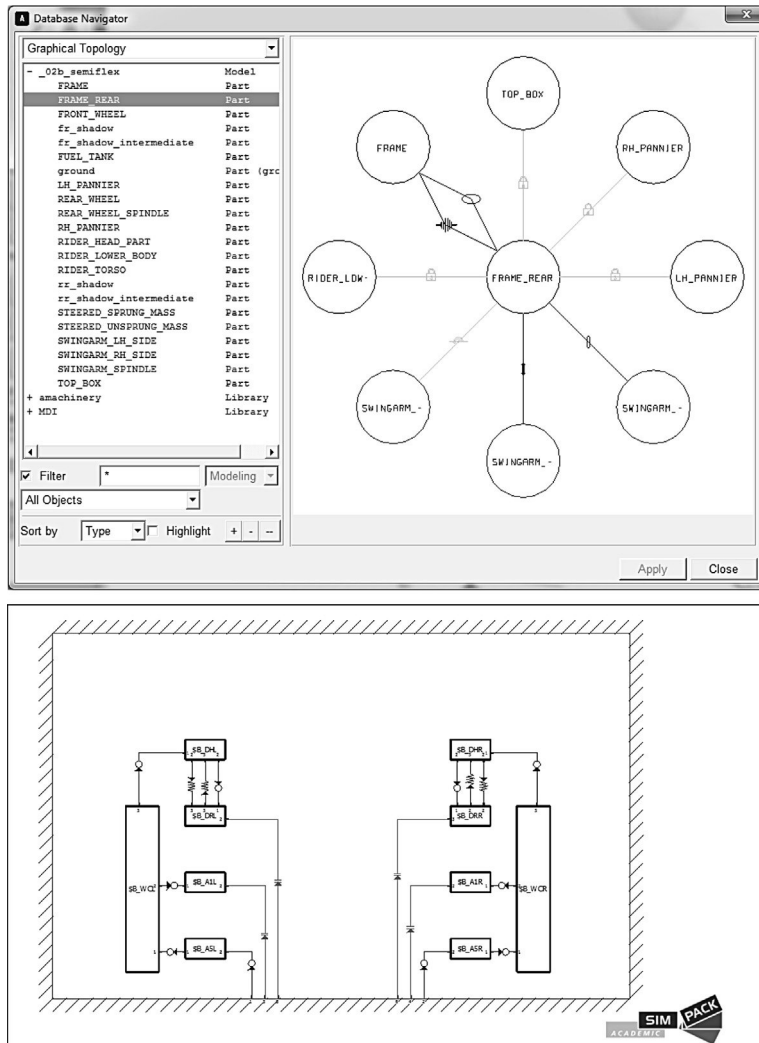
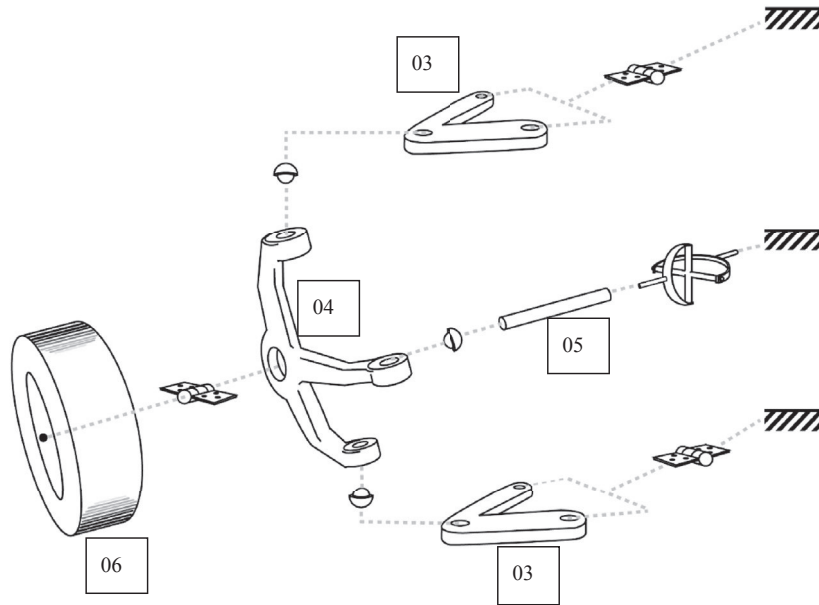


FIGURE 3.7

A Graphical Topology Map as Implemented in Two Different Software Packages. Although somewhat helpful, it still does not assist in assimilating the model in the manner of a really useful mechanism sketch.

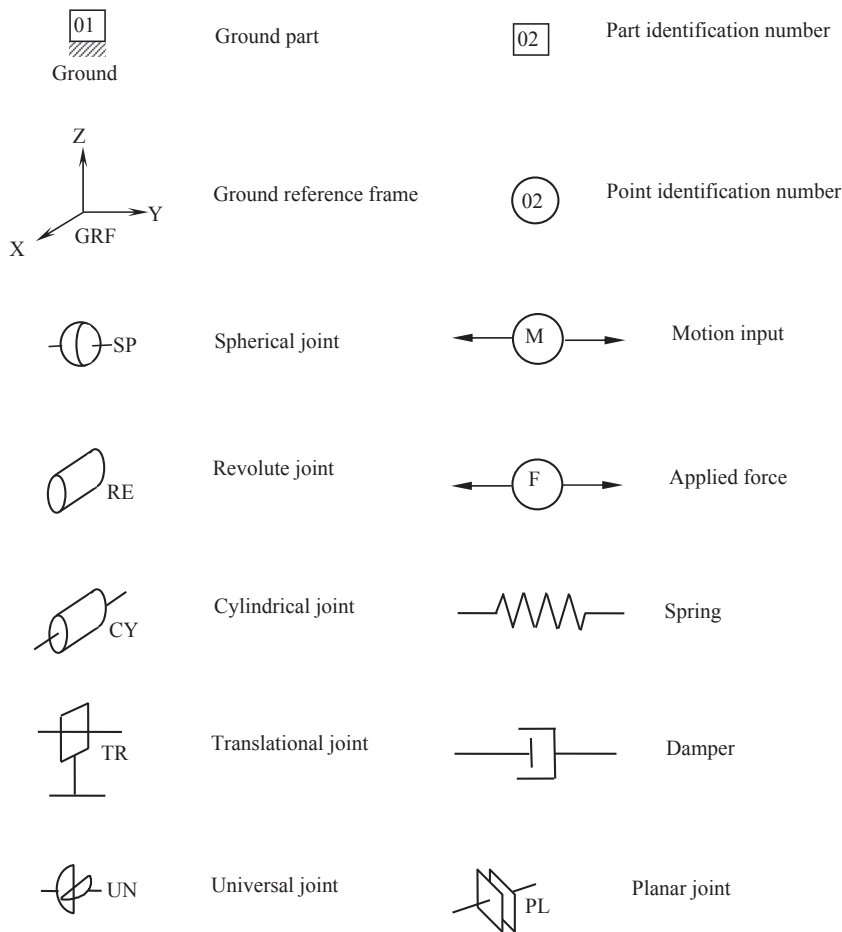
### 3.2.2.1 Ground reference frame

This is by definition the single inertial frame that is considered to be fixed or at 'absolute' rest. Any point defined to be stationary with respect to this frame has zero velocity and acceleration in an absolute sense. The ground reference frame (GRF) is taken to be fixed on a body or part known as the ground part, the physical

**FIGURE 3.8**

Double wishbone suspension system schematic of the kind that it would be helpful if commercial software could produce easily for model audit purposes.

significance of which may vary from model to model. For a single suspension model the ground part may be taken to encompass the points on the vehicle body or sub-frame to which the suspension linkages attach. For a full vehicle model the ground part would relate to the surface of the road used to formulate the contact forces and moments in the tyre model. In addition to providing the single inertial reference, the ground reference can be considered to be the origin of the entire model. As such the absolute coordinates and orientations of all other reference frames and points in the model are measured relative to the GRF. When modelling a complete vehicle, it can be convenient to use the same origin as the master geometric model held in a CAD system. Note that different organisations have different conventions for this master model origin, with some having positive X in the direction of vehicle travel, some with it backward, some having the origin ahead of the vehicle and under the ground, some with the origin at flywheel centre and so on. There is no one universal convention despite the formulation of suggestions for such things by the American Society of Automotive Engineers, for example. Throughout this text the ground part is taken to be the first part in the model and the GRF to be the first frame  $O_1$ . Note that some programs do not explicitly declare a ground part but the existence of a GRF is an implicit acknowledgement that it exists. Practitioners may also describe this frame as a 'global' frame. The exact terminology may vary with different multibody system programs but the notion is identical between them.

**FIGURE 3.9**

Suggested symbols for elements of system schematic.

**3.2.2.2 The local part reference frame**

Each body or part in the system can be considered to have a local part reference frame (LPRF) that moves and changes orientation with the part. The position and orientation of the local part reference system at time zero is defined relative to the GRF. The use of a local axis system on the part may be desirable to facilitate the definition of points on the body by perhaps exploiting the symmetry of the body where the axes of symmetry, at the model definition stage, are not parallel to the axes of the GRF. The explicit declaration of a LPRF is optional and if omitted the LPRF can be considered to be coincident with and parallel to the GRF at the model definition stage. As with the ground part, the fact that an LPRF is not declared does not mean it is not in use behind the scenes.

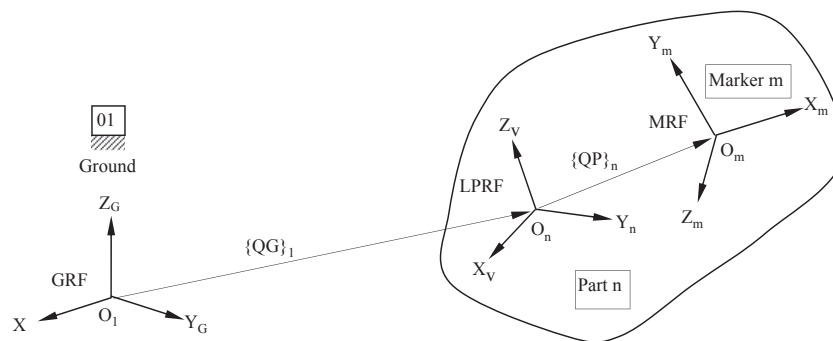
Some practitioners like to declare the LPRF at some convenient point other than the global origin, perhaps at some distinctive feature like one end of a link or else at the component centre of mass. With increased use of product data management tools, it is increasingly likely that a local part origin established as convenient for geometric purposes will become the default formulation for an LPRF in the future. Different industries and different companies in any case have different habits for the definition of LPRF.

With Simpack, the LPRF is defined by the ‘joint’ used to define the part — see [Section 3.2.1](#) — so that the user is not presented with any choice. This can lead to some confusion for practitioners used to the use of global position declarations everywhere, which is one of the advantages of setting the local part and global reference frames coincident during model build.

### 3.2.2.3 Other frames

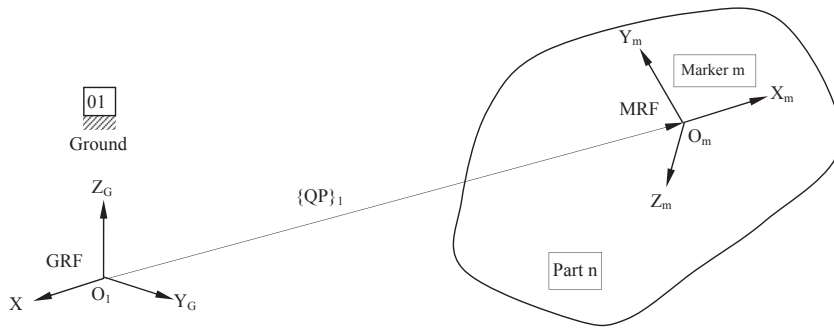
Additional frames of reference are freely used in multibody system models to define the location and orientation of more or less every element in the model. To define a joint, for example, geometrically coincident frames are declared on different parts in the model and an equation declared that, for example, makes the global X-, Y- and Z-displacements of both frames identical in the GRF (a spherical joint). Additional frames are used to define, for example, mass centres, the positions of joints, the ends of springs and the graphical representation of the bodies used for subsequent animations. They may belong to the ground part or any moving part in the system, in which case the frame will move and rotate with the part. In some cases a frame is only required to define the coordinates of a point, such as the end of a spring, where a local definition of the orientation is not important and defaults to the orientation of the GRF. In other instances the orientation of the marker does require definition. An example of this would be the definition of revolute joints for which the axis of rotation must be specified.

The relationship between the three reference frames, in terms of position, is illustrated in [Figure 3.10](#). The position of the LPRF  $O_n$  for any body, in this case part n, is



**FIGURE 3.10**

Relative Position Definition of the GRF, LPRF and Another Frame. GRF, ground reference frame, LPRF, local part reference frame.

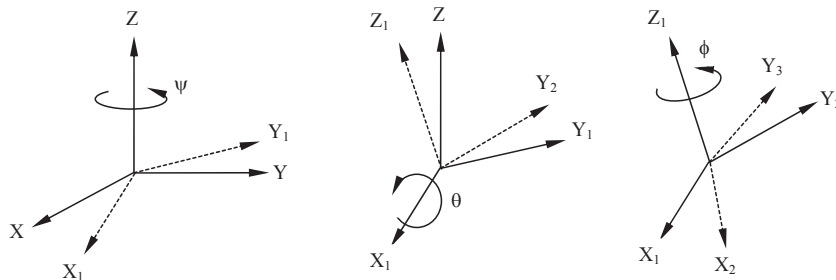
**FIGURE 3.11**

Relative Position Definition of the Additional Frame when the LPRF is Coincident with the GRF. GRF, ground reference frame, LPRF, local part reference frame.

defined using, in MSC ADAMS terminology, a position vector  $\{QG\}_1$ . The position of any markers belonging to part  $n$ , for example marker  $m$  with marker reference frame  $O_m$ , are defined relative to  $O_n$ , using a relative position vector  $\{QP\}_n$ . Note that the  $x$ ,  $y$  and  $z$  components of  $\{QP\}_n$  are resolved parallel to  $O_n$ .

As mentioned earlier, the definition of the LPRF is often taken to be coincident with and parallel to the GRF when setting up the model – either explicitly declared as such or by omitting a declaration, depending on the software in use. The position of the marker reference frame  $O_m$  is then defined relative to the GRF by the position vector  $\{QP\}_1$  as illustrated in Figure 3.11. Note that the  $x$ ,  $y$  and  $z$  components of  $\{QP\}_1$  are now resolved parallel to the GRF  $O_1$ .

There are a number of different methods by which the orientation of one reference frame to another may be established when defining a model. Two commonly used methods are presented here and will be similar to methods used in alternative MBS programs. The first of these methods is a body-fixed 3–1–3 sequence of rotations as shown in Figure 3.12. This Euler angle method involves the definition of three sequential rotations  $\Psi$ ,  $\theta$ , and  $\phi$ . The first rotation  $\Psi$  acts about the

**FIGURE 3.12**

Orientation of a frame by the euler angle method.



z-axis ( $Z$ ) of the initial frame ( $X, Y, Z$ ). The second rotation  $\theta$  is about the new x-axis ( $X_1$ ) of the rotated frame ( $X_1, Y_1, Z$ ). The final rotation  $\phi$  is about the z-axis ( $Z_1$ ) of the second rotated frame ( $X_1, Y_2, Z_1$ ).

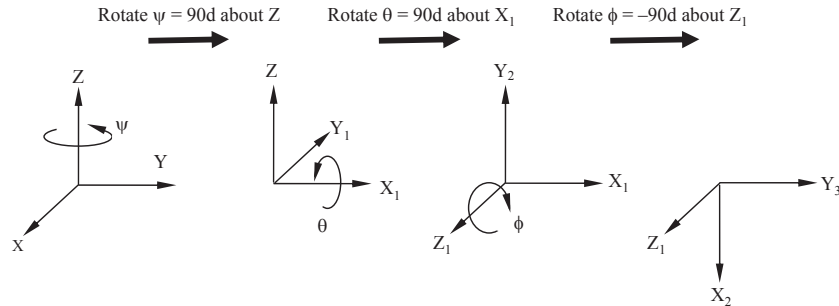
The orientation of the positioned frame is thus the result of the three cumulative rotations and as such these will generally be difficult to visualise. In some cases the required rotations may be generated as output from a CAD package, which will alleviate the problem. In other cases the Euler angles may be straightforward to derive as the positioned frame can be defined for example by combinations of  $90^\circ$  rotations.

The Euler angle method can be used for orientating both the LPRF relative to the GRF and the marker reference frame relative to the LPRF. An example of the use of Euler angles is shown in Figure 3.13 where the  $(\Psi, \theta, \phi)$  sequence is (90d, 90d, -90d), the letter d being used here to denote the use of degrees rather than radians.

An alternative method of orientating a reference frame is referred to as the X-point-Z-point method and involves defining the coordinates of a point that lies on the Z-axis of the positioned frame and another point that lies in the XZ plane of the positioned frame. This is illustrated in Figure 3.14 where this method is used to orientate the LPRF relative to the GRF. The position of the LPRF,  $O_n$ , is defined, as stated earlier, by the vector  $\{QG\}_1$ . The point  $Q$  is coincident with  $O_n$ . The position of  $Z$  is defined by  $\{ZG\}_1$ . The distance of  $Z$  from  $G$  along the Z-axis of  $O_n$  is arbitrary. The position of  $X$  is defined by  $\{XG\}_1$  and may lie anywhere in the XZ plane other than on the Z-axis of  $O_n$ .

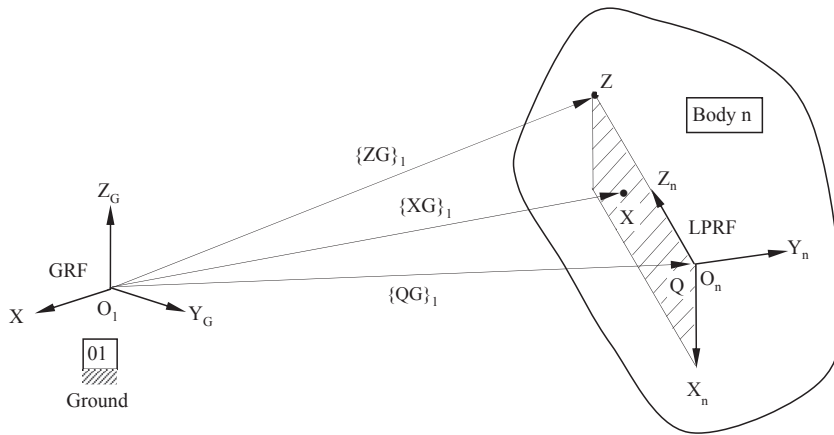
In order to determine the exact orientation of the positioned frame the vector cross product can be applied to first obtain the new Y-axis. The vector cross product of the new Y-axis and the new Z-axis can then be used to find the new X-axis. It will be seen later that if only either the X-axis or Z-axis is important then it is only necessary to specify either  $\{XG\}_1$  or  $\{ZG\}_1$ .

The X-point-Z-point method can also be used to orientate a marker reference frame relative to a LPRF as illustrated in Figure 3.15. The notation is changed using  $QP, XP, ZP$  instead of the  $QG, XG, ZG$  used to orientate the LPRF. It should also be



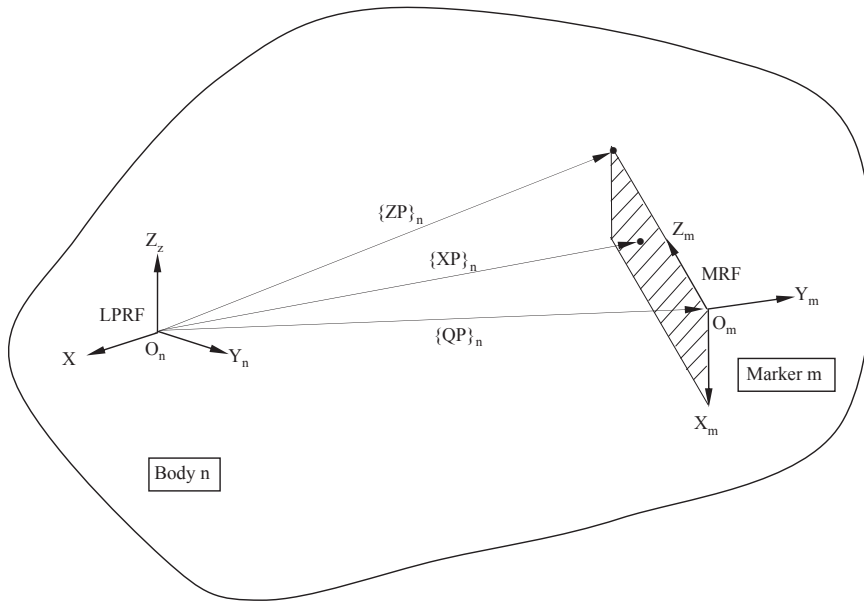
**FIGURE 3.13**

Example application of the euler angle method to orientate a frame.



**FIGURE 3.14**

Orientation of the LPRF Using the X-point-Z-point Method. GRF, ground reference frame, LPRF, local part reference frame.



**FIGURE 3.15**

Orientation of the Marker Reference Frame Using the X-point-Z-point Method. LPRF, local part reference frame.

noted that as with  $\{QP\}_n$  the components of  $\{XP\}_n$  and  $\{ZP\}_n$  would be resolved parallel to the axes of the LPRF  $O_n$ .

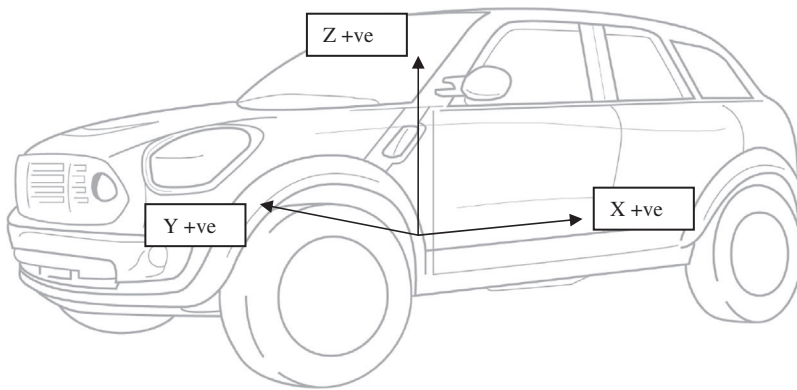
Different software packages implement this in different ways but in general the idea of starting at an origin, specifying a first vector for an axis and then specifying a second vector that is used with two vector product operations is common. In MSC ADAMS, an additional frame can be oriented using 'ORI\_ALONG\_AXIS' if only one vector is given or 'ORI\_IN\_PLANE'. Many current programs allow the arbitrary declaration of any axis, not just the X-point-Z-point logic shown here. It can be seen by inspection that the logic of the operations is identical. Simpack uses dialogue boxes to select which locations are on which axes for the definition of the marker.

This approach has the advantage of being somewhat easier to visualise when compared to the Euler rotation sequence; it is also advantageous in that reference frame axes can be pointed at other locations in the model in a manner that is easy to parameterise and thus have the frames and parts reorient themselves when design variables change the geographical description of the model.

Some packages also allow the definition of direction cosines to orient a reference frame (see Section 3.2.2.1). The direction cosine approach can be visualised as a variation on the X-point-Z-point method, where the length of the vectors specified is unity. Direction cosines are typically given as a  $3 \times 3$  matrix but some consideration will show that the resulting nine pieces of information are in fact redundant, and the minimum required information collapses into the same information as the X-point-Z-point method. Direction cosines are convenient in that a vector quantity can be transformed from one reference frame to another by directly multiplying it by the direction cosines between them, which form the *transformation matrix* as described in Section 2.2.7. A vector may be transformed between two vectors defined with respect to the GRF by transforming one vector back to the GRF and then forward through the direction cosines; again the direction cosine formulation is convenient algorithmically since the backward transformation matrix is simply the transpose of the direction cosine matrix.

For convenience in the subsequent text, the LPRF is taken to be coincident with and parallel to the GRF when setting up the model. The orientation of the marker reference frame  $O_m$  is then defined relative to the GRF. As shown earlier in Figure 3.15, the position vector  $\{QP\}_1$  would define the position of the marker reference frame and similarly  $\{XP\}_1$  and  $\{ZP\}_1$  would now be used to define the orientation. As with  $\{QP\}_1$  the x, y and z components of  $\{XP\}_1$  and  $\{ZP\}_1$  are now resolved parallel to the GRF  $O_1$ . It should be noted that the methods described here have been extended and are more general including the capability to implement parameter-based reference frames.

For the vehicle models described later in this text a consistent approach will be used for the GRF, as shown in Figure 3.16, where the X-axis points back along the vehicle, the Y-axis points to the right of the vehicle and the Z-axis is up. The XZ plane will always be taken to be coincident with the centre line of the vehicle so

**FIGURE 3.16**

Ground reference frame for full vehicle models.

*(Image courtesy of Prodrive).*

as to exploit symmetry when defining, for example, the Y coordinates of left and right suspension systems.

### 3.2.3 Basic model components

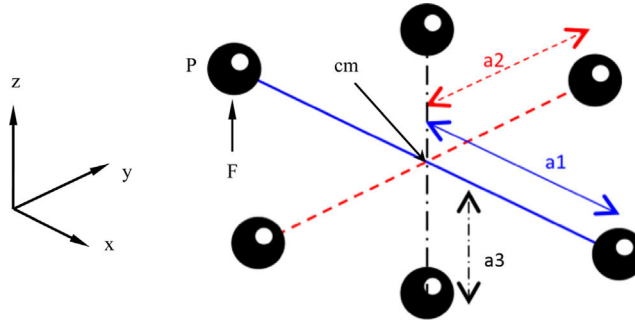
When developing the data set for a model in a MBS analysis program, the following can be considered to be basic model components:

1. Rigid bodies
2. Geometry
3. Constraints equations
4. Forces
5. User-defined algebraic and differential equations

Nonrigid bodies are discussed in principle later in Chapter 6. Constraint equations are meant in the most general sense rather than in any program-specific usage. In the first edition some detailed discussion was given of declaring these items but this information tends to be quite program specific and is thus substantially simplified in this edition.

### 3.2.4 Parts and frames

The part declaration can be used to define any rigid body or lumped mass. For a vehicle suspension, system components such as the control arms and wheel knuckle would typically be modelled as rigid bodies until quite late in the vehicle development process. For other classes of vehicle, such as motorcycles and commercial vehicles, they tend to be substantially influenced by the compliance of the primary structure and so the parts might be made up of multiple parts joined with some

**FIGURE 3.17**

The six particle concept for a rigid body.

kind of compliance matrix. The type of analysis being performed will dictate the amount of data that must be defined for the body. For a dynamic analysis a full definition will be required to include the mass, centre of mass position, the mass moments of inertia, the orientation of the axes about which the mass moments of inertia are measured and any initial translational and angular velocities to be applied to the body. As mentioned earlier it is also possible to define a LPRF to which any markers belonging to the part can be referenced.

Mass moments of inertia are worth some discussion since, while most people can readily estimate and assimilate mass properties, students and new practitioners are sometimes a little slower to grasp the ideas of inertia.

The equations of motion for a particle are easily understood and are declared by Newton's Second Law of motion. Their derivation is given in Section 2.7 of Chapter 2 where we consider the dynamics of a particle, a body for which the motion is restricted to translation without rotation.

Real bodies have nonzero inertia, which is to say they have a reluctance to rotate in the presence of an applied torque.<sup>4</sup> In order to visualise inertia, a real body may be imagined to be represented by six particles, at distances from the mass centre (cm) of  $a_1$ ,  $a_2$  and  $a_3$ , as shown in Figure 3.17. Each particle has one sixth of the mass of the body.

Considering the leftmost particle, point P, in Figure 3.17, it can be seen that a vertical force, F, applied directly to it will attempt to generate an angular acceleration,  $\alpha$ , about the line  $a_2$ . The angular acceleration is related to the resulting linear acceleration, A, by the radius, R, about which the rotation occurs.

$$\alpha = A/R \quad (3.1)$$

<sup>4</sup>A torque is taken in this text to be a generalised force acting to rotate a body about an axis. As such it is distinct from a moment, which is taken in this text to mean a force acting at a distance from an axis, attempting to both translate and rotate a body. A couple, a pair of equal and opposite forces symmetrically distributed about an axis and with no net force, is functionally equivalent to a torque. Other authors may use these terms differently.

In this case the radius  $R$  is equal to the distance  $a_1$ . The applied torque,  $T$ , is the force multiplied by the radius at which it acts

$$T = F \cdot R \quad (3.2)$$

Since  $F = mA$ , we can gain some sense of an equivalent angular quantity for the rotational inertia  $I$

$$\begin{aligned} I &= T/\alpha \\ &= F \cdot R / (A/R) \\ &= F/A \cdot R^2 \\ &= m \cdot R^2 \end{aligned} \quad (3.3)$$

Thus it can be seen that for each of the six particles, their reluctance to rotate is given by  $m$  and  $R^2$ . Considering each axis of rotation in turn and summing the effects of each mass, it can be seen that

$$I_1 = 2 \left( \frac{m}{6} a_2^2 + \frac{m}{6} a_3^2 \right) \quad (3.4)$$

$$I_2 = 2 \left( \frac{m}{6} a_1^2 + \frac{m}{6} a_3^2 \right) \quad (3.5)$$

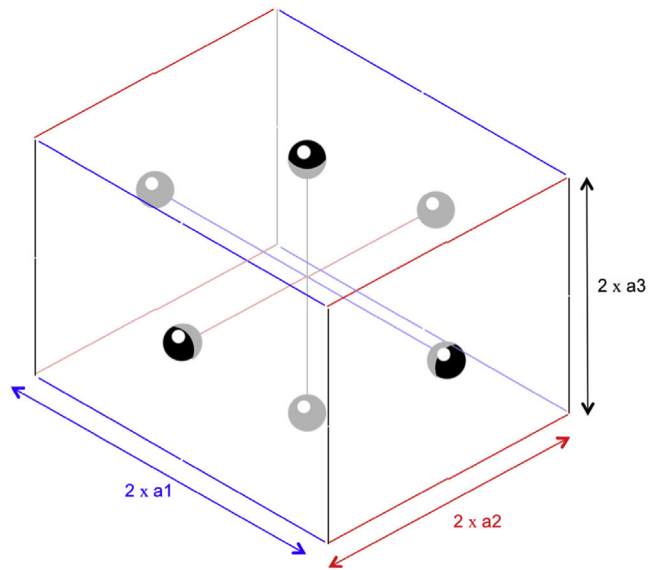
$$I_3 = 2 \left( \frac{m}{6} a_1^2 + \frac{m}{6} a_2^2 \right) \quad (3.6)$$

This description is simply a lumped mass variation on the usually given classical description of mass moments of inertia as described in this text in Chapter 2; however, it is the authors' experience that students more readily grasp some kind of intuitive meaning for inertia using the 'six-mass' method compared to the classical method. It has the added advantage that if students can instinctively estimate the mass and physical dimensions of an object, then they are meaningfully able to estimate inertias and 'sense check' supplied data for units conversion errors and the like.

The principal inertias represent some kind of 'equivalent rectangular prism' — an instinctive box (Figure 3.18) we might draw representing the body as an item of uniform density. There is no guarantee that the directions of the principal inertia vectors line up with our chosen axis system, in which case the contributions of the individual masses must be summed in the same consistent reference frame.

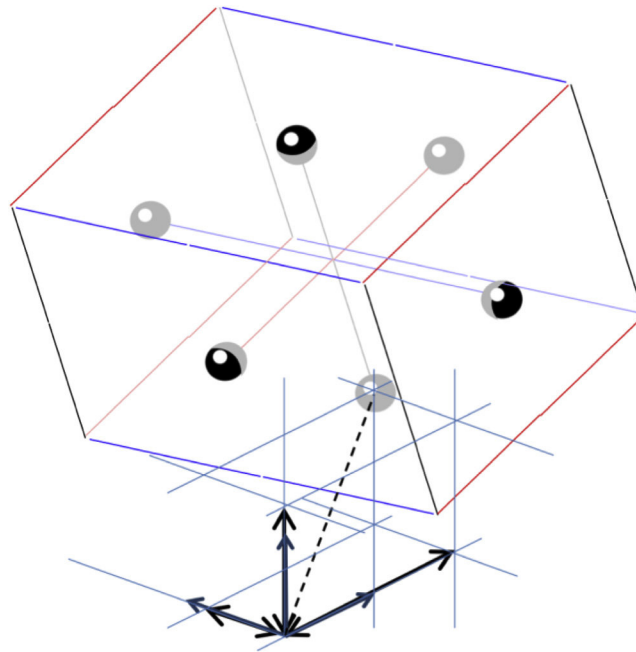
The obvious frame in which to undertake the summation is the global one and this is appealingly simple even when the body is not aligned with the global axis (Figure 3.19).

$$I_{xx} = \sum_{i=1}^6 \frac{m}{6} (y_i^2 + z_i^2) \quad (3.7)$$



**FIGURE 3.18**

A graphical representation of what the 'six-mass' method means for a real object.



**FIGURE 3.19**

The contribution of one of the masses in a global reference frame.

$$I_{yy} = \sum_{i=1}^6 \frac{m}{6} (x_i^2 + z_i^2) \quad (3.8)$$

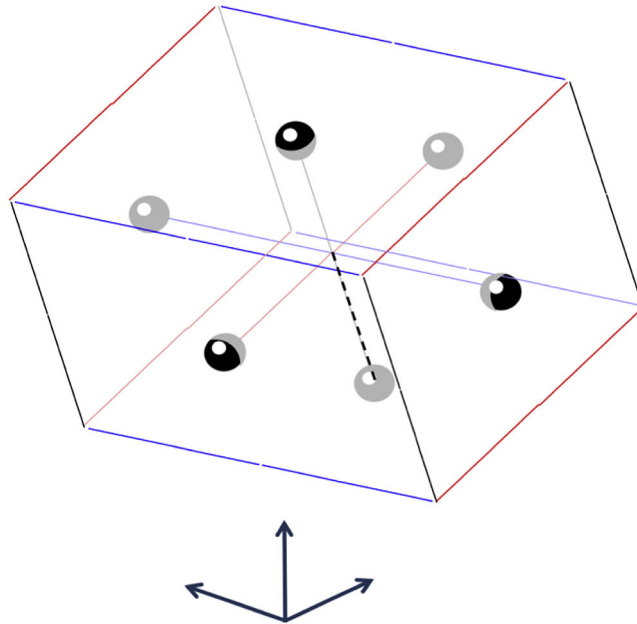
$$I_{zz} = \sum_{i=1}^6 \frac{m}{6} (x_i^2 + y_i^2) \quad (3.9)$$

$$I_{xy} = \sum_{i=1}^6 \frac{m}{6} x_i y_i \quad (3.10)$$

$$I_{xz} = \sum_{i=1}^6 \frac{m}{6} x_i z_i \quad (3.11)$$

$$I_{yz} = \sum_{i=1}^6 \frac{m}{6} y_i z_i \quad (3.12)$$

A more logical frame in which to undertake the summation is one which is positioned at the part mass centre but oriented as the global reference frame, as shown in [Figure 3.20](#).



**FIGURE 3.20**

The contribution of one of the masses in a frame local to the body but oriented as the global reference frame.



$$I_{xx} = \sum_{i=1}^6 \frac{m}{6} ((y_i - y_{CM})^2 + (z_i - z_{CM})^2) \quad (3.13)$$

$$I_{yy} = \sum_{i=1}^6 \frac{m}{6} ((x_i - x_{CM})^2 + (z_i - z_{CM})^2) \quad (3.14)$$

$$I_{zz} = \sum_{i=1}^6 \frac{m}{6} ((x_i - x_{CM})^2 + (y_i - y_{CM})^2) \quad (3.15)$$

$$I_{xy} = \sum_{i=1}^6 \frac{m}{6} (x_i - x_{CM})^2 (y_i - y_{CM})^2 \quad (3.16)$$

$$I_{xz} = \sum_{i=1}^6 \frac{m}{6} (x_i - x_{CM})^2 (z_i - z_{CM})^2 \quad (3.17)$$

$$I_{yz} = \sum_{i=1}^6 \frac{m}{6} (y_i - y_{CM})^2 (z_i - z_{CM})^2 \quad (3.18)$$

There are thus three different possibilities for specifying mass moments of inertia — principals, global and local. This is in contrast with mass, which is simply mass in any reference frame and has a unique position — the mass centre.<sup>5</sup>

The existence of alternative means of specifying gives rise to possibility for errors to trap the unwary. A rigorous and defensible protocol is to always add an additional frame placed at the body mass centre and aligned with the principal axes. Mass moments of inertia can be added directly as their principal components and the opportunity for error is very much reduced.

A second option is to enter a so-called inertia tensor, containing the six moments and products of inertia calculated with respect to the body mass centre.

There are more or less no foreseeable circumstances under which the inertia with respect to some other arbitrary point is relevant. This is a frequent source of difficulty because many CAD systems report it (Figure 3.21), and a frequent habit of students is to retrieve the first piece of data they come to that looks like it might be correct.

In Figure 3.21 the mass centre is reported in Pro-E immediately before the inertia tensor with respect to some other frame. This is an unfortunate consequence of computational convenience (as was seen in Eqns (3.7)–(3.12), the ‘global’

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<sup>5</sup>Note that the mass centre is often referred to colloquially as the ‘centre of gravity’. While this makes no sense (gravity is every atom in the universe acting on every other atom in the universe) it is nevertheless widely understood. Speak to your audience in a language they understand, but know that it really is the mass centre.

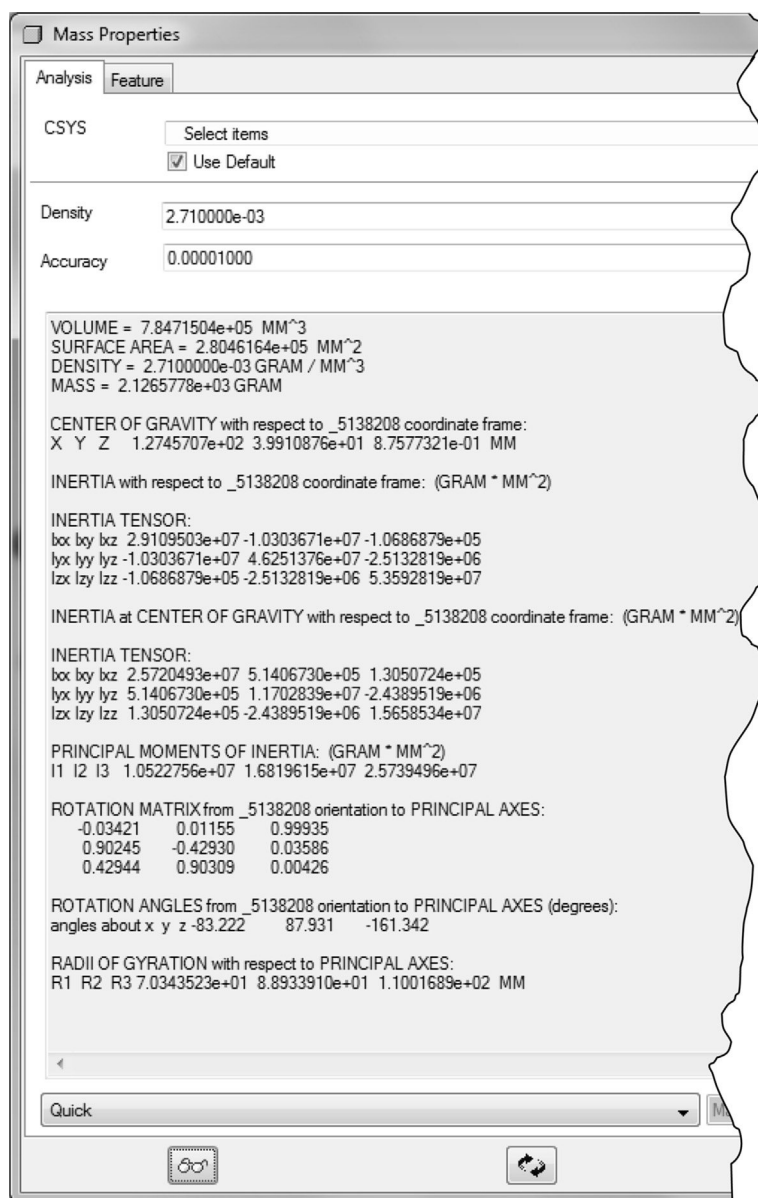


FIGURE 3.21

A typical inertia summary formed from a CAD system.

formulation is substantially simpler and is thus the first one coded by any right-thinking programmer). Further down the printout the quantities of interest are shown. Note the use of centre of gravity for mass centre, as is common. Note also

that the inertia tensor is first reported with respect to the global (in this case the part, \_5138208) frame, and note finally that the units are  $\text{g mm}^2$  where many multibody programs use  $\text{kg mm}^2$  or  $\text{kg m}^2$ .

Because this is such a frequent source of confusion and because the inertia properties of a body determine — along with the applied forces — the applied motion in a very fundamental way, it is worth some investment of effort to develop an intuitive feel for inertia numbers. Note that right at the bottom, the radii of gyration are shown — these are the lengths  $a_1$ ,  $a_2$  and  $a_3$  from Figure 3.17. It is good practice for the practitioner to develop a spreadsheet that can juggle between these forms of information for the purpose of model quality checking and estimating inertia properties when none are known, particularly since there is often a units transition between CAD and multibody software. The forward calculations have already been shown in Eqns (3.4) to (3.6) and the reverse ones can be written by inspection

$$a_1 = \sqrt{\frac{3}{2m}(I_2 + I_3 - I_1)} \quad (3.19)$$

$$a_2 = \sqrt{\frac{3}{2m}(I_1 + I_3 - I_2)} \quad (3.20)$$

$$a_3 = \sqrt{\frac{3}{2m}(I_1 + I_2 - I_3)} \quad (3.21)$$

Immediately above the radii of gyration in the CAD output are some angles that form a rotation sequence. Some multibody programs, such as Simpack, are able to make use of these rotation angles directly in quite a versatile way in specifying the frame to be used for the inertia properties, but some of the older packages like ADAMS are stuck with one particular rotation sequence. Given the existence of six possible Euler rotation sequences, the use of angles is deprecated unless the practitioner is deeply familiar with them and completely confident in which sequence is in use with their chosen package.

The relationship between, for example, the 3–1–3 Euler angles and a Cartesian set of direction cosines is not particularly intuitive. An explanation of these is provided in Section 2.2.6 of Chapter 2.

A further question often posed by students is the difficulty in understanding the globally oriented inertia tensor directly other than a rather indistinct sense that off-diagonal terms mean some sort of misalignment of the principal axis set compared to the global frame. For bodies displaying symmetry about one of the global axis planes, two of the three off-diagonal terms are zero and the inclination of the principal axis set within the plane of symmetry, accepting that the remaining principal axis is normal to the plane of symmetry, was given in Section 2.12

$$\tan 2\theta = \frac{I_{xz}}{\frac{1}{2}(I_{zz} - I_{xx})} \quad (3.22)$$

Note that the numerical conditioning of these expressions is poor, as the axes get closer to the global axes since the denominators collapse to zero.

For a more general three-dimensional case, the principals are found by finding the eigenvectors of the inertia tensor. Recall that the eigenvector problem is solved by subtracting the eigenvalue,  $\lambda$ , from the matrix of interest and setting the determinant to zero

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - \lambda & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - \lambda \end{vmatrix} = 0 \quad (3.23)$$

This results in a cubic equation in  $\lambda$

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (3.24)$$

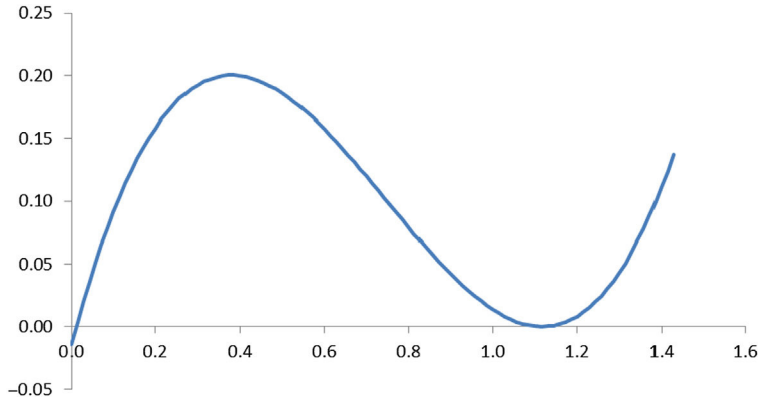
where

$$A = -I_{xx} - I_{yy} - I_{zz} \quad (3.25)$$

$$B = I_{xx} \cdot I_{yy} + I_{xx} \cdot I_{zz} + I_{yy} \cdot I_{zz} - I_{xy}^2 - I_{xz}^2 - I_{yz}^2 \quad (3.26)$$

$$C = -I_{xx} \cdot I_{yy} \cdot I_{zz} + I_{xx} \cdot I_{yz}^2 + I_{yy} \cdot I_{xz}^2 + I_{zz} \cdot I_{xy}^2 + 2 \cdot I_{xy} \cdot I_{xz} \cdot I_{yz} \quad (3.27)$$

This cubic equation produces a graph, such as that shown in Figure 3.22. Note that in Figure 3.22, the curve is typical of a long, thin body with one principal inertia quite low and the other two quite similar to each other. If the body is cylindrical or of square section then the equation has two repeated roots. The roots can be found using a Newton–Raphson iteration or some other solver, such as the linear solver in



**FIGURE 3.22**

The roots (zero crossings) for eqn (3.24), shown graphically.

Excel or Open Office; for awkwardly shaped bodies the numerical conditioning of this problem is poor.

$$\lambda_1 = I_1 \quad (3.28)$$

$$\lambda_2 = I_2 \quad (3.29)$$

$$\lambda_3 = I_3 \quad (3.30)$$

Once the eigenvalues have been found, the eigenvectors can be found in the usual way

$$\begin{bmatrix} I_{xx} - I_i & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I_i & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I_i \end{bmatrix} \begin{bmatrix} x_{i\lambda} \\ y_{i\lambda} \\ z_{i\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.31)$$

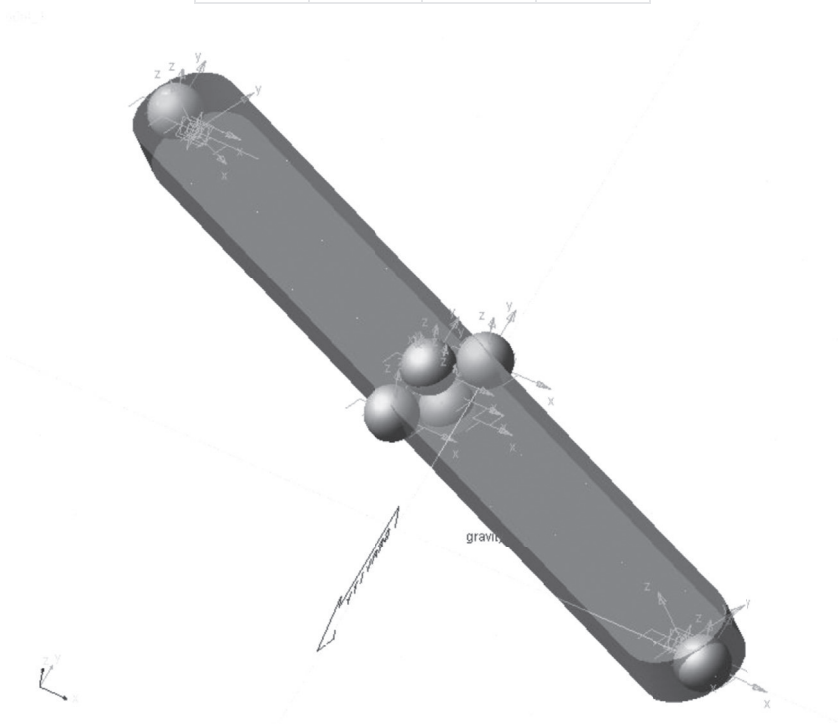
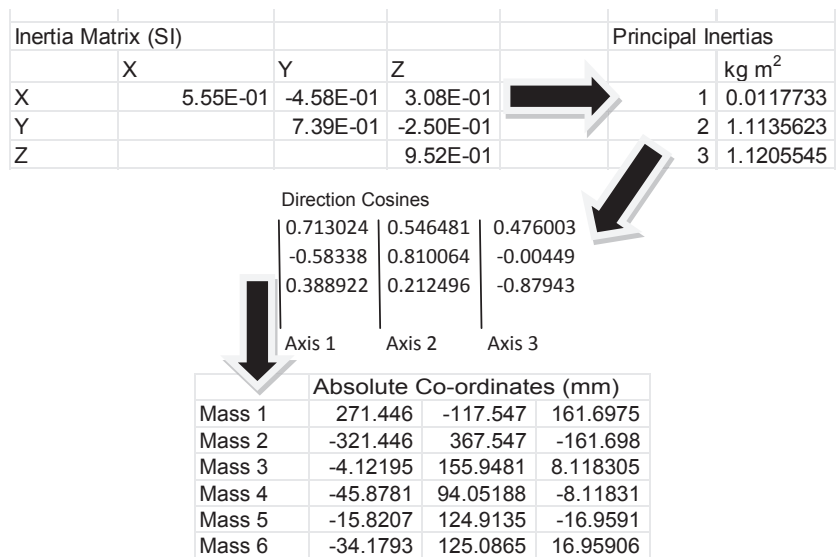
There are an infinite number of possible equivalent eigenvectors, so typically one value, for example  $x_{i\lambda}$ , is set to unity and then a solution found for the other two. This can be performed using Gaussian elimination or again the linear solver in Excel or Open Office. The resulting vector is typically not of unity length and so is usually normalised to give direction cosines in the familiar form.

Note that in [Eqn \(3.31\)](#) the order of the eigenvalues produced is not necessarily ascending, and in general the roots of an equation do not necessarily appear in any conveniently sorted order. There is not any hard and fast convention concerning the reporting and use of principal inertias, with some practitioners using  $I_1$  as the principal visually closest to the global X-axis, some using  $I_1$  as the numerically lowest and some the numerically highest. The lack of a clear dominance of any one convention suggests that it is unimportant as long as the correct vector is associated with the correct principal inertia. As ever, it is important to verify the behaviour of a particular package with an example before any critical decisions are based upon it.

In [Figure 3.23](#) a sample of the entire process from end-to-end is shown. Input for the calculations is the global inertia matrix about the component mass centre, from which the principals and their direction cosines are calculated as has been described. The vectors a1, a2 and a3 are used to position six masses on a new part, which can be seen in the diagram as baubles superimposed on the component.

Data from commercial packages, such as CAD systems, is typically more exact than the illustrative conversion shown here and thus using the principals and their direction cosines at source gives the highest confidence in the resulting data, or else entering the inertia tensor directly is plan B, as previously mentioned.

Another possible source of inertia data is from tests, such as those carried out by the Cranfield Impact Centre or more recent methods advocated by companies, such as Resonic (Klöpfer, 2011). Not all practitioners of such testing are equally competent and when dealing with a new supplier, the ability to vet the measurements for



**FIGURE 3.23**  
Sample Six-Mass Calculations for an Arbitrary Body in MSC ADAMS. The link body is shown along with the six baubles positioned according to the spreadsheet output.

physical plausibility is important; only recently the authors were handed an inertia tensor as part of a commercial measurement contract where the basic test — that all the radii of gyration are not negative as calculated in Eqns (3.19)–(3.21) — was failed by a spectacular margin, requiring an adjustment of over 30% just to become physically possible.<sup>6</sup>

As an alternative to entering inertia data at all, many multibody packages offer a conceptual sketching environment allowing for basic solids — blocks, cylinders, extrusions, frusta, tori — and their Boolean combination to give some visual impression of the parts of the model. Although originally something of a ‘gaffer-dazzler’, the animation of such graphics can give an excellently realistic impression of system motion for communication with a nonexpert audience and also allow debugging when a perfectly valid model has been constructed that is not quite like the model the user *intended* to construct through some oversight or other. These graphical entities, with well-known published solutions for their inertia tensors, can be combined by the software using parallel axis theorem to give an estimated mass and inertia tensor for a component.

It should be noted that many packages presume such entities are solid and also presume they are steel. Anyone who has ever moved a metre of 50 mm radius bar will note that it is a good thing that not many things are actually solid steel and this gives rise to rather excessive properties if used unthinkingly. Since the sketching environment in most multibody packages is nothing like a modern three-dimensional CAD system, a great deal of time can be consumed ineffectively producing ‘concept level’ graphics. The solution is either to reduce the density until a reasonable mass is obtained and accept the accompanying inertia properties, or to learn to estimate mass and physical dimensions and estimate inertia properties as described.

Finally, when specifying a body, the initial velocity conditions are required. A fully safe approach to initial conditions is to start the entire system from rest and accelerate it with applied forces until it reaches the condition of interest. However, this represents a rather large waste of computing resource and it is much better with vehicle systems, for example, to be able to initialise the model with some known running condition, such as travelling in a straight line at a known speed. There are some risks to this approach if not all the bodies are initialised with compatible speeds since large forces are required at the first iteration step to harmonise velocities and solve the constraint equations. This can give rise to some incomprehensible start-up transients in the model. Making sure all parts have the same translational velocity avoids any difficulties; it is recommended that a body be solved for a

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<sup>6</sup>Measurements containing such inaccuracies have a negative value to the organisation because they promote confidence where none should exist, which is sure to become apparent later (and more expensively) in the product development process. This is part of a wider issue of unwarranted overconfidence in measurement results, often summed up with ‘everybody believes a measurement except the man who did it; nobody believes a calculation except the man who did it’.

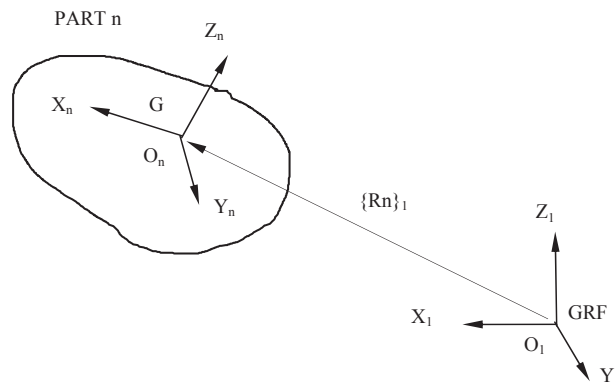
few tenths of a second in free-fall with the required initial velocity to uncover any errors in setting initial velocities, which can otherwise be somewhat subtle to uncover. Setting initial velocity as a model parameter and enforcing the same parametric value on every part is good working practice and enables rapid investigation of the influence of speed on the behaviour of interest in any case.

Rotational initial conditions, for example wheel spin speed, can be set a little more approximately since in real systems, such as an aircraft landing, it can be shown that such events can (and indeed should) be tolerated by the system. In fact, the authors' preference is to leave wheel and driveline components nonrotating again as a debug case and ensure the model solves a 'landing spin-up' case effectively as part of the commissioning process. Initial rotational velocities can be parameterised back to initial translational velocities as a final commissioning step.

It is important to note that initial angular velocity conditions are typically resolved using the axes of the centre of mass marker.

### 3.2.5 Equations of motion for a part

The following sections describe the formulation of the rigid body equations of motion, constraint equations and solution methods used in a general-purpose MBS program, such as MSC ADAMS. The formulations are based on those given by Wielenga (1987) and presented here using the vector methods described in Chapter 2. Although the solvers have been subject to continual development since 1987, the following will serve as an introduction to the typical calculations performed by the software. As a starting point it can be shown that kinematic variables are required to represent the location and orientation of a part with respect to the GRF, as shown in Figure 3.24. Note that here we are using the general form of the position vector  $\{R_n\}_1$  to locate the part rather than the equivalent  $\{QG\}_1$  used in MSC ADAMS terminology.



**FIGURE 3.24**

The Location and Orientation of a Part. GRF, ground reference frame.



The location of any part is specified by a position vector  $\{R_n\}_1$  from the GRF to the centre of mass,  $G$ , of the part. In this case the part is labelled as the  $n$ th part in the system and the GRF is taken to be the first frame  $O_1$ . The components of the vector  $\{R_n\}_1$  are resolved parallel to the axes of the GRF as indicated by the subscript 1. The velocity is obtained using

$$\{V_n\}_1 = \frac{d}{dt}\{R_n\}_1 \quad (3.32)$$

The orientation of the part reference frame is specified by a set of Euler angles  $(\Psi, \phi, \theta)$ . Note that the Euler angles are stored in an order that differs from the sequence  $(\Psi, \theta, \phi)$  used to change the orientation of a reference frame shown in Figure 3.24.

There are three frames of interest during the transformation. The first is the GRF  $(X, Y, Z)$ , which is also frame  $O_1$ . The second is a frame made up of the axes about which each of the rotations takes place. This is known as the Euler-axis frame  $(Z, X_1, Z_1)$  and will be referred to as frame  $O_e$ . Note that this is not a reference frame in the true sense as the three axes are not perpendicular to one another. The third frame is the resulting part frame  $(X_2, Y_3, Z_1)$ . For the  $n$ th part in a system this would be the part frame  $O_n$ . The matrix  $[A_{1n}]$  is the Euler matrix for part  $n$  and performs the transformation from the part frame  $O_n$  to the GRF  $O_1$ .

$$[A_{1n}] = \begin{bmatrix} \cos\Psi \cos\phi - \sin\Psi \cos\theta \sin\phi & -\cos\Psi \sin\phi - \sin\Psi \cos\theta \cos\phi & \sin\Psi \sin\theta \\ \sin\Psi \cos\phi + \cos\Psi \cos\theta \sin\phi & -\sin\Psi \sin\phi + \cos\Psi \cos\theta \cos\phi & -\cos\Psi \sin\theta \\ \sin\theta \sin\phi & \sin\theta \cos\phi & \cos\theta \end{bmatrix} \quad (3.33)$$

Note that the inverse of this matrix  $[A_{n1}]$  is simply the transpose and performs the transformation from the GRF to the part frame. Another matrix  $[B]$  performs the transformation from the Euler-axis frame  $O_e (Z, X_1, Z_1)$  to the part frame  $O_n (X_2, Y_3, Z_1)$ .

$$[B] = \begin{bmatrix} \sin\theta \sin\phi & 0 & \cos\phi \\ \sin\theta \cos\phi & 0 & -\sin\phi \\ \cos\theta & 1 & 0 \end{bmatrix} \quad (3.34)$$

Note that this matrix becomes singular when  $\sin\theta = 0$ . This corresponds to the situation where  $Z$  and  $Z_1$  are parallel and point in the same direction ( $\theta = 0$ ), or parallel and point in the opposite direction ( $\theta = 180^\circ$ ). When this occurs an internal adjustment is used to set up a new part frame where the  $Z_1$ -axis is rotated through  $90^\circ$ . Note also that the  $[B]$  matrix corresponds with an internal reordering of the Euler angles to  $(Z, Z_1, X_1)$ .

For large rotations the set of Euler angles for the  $n$ th part  $\{\gamma_n\}_e = [\Psi_n \phi_n \theta_n]^T$  cannot actually be represented by a vector as indicated here although they can be

considered to make up a set of kinematic orientation variables for the  $n$ th part. An infinitesimal change in orientation in the part frame  $O_n$  can, however, be represented by a vector, which will be denoted  $\{\delta\gamma_n\}_n$ . In a similar manner an infinitesimal change in the Euler angles can be represented by a vector  $\{\delta\gamma_n\}_e$ . The angular velocity vector for the part in the local part frame can also be specified by  $\{\omega_n\}_n$ . MSC ADAMS also requires the components of these vectors in the Euler-axis frame  $O_e$ . The angular velocity in the Euler-axis frame is simply the time derivative of the Euler angles.

$$\{\omega_n\}_e = \frac{d}{dt} \{\gamma_n\}_e \quad (3.35)$$

The transformation between the part frame and the Euler-axis frame is established using the  $[B]$  matrix.

$$\{\delta\gamma_n\}_n = [B] \{\delta\gamma_n\}_e \quad (3.36)$$

$$\{\omega_n\}_n = [B] \{\omega_n\}_e \quad (3.37)$$

In summary, there are now a set of kinematic position and velocity variables for the  $n$ th part with components measured in the GRF and also a set of orientation and angular velocity variables measured about the Euler-axis frame

$$\{R_n\}_1 = [R_{nx} \ R_{ny} \ R_{nz}]^T \quad (3.38)$$

$$\{V_n\}_1 = [V_{nx} \ V_{ny} \ V_{nz}]^T \quad (3.39)$$

$$\{\gamma_n\}_e = [\psi_n \ \phi_n \ \theta_n]^T \quad (3.40)$$

$$\{\omega_n\}_e = [\omega_n\psi \ \omega_n\phi \ \omega_n\theta]^T \quad (3.41)$$

There is also a set of kinematic equations associated with the part, which may be simply stated as

$$\{V_n\}_1 = \frac{d}{dt} \{R_n\}_1 \quad (3.42)$$

$$\{\omega_n\}_e = \frac{d}{dt} \{\gamma_n\}_e \quad (3.43)$$

The remaining part variables and equations are those obtained by considering the equations of motion for a rigid body. Each part can be considered to have a set of six generalised coordinates given by

$$q_j = [R_{nx}, R_{ny}, R_{nz}, \psi_n, \theta_n, \phi_n] \quad (3.44)$$

The translational coordinates are the translation of the centre of mass measured parallel to the axes of the GRF, while the rotational coordinates are provided by the Euler angles for that part. For any part the translational forces are therefore summed in the X, Y and Z directions of the GRF while the summation of moments takes place

at the centre of mass and about each of the axes of the Euler-axis frame. Using a form of the Lagrange equations this can be shown as

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j + \sum_{i=1}^n \frac{\partial \Phi_i}{\partial q_j} \lambda_i = 0 \quad (3.45)$$

The kinetic energy  $T$  is expressed in terms of the generalised coordinates  $q_j$  and is given by

$$T = \frac{1}{2} \{Vn\}_1^T m \{Vn\}_1 + \frac{1}{2} \{\omega n\}_e^T [B]^T [I_n] [B] \{\omega n\}_e \quad (3.46)$$

The mass properties are specified by  $m$  which is the mass of the part and  $[I_n]$  which is the mass moment of inertia tensor for the part and given by

$$[I_n] = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad (3.47)$$

The terms  $\Phi$  and  $\lambda$  represent the reaction force components acting in the direction of the generalised coordinate  $q_j$ . The term  $Q_j$  represents the sum of the applied force components acting on the part and in the direction of the generalised coordinate  $q_j$ . The equation can be simplified by introducing a term for the momenta  $P_j$  associated with motion in the  $q_j$  direction, and a term  $C_j$  to represent the constraints

$$P_j = \frac{\partial T}{\partial \dot{q}_j} \quad (3.48)$$

$$C_j = \sum_{i=1}^n \frac{\partial \Phi_i}{\partial q_j} \lambda_i \quad (3.49)$$

This results in the equation

$$\dot{P}_j - \frac{\partial T}{\partial q_j} - Q_j + C_j = 0 \quad (3.50)$$

By way of example, consider first the equations associated with the translational coordinates. The generalised translational momenta  $\{Pn_t\}_1$  for the part can be obtained from

$$\{An\}_1 = \frac{d}{dt} \{Vn\}_1 \quad (3.51)$$

$$\{Pn_t\}_1 = \partial T / \partial \{Vn\}_1 = m \{Vn\}_1 \quad (3.52)$$

$$\frac{d}{dt} \{Pn_t\}_1 = m \{An\}_1 \quad (3.53)$$

This results in  $\{\mathbf{A}_n\}_1$ , as the acceleration of the centre of mass for that part. It should also be noted that the kinetic energy is dependent on the velocity but not the position of the centre of mass,  $\partial T/\partial\{\mathbf{R}_n\}_1$  is equal to zero. We can now write the equation associated with translational motion in the familiar form

$$m\{\mathbf{A}_n\}_1 - \sum\{\mathbf{F}_{nA}\}_1 + \sum\{\mathbf{F}_{nC}\}_1 = 0 \quad (3.54)$$

where  $\{\mathbf{F}_{nA}\}_1$  and  $\{\mathbf{F}_{nC}\}_1$  are the individual applied and constraint reaction forces acting on the body. The rotational momenta  $\{\mathbf{P}_{nr}\}_e$  for the part can be obtained from

$$\{\mathbf{P}_{nr}\}_e = \partial T/\partial\{\omega_n\}_e = [\mathbf{B}]^T [\mathbf{I}_n] [\mathbf{B}]\{\omega_n\}_e \quad (3.55)$$

We can now write the equations associated with rotational motion in the form

$$\{\mathbf{P}_{nr}\}_e - \partial T/\partial\{\gamma_n\}_e - \sum\{\mathbf{M}_{nA}\}_e + \sum\{\mathbf{M}_{nC}\}_e = 0 \quad (3.56)$$

$$\{\mathbf{P}_{nr}\}_e = [\mathbf{B}]^T [\mathbf{I}_n] [\mathbf{B}]\{\omega_n\}_e \quad (3.57)$$

In this case  $\{\mathbf{M}_{nA}\}_e$  and  $\{\mathbf{M}_{nC}\}_e$  are the individual applied and constraint reaction moments acting about the Euler-axis frame at the centre of mass of the body. Introducing the equation above for the rotational momenta introduces an extra three variables and equations for each part.

The 15 variables for each part are

$$\{\mathbf{R}_n\}_1 = [\mathbf{R}_{nx} \ \mathbf{R}_{ny} \ \mathbf{R}_{nz}]^T \quad (3.58)$$

$$\{\mathbf{V}_n\}_1 = [\mathbf{V}_{nx} \ \mathbf{V}_{ny} \ \mathbf{V}_{nz}]^T \quad (3.59)$$

$$\{\gamma_n\}_e = [\psi_n \ \phi_n \ \theta_n]^T \quad (3.60)$$

$$\{\omega_n\}_e = [\omega\psi_n \ \omega\phi_n \ \omega\theta_n]^T \quad (3.61)$$

$$\{\mathbf{P}_{nr}\}_e = [\mathbf{P}\psi_n \ \mathbf{P}\phi_n \ \mathbf{P}\theta_n]^T \quad (3.62)$$

The 15 equations for each part are

$$\{\mathbf{V}_n\}_1 = \frac{d}{dt}\{\mathbf{R}_n\}_1 \quad (3.63)$$

$$\{\omega_n\}_e = \frac{d}{dt}\{\gamma_n\}_e \quad (3.64)$$

$$\{\mathbf{P}_{nr}\}_e = [\mathbf{B}]^T [\mathbf{I}_n] [\mathbf{B}]\{\omega_n\}_e \quad (3.65)$$

$$m\{\mathbf{A}_n\}_1 - \sum\{\mathbf{F}_{nA}\}_1 + \sum\{\mathbf{F}_{nC}\}_1 = 0 \quad (3.66)$$

$$\{\mathbf{P}_{nr}\}_e - \partial T/\partial\{\gamma_n\}_e - \sum\{\mathbf{M}_{nA}\}_e + \sum\{\mathbf{M}_{nC}\}_e = 0 \quad (3.67)$$

### 3.2.6 Basic constraints

Constraints are used to restrict the motion of parts. There are a number of modelling elements that can be used to do this, and the constraint may restrict the absolute motion of a body relative to the ground or the relative motion between interconnected parts. Constraints can be considered to be of two types:

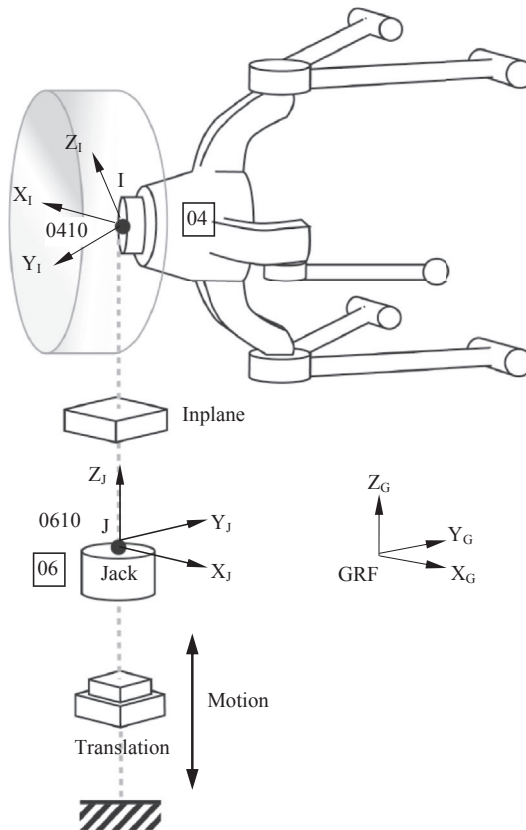
1. Holonomic constraints are those that are dependent on restricting displacement and result in algebraic equations.
2. Nonholonomic constraints are those where a velocity-dependent motion is enforced and result in differential equations.

There are a wide range of constraint elements available ranging from joint primitives that can constrain combinations of individual DOF between bodies through mechanical type joints and gear elements, to higher pair constraints such as those constraining a point to lie on a curve. The examples shown here are restricted to those that are used to support the examples used in this text.

A typical example of a joint primitive is the use of an inplane type of constraint that restricts the motion of a point on one body to remain in a plane on another body. A typical example of this is given later when the vertical motion of individual suspension models is controlled by using a jack to impart motion to the wheel centre as shown in [Figure 3.25](#).

It is quite common in multibody system programs to refer to joint frames as the 'I' frame and the 'J' frame. The reasons for this are largely traditional and centre around the use of the letters from I onwards being used for implicitly defined integer variables in FORTRAN when taken as the first character of a variable name. In this example the I frame is defined to belong to the wheel knuckle and is located at the wheel centre. The J frame is defined to belong to the jack part and will move vertically with that part according to other constraints that control its motion. The orientation of the J frame must be defined in this case so that the x and y axes define the surface plane of the jack in which the I frame will be constrained to remain. For clarity the jack in [Figure 3.25](#) is shown below the wheel but in actual fact it would be defined to locate the xy plane of J at the wheel centre. The orientation of the I frame is not important, unless used for other additional purposes, but the J frame must be defined as shown. In this case the z direction of the J frame is parallel to the z-axis of the GRF so no further definition is required to change the orientation of the J frame. If this was not the case the ZP method is often the most convenient method to orientate the z-axis of the J frame normal to the surface of the plane.

In this case the I frame on the wheel is identified as 0410 and the J frame on the jack is defined as 0610. This joint primitive would constrain the single translational DOF between the I frame and the J frame in the direction normal to the xy plane of the jack. In other words the I frame has 5 DOF relative to the top surface of the jack defined by the xy plane of the J frame. In MSC ADAMS this would be six equations of motion and one constraint equation, in Simpack it would be simply the five equations of interest. The I frame can translate in the J frame X- and Y-directions and is free to rotate about all axes. There are further types of joint primitive in addition to



**FIGURE 3.25**

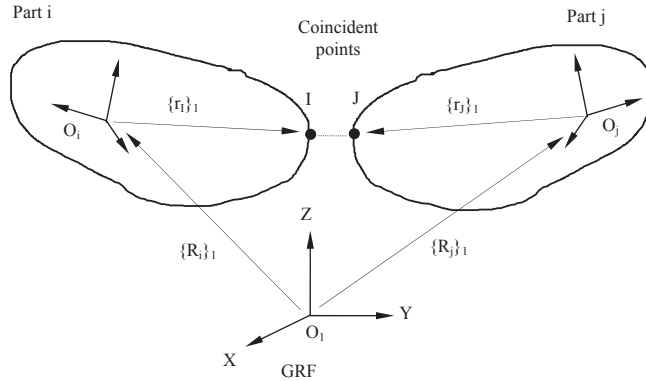
### Application of Inplane joint primitive.

the inplane type introduced here. Others, for example, can constrain a marker on one body to follow a line defined on another body or can constrain the orientation of two markers on different bodies to remain parallel wherever those markers may be in the system model; these concepts are common to most multibody packages.

The various joints and joint primitives can be developed using combinations of four basic constraint elements. For each constraint the resulting forces and moments need to be added into the force and moment balance for a part. Consider first a basic *atpoint* constraint element shown in [Figure 3.26](#), which constrains a point I on one part to remain at the same location in space as a point J on another part, but does not prevent any relative rotation between the two points.

This constraint can be represented by a vector constraint equation working in coordinates parallel to the axes of the GRF.

$$\{\Phi_a\}_1 = (\{R_i\}_1 + \{r_l\}_1) - (\{R_j\}_1 + \{r_j\}_1) = \{0\} \quad (3.68)$$

**FIGURE 3.26**

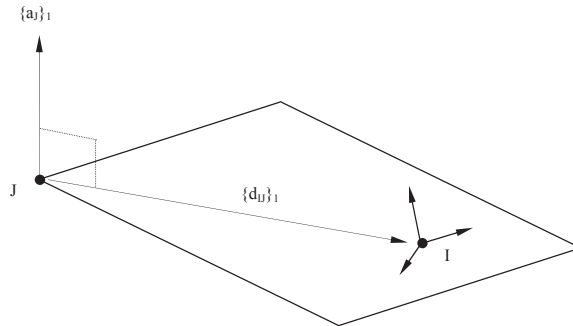
Atpoint constraint element.

This expression may be simplified by introducing a vector term  $\{d_{IJ}\}_1$  to represent the constrained displacement between the I and the J marker.

$$\{d_{IJ}\}_1 = (\{R_i\}_1 + \{r_i\}_1) - (\{R_j\}_1 + \{r_j\}_1) \quad (3.69)$$

The reaction force on part i can be represented by the vector  $\{\lambda\}_1$  with a moment given by  $\{r_i\}_1 \times \{\lambda\}_1$ . Applying Newton's Third Law the reaction force on part j can be represented by the vector  $-\{\lambda\}_1$  with a moment given by  $-\{\lambda\}_1 \times \{r_j\}_1$ . In order to complete the calculation, the contribution to the term  $\sum \{Mn_C\}_e$  in Eqn (3.67), a transformation of the moments into the coordinates of the part Euler-axis frame is required. For part i this would be achieved using  $[B_i]^T \{r_i\}_i \times [A_{i1}] \{\lambda\}_1$ . For part j this would be achieved using  $-[B_j]^T \{r_j\}_j \times [A_{j1}] \{\lambda\}_1$ .

The second basic constraint element constrains a point on one part to remain fixed within a plane on another part and is known as the *inplane* constraint or joint primitive, an application of which was shown earlier in Figure 3.25. As such it removes 1 DOF out of the plane as shown in Figure 3.27.

**FIGURE 3.27**

Inplane constraint element.

The plane is defined by a unit vector  $\{a_j\}_1$ , fixed in part j and perpendicular to the plane. The I marker belonging to part i is constrained to remain in the plane using the vector dot (scalar) product to enforce perpendicularity.

$$\{d_{IJ}\}_1 \bullet \{a_j\}_1 = 0 \quad (3.70)$$

Expanding this using the definition given for  $\{d_{IJ}\}_1$  in Eqn (3.38) gives the full expression for the constraint  $\Phi_d$ .

$$\Phi_d = [(\{R_i\}_1 + \{r_i\}_1) - (\{R_j\}_1 + \{r_j\}_1)] \bullet \{a_j\}_1 = 0 \quad (3.71)$$

This constraint can be represented by a vector constraint equation working in coordinates parallel to the axes of the GRF. The magnitude of the reaction force corresponding to this constraint can be represented by a scalar term  $\lambda_d$ . The reaction force on part i can be represented by the vector  $\{a_j\}_1 \lambda_d$  with a moment given by  $\{r_i\}_1 \times \{a_j\}_1 \lambda_d$ . Applying Newton's Third Law again the reaction force on part j can be represented by the vector  $-\{a_j\}_1 \lambda_d$ . The moment contribution to part j is given by  $-(\{r_j\}_1 + \{d_{IJ}\}_1) \times \{a_j\}_1 \lambda_d$ .

Expanding this again using the definition given for  $\{d_{IJ}\}_1$  in Eqn (3.69) gives  $-(\{R_i\}_1 + \{r_i\}_1 - \{R_j\}_1) \times \{a_j\}_1 \lambda_d$ . In order to complete the calculation, the contribution to the term  $\sum \{Mn_C\}_e$  in Eqn (3.36), a transformation of the moments into the coordinates of the part Euler-axis frame is needed.

For part i this would be achieved using  $[B_i]^T \{r_i\}_i \times [A_{ij}] \{a_j\}_1 \lambda_d$ .

For part j this would be achieved using  $[B_j]^T \{a_j\}_j \times [A_{ji}](\{R_i\}_1 + [A_{ji}]\{r_i\}_i - \{R_j\}_1) \lambda_d$ .

The third basic constraint element constrains a unit vector fixed in one part to remain perpendicular to a unit vector located in another part and is known as the *perpendicular* constraint. The constraint shown in Figure 3.28 is defined using a unit vector  $\{a_j\}_1$  located at the J marker in part j and a unit vector  $\{a_i\}_1$  located at the I marker belonging to part i.

The vector dot (scalar) product is used to enforce perpendicularity, as shown in Eqn (3.72).

$$\Phi_p = \{a_i\}_1 \bullet \{a_j\}_1 = 0 \quad (3.72)$$

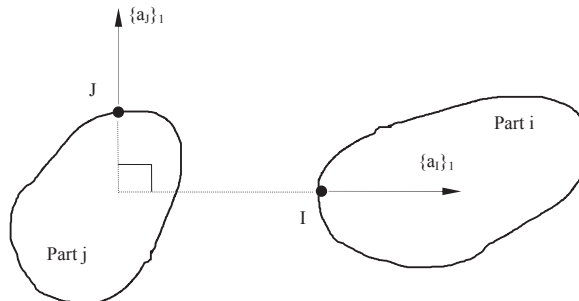


FIGURE 3.28

Perpendicular constraint element.



The constraint can be considered to be enforced by equal and opposite moments acting on part i and part j. The constraint does not contribute any forces to the part equations, but does include the scalar term  $\lambda_p$  in the formulation of the moments. The moment acting on part i is given by  $\{a_i\}_1 \times \{a_j\}_1 \lambda_p$ . Applying Newton's third law the moment acting on part j is given by  $-\{a_i\}_1 \times \{a_j\}_1 \lambda_p$ . The moments must be transformed into the coordinates of the part Euler-axis frame.

For part i this would be achieved using  $[B_i]^T \{a_i\}_1 \times [A_{ij}]\{a_j\}_j \lambda_p$ .

For part j this would be achieved using  $[B_j]^T \{a_j\}_j \times [A_{ji}]\{a_i\}_i \lambda_p$ .

The fourth and final basic constraint element is the *angular* constraint, which prevents the relative rotation of two parts about a common axis. The constraint equation is:

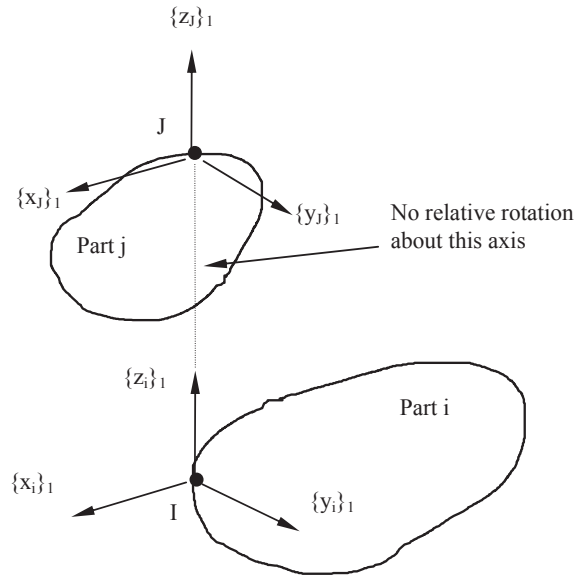
$$\Phi_\alpha = \tan^{-1} ( \{x_i\}_1 \bullet \{y_j\}_1 / \{x_i\}_1 \bullet \{x_j\}_1 ) = 0 \quad (3.73)$$

In applying this constraint, it is assumed that other system constraints will maintain the z-axes of the two parts to remain parallel as shown [Figure 3.29](#).

The moment acting on part i is given by  $\{z_i\}_1 \lambda_\alpha$  and on part j by  $-\{z_j\}_1 \lambda_\alpha$ . Transforming into the Euler axis system for each part gives a moment in the coordinate system for part i equal to  $[B_i]^T \{z_i\}_i \lambda_\alpha$  and on part j by  $-[B_j]^T \{z_j\}_j \lambda_\alpha$ .

The equations associated with each of the four basic constraint elements are summarised in [Table 3.1](#).

The force and moment contributions to each part in the generalised coordinates are summarised in [Table 3.2](#) and [Table 3.3](#).



**FIGURE 3.29**

Angular constraint element.

**Table 3.1** Basic Constraint Element Equations

Constraint	Full Equation	Abbreviated Form
Atpoint	$\{\Phi_a\}_1 = (\{R_i\}_1 + \{r_i\}_1) - (\{R_j\}_1 + \{r_j\}_1)$	$\{d_{ij}\}_1$
Inplane	$\Phi_d = [(\{R_i\}_1 + \{r_i\}_1) - (\{R_j\}_1 + \{r_j\}_1)] \cdot \{a_{ij}\}_1$	$\{d_{ij}\}_1 \cdot \{a_{ij}\}_1$
Perpendicular	$\Phi_p = \{a_i\}_1 \cdot \{a_j\}_1$	$\{a_i\}_1 \cdot \{a_j\}_1$
Angular	$\Phi_\alpha = \tan^{-1} (\{x_i\}_1 \cdot \{y_j\}_1 / \{x_j\}_1 \cdot \{y_i\}_1)$	$\alpha_{ij}$

**Table 3.2** Force Contributions for Basic Constraint Elements

Constraint	Part I Force	Part J Force
Atpoint	$\{\lambda\}$	$-\{\lambda\}$
Inplane	$[A_{ij}]\{a_{ij}\}_j \lambda_d$	$-[A_{ij}]\{a_{ij}\}_j \lambda_d$
Perpendicular	0	0
Angular	0	0

**Table 3.3** Moment Contributions for Basic Constraint Elements

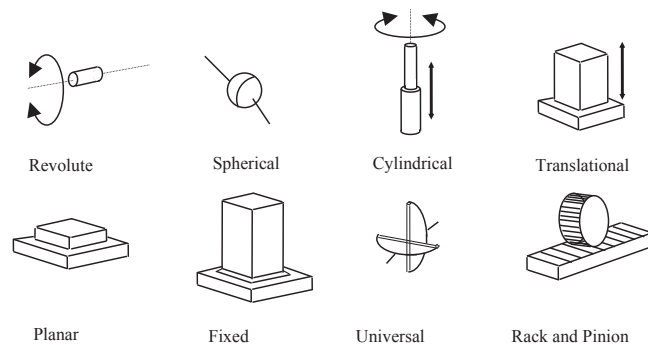
Constraint	Part I Moment	Part J Moment
Atpoint	$[B_j]^T \{r_i\}_i \times [A_{i1}] \lambda$	$[B_j]^T \{r_j\}_j \times [A_{j1}] \lambda$
Inplane	$[B_j]^T \{r_i\}_i \times [A_{ij}]\{a_{ij}\}_j \lambda_d$	$[B_j]^T \{a_{ij}\}_j \times [A_{j1}] (\{R_i\}_1 + [A_{i1}]\{r_i\}_i - \{R_j\}_1) \lambda_d$
Perpendicular	$[B_j]^T \{a_i\}_i \times [A_{ij}]\{a_{ij}\}_j \lambda_p$	$[B_j]^T \{a_{ij}\}_j \times [A_{ij}]\{a_{ij}\}_i \lambda_p$
Angular	$[B_j]^T \{z_i\}_i \lambda_\alpha$	$-[B_j]^T \{z_j\}_j \lambda_\alpha$

### 3.2.7 Standard joints

As stated there are a number of mechanical type joints that may be used to constrain the motion of bodies. Examples of some of the most commonly used joints are shown in [Figure 3.30](#).

Of the joints shown in [Figure 3.30](#) the spherical, revolute, translational, cylindrical and universal will figure most prominently in this text, particularly with regard to the modelling of suspension systems. The concept of an I frame on one part connecting to a J frame on another part is used again for joint elements.

For a spherical joint the I frame and J frame are defined to be coincident at the centre of the joint, but the orientation of the frames is irrelevant. For other joints, such as the revolute, cylindrical and translational, it is necessary not only to position the joint through the coordinates of the I and J frame but also to define the orientation

**FIGURE 3.30**

Examples of commonly used joint constraints.

of the axis associated with the mechanical characteristics, rotation and/or translation, of the joint. The method used in MSC ADAMS is to use the local z-axis of the markers to define the axis, the most convenient method of doing this often being to define a ZP parameter for each frame. For the universal joint, the axes of the spindles need to be defined perpendicular to one another. For this joint the I and J frame are defined to be coincident with the z-axis of each orientated to suit the axis of the spindle on the side of the joint associated with the body to which the marker belongs.

The vector equations that have been derived earlier for the basic constraints can be combined to generate constraint equations for the standard joints. A spherical joint, for example, fulfils exactly the same function as an atpoint joint primitive. Examples of constraint equations for some commonly used joints are shown in [Table 3.4](#).

It is also possible to define initial conditions associated with a joint, such as a revolute, translational or cylindrical. These are defined to be translational or rotational according to the characteristics of the associated joint. A cylindrical joint,

**Table 3.4** Joint Constraints in MSC ADAMS

Joint Type	Constraints			Abbreviated Equation
	Trans'	Rot'	Total	
Spherical	3	0	3	$\{d_{ij}\}_1 = 0$
Planar	1	2	3	$\{z_i\}_1 \cdot \{x_j\}_1 = 0, \{z_i\}_1 \cdot \{y_j\}_1 = 0, \{d_{ij}\}_1 \cdot \{z_j\}_1 = 0$
Universal	3	1	4	$\{d_{ij}\}_1 = 0, \{z_i\}_1 \cdot \{z_j\}_1 = 0$
Cylindrical	2	2	4	$\{z_i\}_1 \cdot \{x_j\}_1 = 0, \{z_i\}_1 \cdot \{y_j\}_1 = 0, \{d_{ij}\}_1 \cdot \{x_j\}_1 = 0, \{d_{ij}\}_1 \cdot \{y_j\}_1 = 0$
Revolute	3	2	5	$\{d_{ij}\}_1 = 0, \{z_i\}_1 \cdot \{x_j\}_1 = 0, \{z_i\}_1 \cdot \{y_j\}_1 = 0$

for example, may be defined to have an initial translational velocity of zero and a starting displacement of 100mm in the translational direction.

Initial conditions can be enforced at the start of analysis but released once the simulation commences, in other words after time equals zero in some multibody packages. For translational, revolute and cylindrical joints it is also possible to constrain the movement of the joint during the simulation using an applied motion statement.

For the joint referenced it is necessary to define a functional equation, normally only dependent on time, which controls the movement of the I frame relative to the J frame at the associated joint. It can be readily imagined that the functional equation can be extended to encompass more complex formulations using a library of off-the-shelf mathematical functions and expressions of the type associated with an engineering or scientific programming software. Newcomers to MBS analysis often find the concept of a defined motion being a constraint difficult to grasp as the modelling element involves movement. The motion statement here constrains the associated DOF at the joint. The movement defined by the function is enforced and cannot be altered by, for example, changes to the mass properties of the bodies or the introduction of external forces. It should also be noted that where a motion is applied to a joint it would be inconsistent to specify initial conditions for the DOF associated with the motion at the joint.

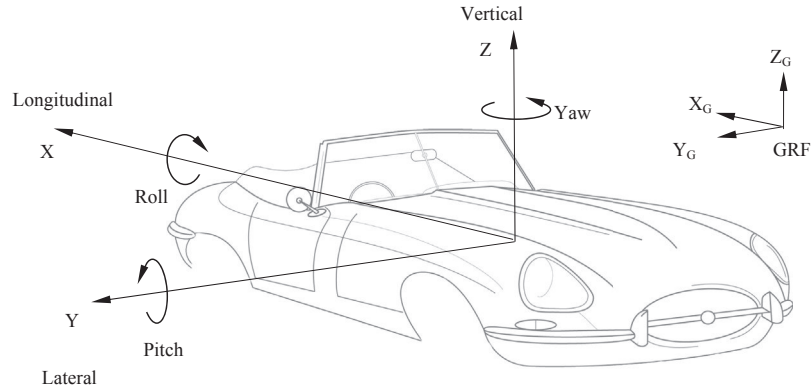
Another constraint element that will be used in this text is a coupling equation and is used to constrain, or couple, the movement of two or three joints by applying scale factors. One possible application of a coupling equation in this text is to represent the mechanical behaviour of a steering gear and so define the ratio between the rotation of the steering column and the rack. A more detailed discussion of this is given later in Chapter 6.

Note that the ordering and orientation of the frames defining the coupled joints is critical if the correct physical representation of the system is to be obtained.

The coupling equation does not take into account mechanical features, such as play in the joints or backlash, and as such does not model the reaction forces within the real mechanism. The coupling equation also does not consider one joint to drive the other. This will be a function of other forces defined elsewhere in the system model. As with the motion statement a DOF is lost to the system, as the coupling equation has enforced a kinematic relationship between the motion at joints that cannot be changed by external forces.

### 3.2.8 Degrees of freedom

Having introduced the modelling of rigid bodies and constraints, it is now possible to describe the determination of the degrees of freedom (DOF) in a mechanical system. The concepts are very similar between ADAMS-style multibody models and Simpack-style multibody models; the DOF are brought into being by the declaration of bodies and constrained out of existence by constraint equations. A free-floating rigid body in three-dimensional space will have 6 DOF, as shown in [Figure 3.31](#).

**FIGURE 3.31**

Degrees of Freedom Associated with an Unconstrained Rigid Body. GRF, ground reference frame.

For the vehicle body shown here and for the handling simulations that will be discussed later the body will have no direct constraint connecting it to the ground part. The only influence of the ground part will be through the forces and moments generated by one or more tyre models. Aerodynamic effects are typically modelled as stand-alone free-body forces and thus do not have any interaction with the ground part. For the axis system shown here, the vehicle will have DOF associated with translational motion in the longitudinal direction X, the lateral direction Y and the vertical direction Z. The rotational motions will involve roll about the x-axis, pitch about the y-axis and yaw about the z-axis. For vehicles such as ships and aircraft the terms surge, sway and heave are terms used to describe the translational motions but these are not commonly used in vehicle dynamics.<sup>7</sup> It should also be noted that for the examples in this text the x-axis is taken to point towards the rear of the vehicle, where in other texts discussing vehicle motion this is often forward to be consistent with the normal direction of travel.

For any multibody system model it is important that the analyst can determine and understand the total DOF in the system. For an ADAMS-style model, this can be achieved by using the Chebychev–Grübler–Kutzbach criterion (sometimes referred to as the ‘Gruebler equation’ or ‘Gruebler count’)

$$\text{Total DOF} = 6 \times (\text{Number of parts} - 1) - (\text{Number of constraints}) \quad (3.74)$$

The parts count in Eqn (3.74) is reduced by one to account for the fact that the nonmoving ground is counted as a part in the system. The DOF removed by typical

<sup>7</sup>It is noted that the items referred to as ‘anti-roll bars’ in UK usage are sometimes known as ‘anti-sway bars’ or just ‘sway bars’ in US English.

**Table 3.5** Degrees of Freedom Removed by Constraint Elements

Constraint Element	Translational Constraints	Rotational Constraints	Coupled Constraints	Total Constraints
Cylindrical joint	2	2	0	4
Fixed joint	3	3	0	6
Planar joint	1	2	0	3
Rack-and-pinion joint <sup>a</sup>	0	0	1	1
Revolute joint	3	2	0	5
Spherical joint	3	0	0	3
Translational joint	2	3	0	5
Universal joint	3	1	0	4
Atpoint joint primitive <sup>b</sup>	3	0	0	3
Inline joint primitive	2	0	0	2
Inplane joint primitive	1	0	0	1
Orientation joint primitive	0	3	0	3
Parallel joint primitive	0	2	0	2
Perpendicular joint primitive	0	1	0	1
Motion (translational)	1	0	0	1
Motion (rotational)	0	1	0	1
Coupler <sup>c</sup>	0	0	1	1

<sup>a</sup> A rack-and-pinion joint is a specific type of coupler.  
<sup>b</sup> A Atpoint primitive is functionally identical to a spherical joint.  
<sup>c</sup> Couplers may actually couple more than one joint, as in the case of a differential gear represented with three joints and a coupler.

constraint elements are summarised in Table 3.5 and may be used to complete the calculation. The method used in Simpack is demonstrated in Table 3.6.

For a Simpack-Style model, a version of the formula that is conceptually identical is necessary but differs in detail.

$$\text{TOTAL DOF} = (\text{Number of Joint DoF}) - (\text{Number of Constrained DoF}) \quad (3.75)$$

At this stage, it is necessary to introduce the subject of redundant constraints and over constraint checking. As a starting point consider a typical application in a suspension that involves the modelling of the mounts attaching a control arm or wish-bone to the body or chassis. This is illustrated in Figure 3.32, where in the real system the suspension arm is attached by two rubber bush mounts. As such the suspension arm is elastically mounted and if modelled in this way loses none of its 6 DOF to the mounts. The geometry and alignment of the bushes might suggest that the rigid body modelling approach might utilise a revolute attachment at each bush location. This in fact would introduce redundant constraints. A single revolute would fix the suspension arm and only allow it to rotate about the axis of the

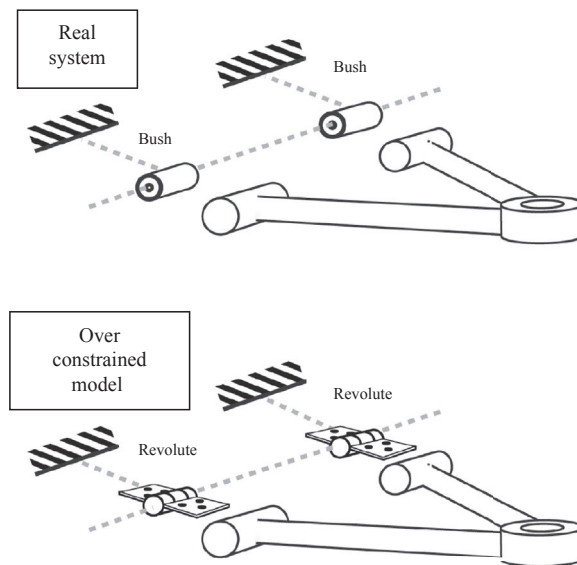
**Table 3.6** Joint DOF and Constrained DOF for Use with Simpack

Joint	Translational DOF	Rotational DOF	Coupled DOF	Total DOF
Free joint	3	3	0	6
Cylindrical joint	1	1	0	2
Fixed joint	0	0	0	0
Planar joint	2	1	0	3
Revolute joint	0	1	0	1
Spherical joint	0	3	0	3
Translational joint	1	0	0	1
Universal joint	0	2	0	2

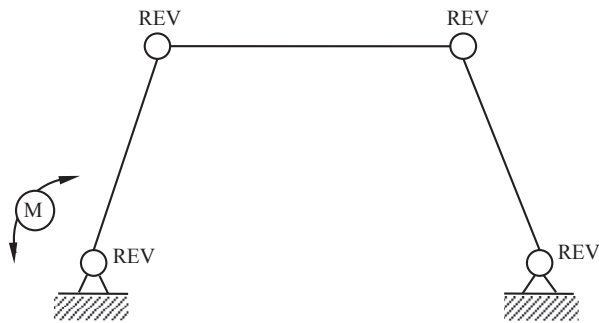
  

Constraint	Translational Constraints	Rotational Constraints	Coupled Constraints	Total Constraints
Rack-and-pinion constraint	0	0	1	1
Atpoint constraint	3	0	0	3
Inline constraint	2	0	0	2
Inplane constraint	1	0	0	1
Orientation constraint	0	3	0	3
Parallel constraint	0	2	0	2
Perpendicular constraint	0	1	0	1
Cylindrical constraint	2	2	0	4
Fixed constraint	3	3	0	6
Planar constraint	1	2	0	3
Revolute constraint	3	2	0	5
Spherical constraint	3	0	0	3
Translational constraint	2	3	0	5
Hooke constraint	3	1	0	4
Motion (translational)	1	0	0	1
Motion (rotational)	0	1	0	1
Coupler	0	0	1	1

*DOF, degrees of freedom.*

**FIGURE 3.32**

Redundant constraints in a suspension model.

**FIGURE 3.33**

Overconstrained Four-Bar Linkage Problem. REV, revolute, M, motion

freedom. Introducing the second revolute constraint only replicates the function of the first and would introduce in this case five redundant constraints.

Taking this a step further, consider the classical four-bar linkage problem shown in Figure 3.33. The mechanism forms a single loop comprising three moving parts and the fixed ground part. Revolute joints are used to constrain two of the parts to the ground and a motion is applied at one of these to impart movement to the system. An intuitive modelling attempt might consider all four connections as revolute joints as the likelihood is that in the real mechanism they would all appear similar.



On the basis of this modelling approach it is possible to apply the Greubler equation given in Eqns (3.74) and (3.75) and obtain the total DOF in the system based on the calculation tabulated below.

ADAMS Style				Simpack Style			
Parts	$6 \times (4-1) =$	18		Joints	$3 \times 1 =$	3	
Revolutes	$-5 \times 4 =$	-20		Constraints	$-5 \times 1 =$	-5	
Motion	$-1 \times 1 =$	-1		Motion	$-1 \times 1 =$	-1	
			<hr/>				<hr/>
Total DOF		$= -3$		Total DOF		$= -3$	

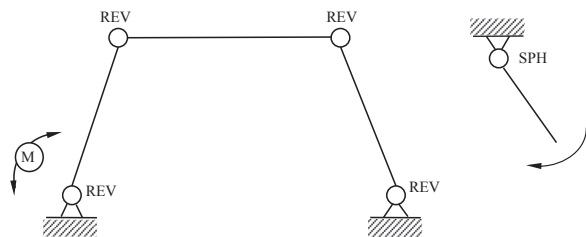
The total sum of the DOF for this system is negative, which is physically meaningless but has resulted through the selection of the joints and the introduction of redundant constraints. General-purpose programs will often identify when redundant constraints have been applied and use some automated algorithm to remove them. However, it may be that the constraint equations removed were at a point where connection forces were of interest; removing the constraint equation sets the forces to zero and reports them as such. The model looks correct and so a cursory examination of the results would appear plausible. This is the worst kind of error, the so-called ‘plausible-but-wrong’ error. For this reason, redundant constraints should be approached with a great deal of caution. It is always preferable for the analyst to set the model up in such a way that it is not overconstrained and the DOF in the system are fully understood.

As a first step in rectifying the problem shown in Figure 3.33 it might seem possible to overcome the problem by adding in extra bodies and constraints that result in an overall balance for the system as shown in Figure 3.34.

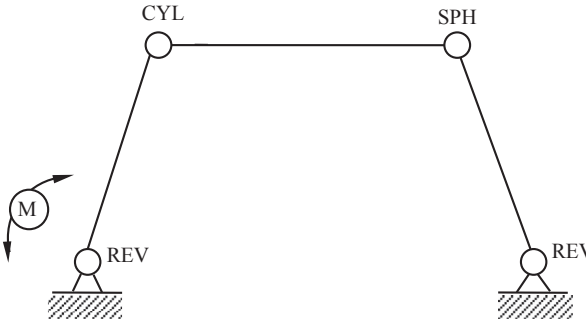
The addition of an extra body attached to the ground by a spherical joint appears to add the extra 3 DOF needed to at least obtain a 0 DOF model that would be used for a kinematic analysis as tabulated below.

ADAMS Style				Simpack Style			
Parts	$6 \times (5-1) =$	24		Joints	$3 \times 1 +$		
Revolutes	$-5 \times 4 =$	-20			$1 \times 3 =$	6	
Spherical	$-3 \times 1 =$	-3		Revolutes	$-5 \times 1 =$	-5	
Motion	$-1 \times 1 =$	-1		Motion	$-1 \times 1 =$	-1	
			<hr/>				<hr/>
Total DOF		$= 0$		Total DOF		$= 0$	

Although it appears the problem has been solved there are still redundant constraints in the system. In balancing the DOF in the model, it is necessary that not only is the overall system not overconstrained but also any individual loops within the model. A possible solution in this case is to select the joints shown in Figure 3.35.



**FIGURE 3.34**  
Overconstrained Loop in a System Model. REV, revolutes, M, motion, SPH, spherical.



**FIGURE 3.35**  
Zero Degree of Freedom Four-Bar Linkage Model. REV, revolutes, M, motion, SPH, spherical, CYL, cylindrical.

For the joints selected now the result is a 0 DOF model that would be used for a kinematic analysis. The DOF balance is tabulated below.

ADAMS Style			Simpack Style		
Parts	$6 \times (4-1) =$	18	Joints	$1 \times 1 +$	
Revolutes	$-5 \times 2 =$	-10		$2 \times 1 +$	
Spherical	$-3 \times 1 =$	-3		$3 \times 1 =$	6
Cylindrical	$-4 \times 1 =$	-4	Revolute	$-5 \times 1 =$	-5
Motion	$-1 \times 1 =$	-1	Motion	$-1 \times 1 =$	-1
Total DOF	$=$	0	Total DOF	$=$	0

The cylindrical joint is used to prevent an unwanted DOF that would result in the central link spinning about its own axis. In the authors' experience, formulation of linkage models often stumbles on either redundant constraints or the inadvertent inclusion of link-spin DOF.

### 3.2.9 Force elements

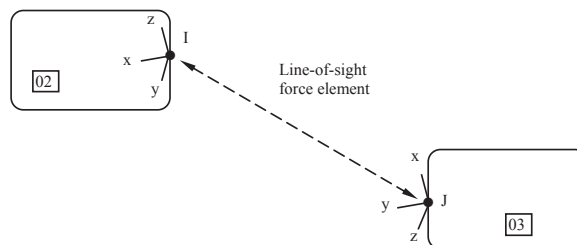
There are two fundamental types of force element that may be defined in a MBS model. The first of these are force elements that can be considered internal to the system model and to involve the effects of compliance between bodies. Examples of these include springs, dampers, rubber bushes and roll bars. As such these forces involve a connection between two bodies and due to the principle of Newton's Third Law are often referred to as action-reaction forces. For the translational class of force elements used to define springs and dampers the force will act along the line between two markers that define the ends of the element and as such this form of definition is referred to as the line-of-sight method.

The second type of force is the one that is external and applied to the model. Examples of these include gravitational forces, aerodynamic forces and any other external force applied to the model where the reaction on another body is not required. As such they may be referred to as action-only forces. The forces generated by a tyre model and input through the wheel centres into a full vehicle model can also be considered to be this type of force. These forces may be translational or rotational and as they require the definition of a magnitude, line of action and sense, the method of definition is referred to as the component method.

The definition of line-of-sight forces is illustrated in Figure 3.36, which shows a force acting along the line of sight between two points, an I and a J frame, on two separate parts. The forces acting on the I and the J frames are equal and opposite. As the line of the force is defined entirely by the location of the I and the J frames, the orientation of these is not relevant when defining the force.

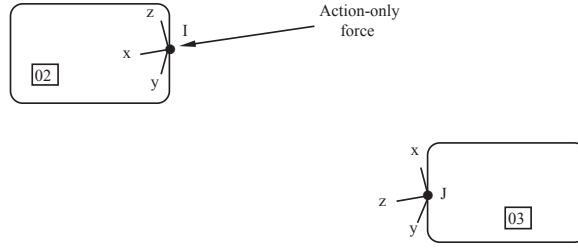
The component method applies to translational action-only forces, where the direction and sense of the force must be defined and to rotational forces, where the axis about which the torque acts is required. In MSC ADAMS, it is the z-axis of the J frame that is used to define the direction and sense of a translational action-only force. In the Simpack interface, a more general method is available for defining the line of action of the force, which can use broadly any frame in the model. The force acts on the I frame as shown in Figure 3.37, and there is no reaction on the J frame.

Rotational forces may also be defined to be action-reaction or action-only. In either case the torque produced is assumed to act on the I frame. For an



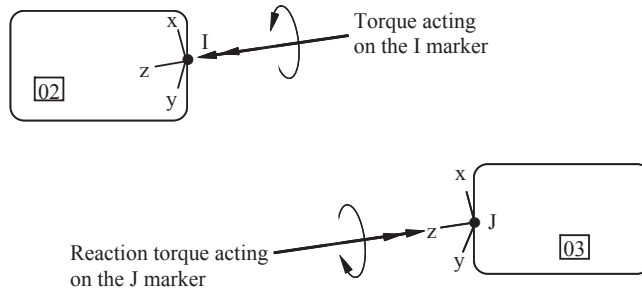
**FIGURE 3.36**

Line-of-sight force element.



**FIGURE 3.37**

Action-only force.



**FIGURE 3.38**

Action-reaction torque.

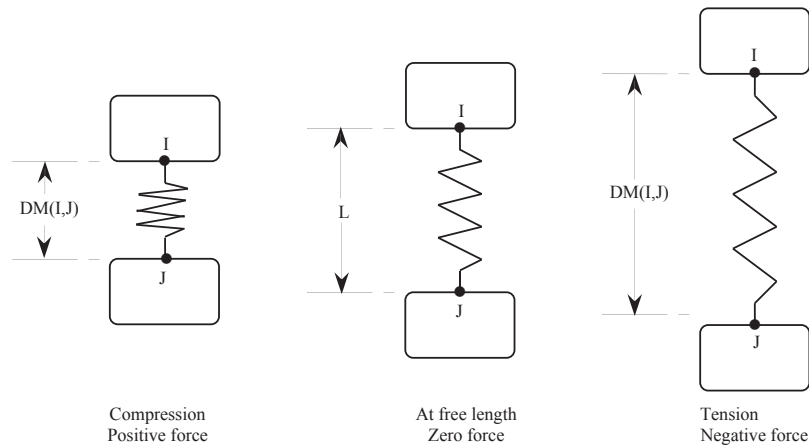
action-only torque it is again the z-axis of the J frame that is used, in this case to define the axis of rotation. If the torque is action-reaction, the z-axes of the I and J frames must be parallel and point in the same direction, as shown in Figure 3.38.

Considering next the definition of translational spring elements, we can start with a definition that is linear and introduces the use of system variables for the formulation of a force. As can be seen in Figure 3.39, the formulation of the spring force will be dependent on the length of the spring. This is made available through a system variable defined here as  $DM(I, J)$ , which represents the scalar magnitude of the displacement between the I and the J frames at any point in time during the simulation. The spring force  $F_S$  is initially defined here to be linear using

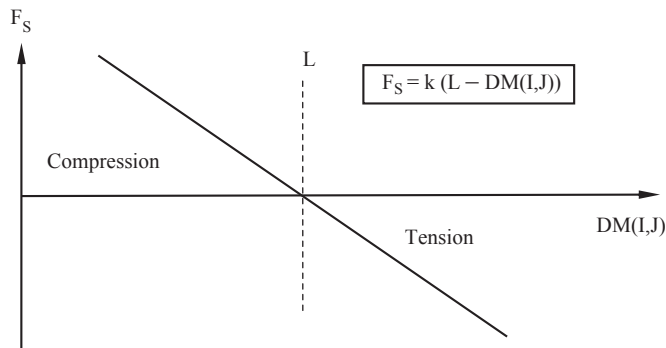
$$F_S = k (DM(I, J) - L) \quad (3.76)$$

where  $k$  is the spring stiffness and  $L$  is the free length of the spring, at zero force.

The equation used in Eqn (3.76) to determine  $F_S$  follows the required convention that the scalar value of force produced is positive when the spring is in compression, zero when it is at its free length and negative when it is in tension. This should not be confused with the components of any reaction force recovered at the I or the J frame. The sign of these will be dependent not only on the state of the spring but also the orientation of the spring force line of action and the reference frame in which the components are being resolved. The formulation of the spring force  $F_S$  is shown graphically in Figure 3.40.

**FIGURE 3.39**

Spring in compression, at free length and in tension.

**FIGURE 3.40**

Formulation of a linear spring force.

Note that the formulation used here produces a force that is positive in compression and negative in tension. This is opposite to the convention used for stresses in stress analysis and finite element programs.

An alternative form of definition that could be used with most general-purpose programs is a built-in spring element. General-purpose programs often include such built-in elements but they are something of a two-edged sword. There can be a colossal escalation in difficulty between using a built-in element and using a more general form by referring to system variables. This develops something of an institutionalised mentality in many student users that discourages them from doing what needs doing — perhaps raising the deflection to the power 1.1 to simulate a progressively wound spring — by encouraging them take the path of least resistance. It also has the effect of making the spring model seem like something sealed up and inaccessible instead of revealing it as just a force applied to a body as a function of

the relative displacement of two markers. This can conceivably (and does, observably, in a teaching environment) prevent the students from seeing the essential brilliance of Newton's work — that acceleration is force over mass and everything flows from that.

In a similar manner to the definition of a spring force the representation of a damper force will involve using the line-of-sight method to formulate an action–reaction force between an I frame on one part and a J marker on another part. We will again start with the linear case where we formulate a damper force  $F_D$  using

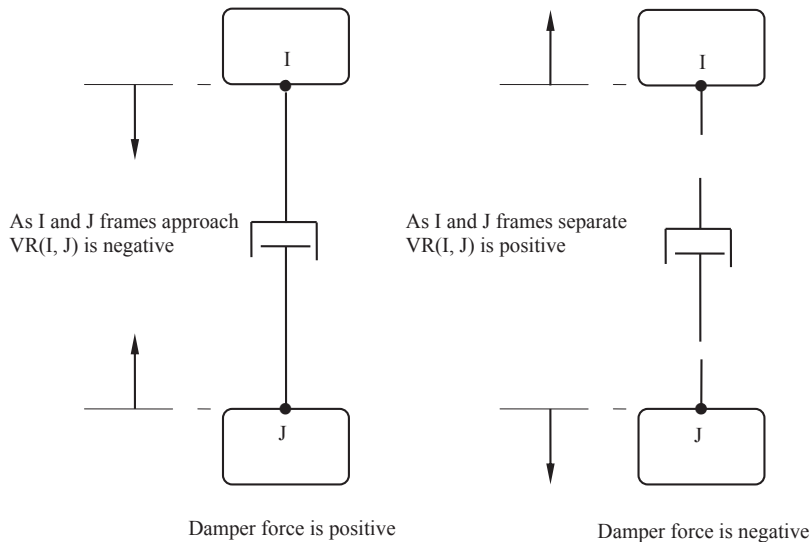
$$F_D = -c \cdot VR(I, J) \quad (3.77)$$

where

$VR(I, J)$  = radial line-of-sight velocity between I and J marker

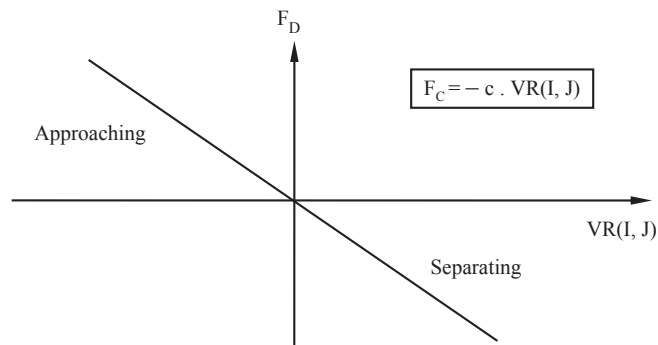
$c$  = damping coefficient

Since the force generated in a damper is related to the sliding velocity acting along the axis of the damper we introduce another system variable  $VR(I, J)$  that will take a positive sign when the markers are separating, as in suspension rebound, and a negative sign when the markers are approaching, as when a suspension moves upwards relative to the body in bump. It may be thought of as a radial velocity in a spherical coordinate system centred on the I frame and measuring the location of the J frame. The formulation of the damper force  $F_D$  in Eqn (3.77) is such that the damper forces are consistent with those of a spring. The force generated is positive in a repelling mode and negative in an attracting mode as illustrated in Figure 3.41.



**FIGURE 3.41**

Sign convention for damper forces and velocities.

**FIGURE 3.42**

Formulation of a linear damper force.

The formulation of the damper force  $F_S$  is shown graphically in Figure 3.42.

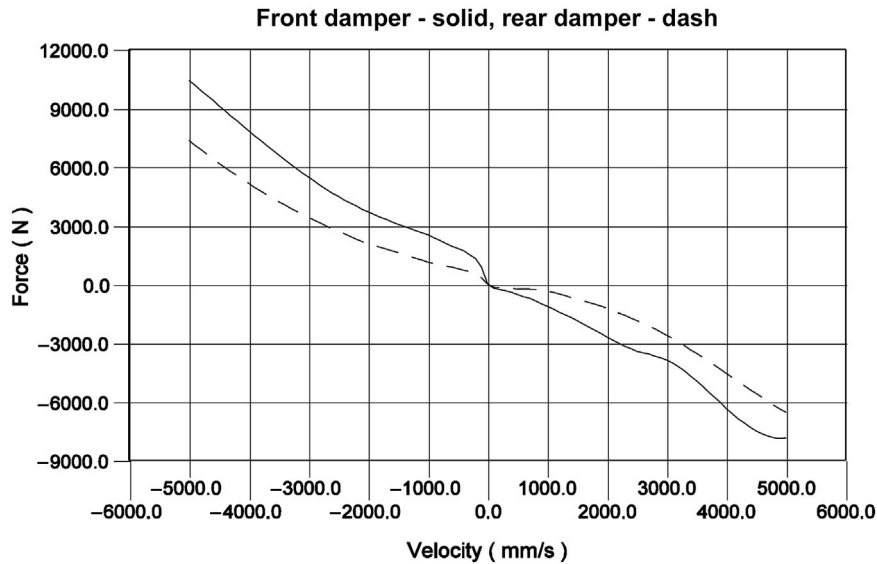
The definitions of spring and damper forces so far have been based on the assumption that the force element can be modelled as linear. This can be extended to consider the modelling of a nonlinear element. The example used will be based on the front and rear dampers for a typical road vehicle.

The nonlinear damper forces are defined in general-purpose programs using xy data sets, where the x values represent the velocity in the damper,  $VR(I, J)$ , and the y values are the force. During the analysis the force values are extracted using, for example, a cubic spline fit. The damper forces are not only nonlinear but are also asymmetric, having different properties in bump and rebound. Example curves for front and rear dampers are shown in Figure 3.43.

The cubic spline used here is based on a cubic curve fitting method (Forsythe et al., 1977). Note that although the function is used here to fit values to xy pairs of data it is also possible to use the function to fit values to three-dimensional xyz data sets of the type used for carpet plots. In these cases, MSC ADAMS uses a cubic interpolation method to first interpolate with respect to the x independent variables and then uses linear interpolation between curves of the second z independent variables. This is covered in more detail in Chapter 5 when the interpolation method is described for use with tyre models.

Other spline methods are available, notably the Akima spline method, in many multibody packages. When the data to be represented show a smooth and unchallenging characteristic, the methods are broadly equivalent. If the characteristic is more aggressively variable then spline methods often produce a result that is not what might have been expected; Figure 3.44 shows the intuitively expected result and a cubic fit to the datapoints. For this reason with spline data, the package in use should always be interrogated visually to see what it has fitted to the data passed to it. Graphing the data in Excel or Open Office cannot always be relied upon to reveal what the program itself will deliver.

An example of the spline data set used here to represent the nonlinear damper force follows. In this case, the velocities are defined in mm/s on the x-axis and

**FIGURE 3.43**

Nonlinear force characteristics for front and rear dampers.

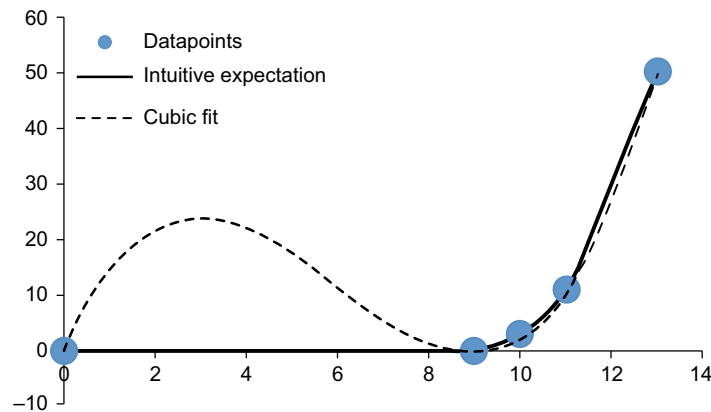
the values of force (N) are returned on the y-axis. It is important that the data set has sufficient range at both the top and bottom end to encompass the conditions during the simulation. Should the independent variable used on the x-axis reach values outside of the range of the spline data then the program will have to extrapolate values that can lead to unreliable results. Some general-purpose codes have a flag so as not to extrapolate with the last known cubic fit but rather to perform a linear extrapolation off the end of the data set.

SPLINE/1

```
X = -5000, -3150, -2870, -2450, -2205, -1925, -1610, -1260, -910,
-630, -470, -400, -350, -300, -250, -230, -200, -190, -160, -120,
-80, -55, -40, -20, -10, -1, -0.1, 0, 0.3, 3, 30, 40, 60, 80, 100,
200, 250, 400, 490, 770, 1050, 1330, 1820, 2060, 2485, 2590, 2730,
2835, 2940, 3080, 5000
Y = 10, 425, 5800, 5200, 4400, 4000, 3600, 3200, 2800, 2400, 2000,
1800, 1700, 1600, 1500, 1400, 1350, 1310, 1290, 1200, 1000, 700,
400, 210, 80, 40, 4, 0.4, 0, -1, -10, -100, -123, -150, -182, -200,
-260, -300, -400, -500, -800, -1200, -1600, -2400, -2800, -3400,
-3500, -3600, -3700, -3800, -4000, -7840
```

Although the method of modelling a nonlinear force element has been demonstrated here using a damper, the method is equally applicable to other force elements, such as springs and rubber bushes. For road springs it is usually sufficient to model these as linear but should a nonlinear formulation be required, the same approach is used with the magnitude of the displacement in the spring, DM(I, J),



**FIGURE 3.44**

When maths and intuition do not match.

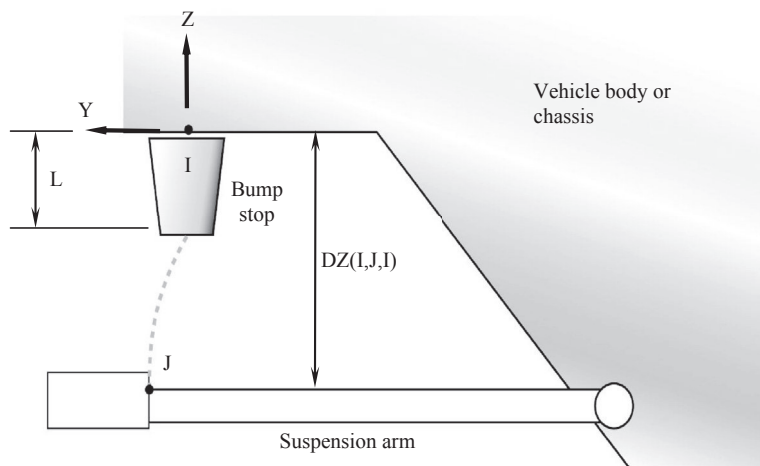
being the independent  $x$  variable. The modelling of nonlinear bush characteristics is dealt with later in this chapter.

Before moving on from the subject of springs and dampers it should be noted that it is also possible to define, in addition to applied torques, rotational springs and dampers. This will be dealt with in Chapter 6, where a rotational spring damper is used to idealise the characteristics of a suspension system in a full vehicle model based on axles that rotate, relative to the vehicle body, about the roll centres of the suspension systems.

Certain types of analysis may also require a suspension model to include the force characteristics of bump and rebound stops. For simulations such as a vehicle traversing off-road terrain, or the road wheel striking a pothole, the bump and rebound stops will need to be modelled. This may be particularly relevant when the vehicle or suspension model is being used to predict the distribution of force as inputs to finite element models.

If we consider the case of a bump stop, as illustrated in Figure 3.45, it can be seen that we face a new modelling problem. The force element needs now to represent the nonlinear problem of a gap, which closes so that the force must not only include the nonlinear characteristics of the rubber on contact but also be able to switch on and off as the gap closes and opens. The approach used to model the bump or rebound stop will depend on its location. The approach used here might need modification for a rebound stop built into a suspension strut.

In this example, the  $J$  frame is taken to belong to the vehicle body or chassis and the  $I$  frame is located at a point on the suspension that would strike the face of the bump stop. The point at which contact is established can be found here by comparing the  $z$  component of the displacement of the  $I$  frame from the  $J$  frame resolved parallel to the axes of the  $I$  frame. This will be described here as  $DZ(I, J, I)$ . Note that the third frame in brackets has been added to the system variable  $DZ$  to indicate that the component of displacement is resolved using this co-ordinate system rather than by default the  $GRF$ .

**FIGURE 3.45**

Modelling a bump stop contact force.

A first attempt at modelling this force could be achieved with the following function statement. In this example 0206 is the I frame, 0307 is the J frame, the length of the bump stop is 50 mm and a linear stiffness for the bump stop of 300 N/mm is used:

```
FUNCTION = IF(DZ(0206,0307,0206)-50: 300 * (50 - DZ(0206,0307,0206)), 0, 0)
```

This example has introduced an arithmetic IF that allows us to conditionally program the value of the FUNCTION and hence the force returned by this statement. The format used here is:

```
IF (expression 1: expression 2, expression 3, expression 4)
```

Expression 1 is evaluated and the value obtained is used to determine which expression is used to evaluate the FUNCTION as follows:

```
IF expression 1 < 0 then the FUNCTION = expression 2
IF expression 1 = 0 then the FUNCTION = expression 3
IF expression 1 > 0 then the FUNCTION = expression 4
```

Note that the arithmetic is in common use in multibody programs and is different from the three-expression IF used inside Excel or Open Office. In this case expression 1 is  $DZ(0206,0307,0206)-50$ . Clearly when this is greater than zero the gap is open and so the calculated force from expression 4 is zero. When expression 1 is equal to zero the I marker is just making contact with the face of the bump stop but no deformation has taken place and so the force is still zero. When expression 1 is less than zero then the contact force is generated using

$300 \cdot (50 - \text{DZ}(0206, 0307, 0206))$ ). Note that this has been programmed to ensure that a positive value is generated for the bump stop as it compresses.

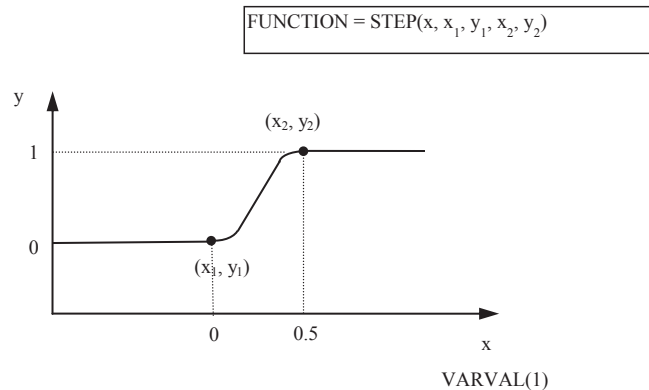
Although the method introduced here might work it is possible that it will cause problems during the numerical integration of the solution. This is because the arithmetic IF causes the force to switch on and off instantaneously about the point of contact. Such modelling is usually undesirable and some method is needed to improve the formulation by ‘smoothing’ the transition as contact occurs. This could be achieved by introducing another feature known as a STEP FUNCTION as follows:

```
VARIABLE/1, FUNCTION = 50 - DZ(0206, 0307, 0206)
FUNCTION = STEP( VARVAL(1), 0, 0, 0.5, 1) * (300 * (50 - DZ(0206, 0307, 0206)))
```

Although referred to here as a step, users of similar MBS software packages may also think of it as a ramp since the change in value returned by the function is not an instantaneous step change. The STEP function used here uses a cubic polynomial to smooth the transition from one state to another, as shown in Figure 3.47. The interval for the transition from the first to the second values can be quite short but when the integrator proceeds very slowly, the STEP form remains continuously differentiable – a key requirement for the successful convergence and completion of the integration step.

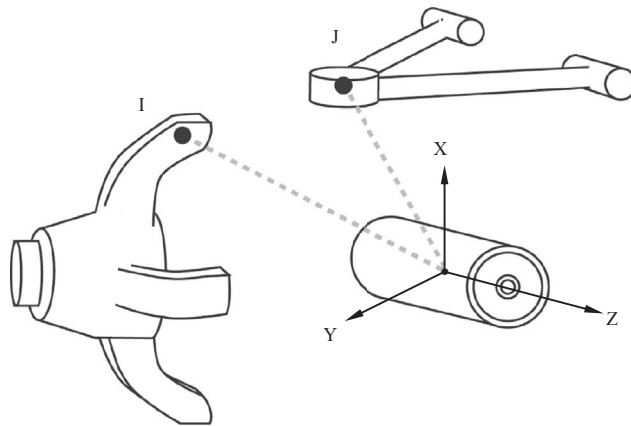
This example also introduces the use of VARIABLES that can be used to program equations and substitute the returned value, VARVAL(id), into another FUNCTION. In this case, the line VARIABLE/1... is used to program the deformation of the bump stop. The value of this, VARVAL(1) is used to define the variable on the x-axis that is used to step from one state to another. In this the step function is used to smooth the formulation of the contact force on between 0 and 0.5 mm of bump stop deformation. Note that not every package makes the use of solution variables as straightforward as this.

Additional functions often exist, such as a penalty stiffness modelling function which might be used to switch on a contact force or to extend the description

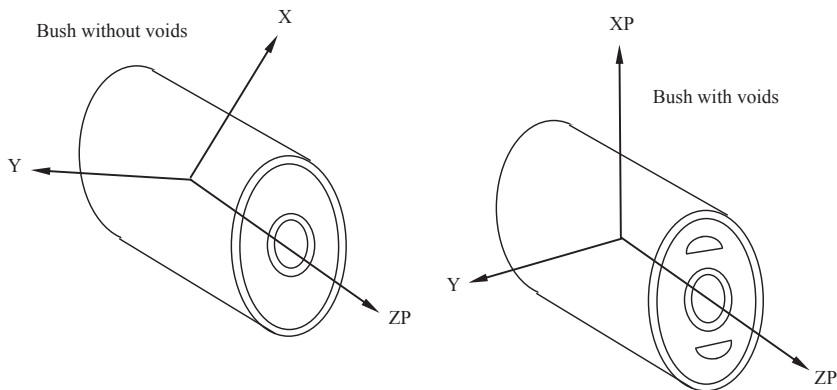


**FIGURE 3.46**

Step function for a bump stop force.

**FIGURE 3.47**

Modelling of suspension bushes.

**FIGURE 3.48**

Orientation of bush axis system.

from the initial linear one used here to a nonlinear model. Another consideration here is that the force formulation takes no account of the possibility that the solution could find a point where the I frame actually moves through the bump stop and past the J frame into the vehicle body. Although this makes no sense physically there is nothing in the formulation to take account of this and should it happen the force would actually reverse direction leading to a probable failure of the solution. Clever programming can take this into account. In addition to improving the description this could include, for example, building a sensor into the model to stop the simulation should this be about to occur, allowing the analyst to investigate the problem further. Alternatively, the I frame could be positioned substantially above its intuitive location and used with a progressively stiffening formulation to give rise to such large forces that the solution is unlikely to pass the point of metal-to-metal contact, but

if it momentarily does (perhaps during the iteration associated with an integration step) then the sense of the solution remains and the suspension arm is encouraged to accelerate back towards its real working space. Such an approach also guards against the unpredictable direction in which line-of-sight bump stop forces may act when the two markers are close to each other but undergoing lateral movements, perhaps on suspension elastomer compliances as described next.

The various elastic bushes or mounts used throughout a suspension system to isolate vibration may be represented initially by six linear uncoupled equations based on stiffness and damping. As with a joint, a bush connects two parts using an I frame on one body and a J frame on another body (Figure 3.47). These frames are normally taken to be coincident when setting up the model but it will be seen from the formulation presented here that any initial offset, either translational or rotational, would result in an initial preforce or torque in the bush. This would be in addition to any initial value for these that the user may care to define.

The general form of the equation for the forces and torques generated in the bush is given in Eqn (3.78)

$$\{F_{ij}\}_j = -[k] \{d_{ij}\}_j - [c] \{v_{ij}\}_j + \{f_{ij}\}_j \quad (3.78)$$

where

$\{F_{ij}\}_j$  is a column matrix containing the components of the force and torque acting on the I frame from the J frame;

$[k]$  is a square stiffness matrix where all off-diagonal terms are zero;

$\{d_{ij}\}_j$  is a column matrix containing the components of the displacement and rotation of the I frame relative to the J frame;

$[c]$  is a square damping matrix where all off-diagonal terms are zero;

$\{v_{ij}\}_j$  is a column matrix of time derivatives of the terms in the  $\{d_{ij}\}_j$  matrix;

$\{f_{ij}\}_j$  is a column matrix containing the components of the pre-force and pre-torque applied to the I frame.

Expanding Eqn (3.79) leads to the following set of uncoupled equations presented in matrix form as

$$\begin{bmatrix} F_x \\ F_y \\ F_z \\ T_x \\ T_y \\ T_z \end{bmatrix} = - \begin{bmatrix} k_x & 0 & 0 & 0 & 0 & 0 \\ 0 & k_y & 0 & 0 & 0 & 0 \\ 0 & 0 & k_z & 0 & 0 & 0 \\ 0 & 0 & 0 & k_{tx} & 0 & 0 \\ 0 & 0 & 0 & 0 & k_{ty} & 0 \\ 0 & 0 & 0 & 0 & 0 & k_{tz} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} c_x & 0 & 0 & 0 & 0 & 0 \\ 0 & c_y & 0 & 0 & 0 & 0 \\ 0 & 0 & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{tx} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{ty} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{tz} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \\ f_z \\ t_x \\ t_y \\ t_z \end{bmatrix} \quad (3.79)$$

It should be noted that all the terms in Eqns (3.78) and (3.79) are referenced to the J frame and that the equilibrating force and torque acting on the J frame is determined from

$$\{F_{ji}\}_j = -\{F_{ij}\}_j \quad (3.80)$$

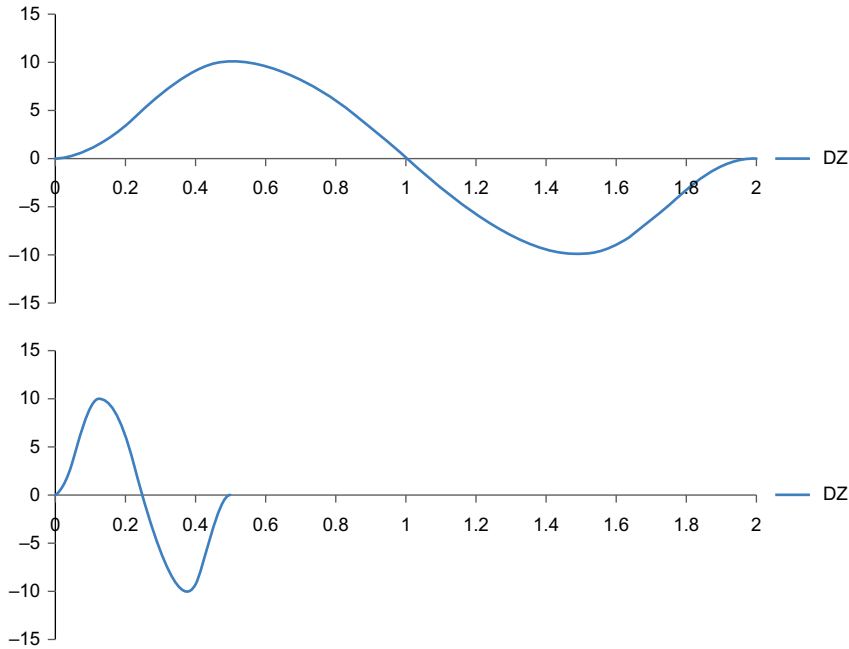
$$\{T_{ji}\}_j = -\{T_{ij}\}_j - \{d_{ij}\}_j \times \{F_{ij}\}_j \quad (3.81)$$

The shape and construction of the bush will dictate the manner in which the I frame and J frame are set up and orientated. This is illustrated in Figure 3.48, where it can be seen that for the cylindrical bush where there are no voids the radial stiffness is constant circumferentially. For this bush it is only necessary to ensure the z-axes of the I and J frame are aligned with the axis of the bush. For the bush with voids the radial stiffness will require definition in both the x- and y-directions as shown.

As with springs, it is possible to formulate a linear, viscously damped implementation of a suspension bush using system variables as described.

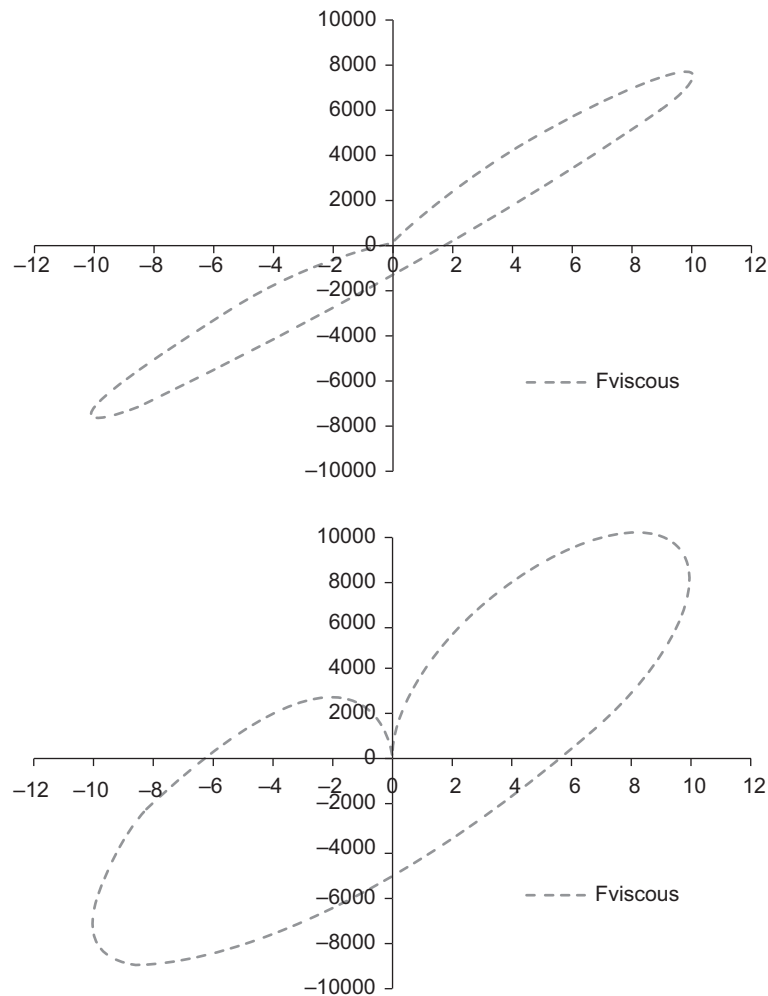
For a sinusoidal excitation, a linear combination of displacement and velocity will draw a simple Lissajous figure — an oval — with the viscous model. In a real test, a start-up and shutdown transient will be included, as shown in Figure 3.49 where the first and last quarter cycles are described using the expression:

$$DZ=0.5 \cdot (1 - \cos(2 \cdot \pi \cdot \text{time} \cdot \text{frequency})) \quad (3.82)$$



**FIGURE 3.49**

A comparison of test-style inputs at 0.5 and 2 Hz showing the start-up and shutdown transients in both cases.

**FIGURE 3.50**

A linear stiffness and viscous damping description of a bush at 0.5 Hz (top) and 2 Hz (bottom).

Using a simple linear stiffness and viscous damping gives a plausible curve at low frequency, with a characteristic ‘top lip’ shape to the upper part of the trace — [Figure 3.50](#).

However, it can be seen in the bottom of [Figure 3.50](#) that the frequency sensitivity of the bush is somewhat extraordinary and it may readily be appreciated that either forces are overrepresented at high frequencies or that energy dissipation is underrepresented at low frequencies. A viscous representation is simply not

adequate for elastomer elements. They rarely behave in a velocity-dependent manner, and are often better described with a hysteretic formulation.

The essence of hysteretic behaviour is that it is broadly independent of frequency. This is in strong contrast to viscous behaviour which, being proportional to velocity, varies directly with frequency — a doubling of frequency results in a doubling of the energy dissipated.

The key to this is a behaviour not dissimilar to Coulomb friction. Strictly, Coulomb friction is defined thus

$$V < 0, F = \mu N \quad (3.83)$$

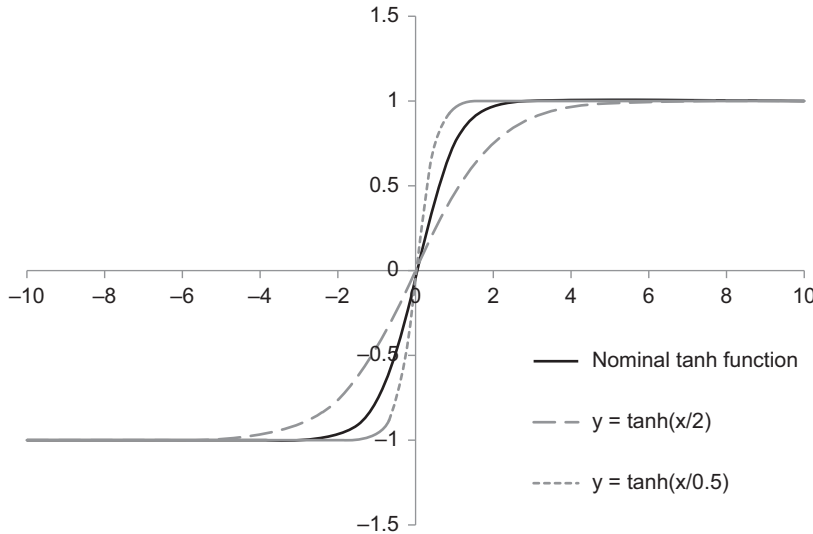
$$V = 0, F = 0 \quad (3.84)$$

$$V > 0, F = -\mu N \quad (3.85)$$

However, implementing this in a continuously differentiable manner is not possible directly. A solution used by many practitioners is the TANH function, the hyperbolic tangent

$$\tanh(x) = (1 - e^{-2x}) / (1 + e^{-2x}) \quad (3.86)$$

The derivation of the hyperbolic tangent is somewhat arcane but the hyperbolic tangent looks quite like a Coulomb function except for not having a discontinuity. It is illustrated in [Figure 3.51](#).



**FIGURE 3.51**

The tanh function can be used to give a Coulomb-like transition in force with no velocity dependence if velocity is used as the input variable.



Hysteretic damping is often described by the expression:

$$F_{\text{hysteretic}} = kx (1+i\eta) \quad (3.87)$$

where

- k is a spring constant;
- x is deflection;
- i is the square root of  $-1$ ;
- $\eta$  is a hysteretic damping constant.

Something resembling this can be implemented in a time domain model in the following form:

$$F_{\text{hystSimple}} = k \cdot DZ + F_h \cdot \tanh(VZ/V_{\text{ref}}) \quad (3.88)$$

where

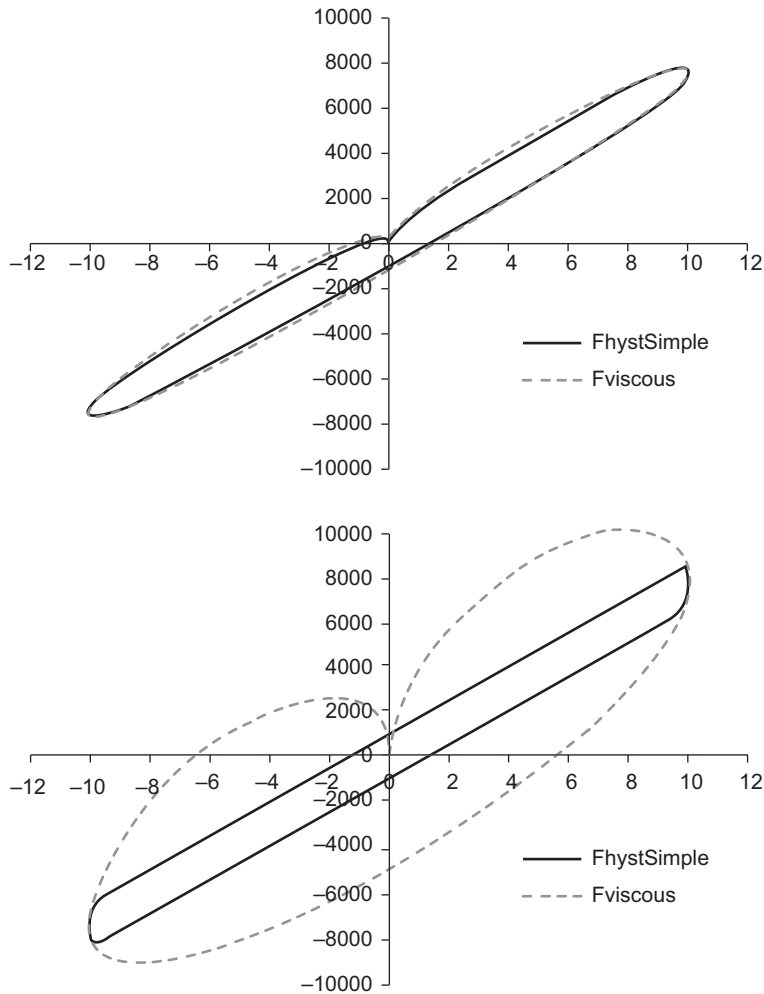
- k is a spring constant;
- DZ is deflection in the multibody model;
- $F_h$  is the size of the hysteretic force;
- VZ is velocity in the multibody model;
- $V_{\text{ref}}$  is the velocity when the full hysteresis effect is present.

It can be seen (Figure 3.52) that with access to a function library (most solvers will support the *tanh* function) and the same two states as discussed previously, quite a different character of behaviour can be produced to the viscous description; even this simplest hysteretic representation is substantially less sensitive to frequency. There is nothing to prevent the elastic term in the equation ( $k \cdot DZ$ ) being represented with a polynomial expression or a nonlinear spline.

Since historically multibody packages were primarily used for low frequency dynamics up to about 15 Hz then the omission of a hysteretic formulation was understandable although arguably disappointing. As can be seen, though, this can be remedied with a relatively simple function expression.

However, with detailed harshness modelling using flexible component representations and expectations of high fidelity simulations to much higher frequency, there are now a selection of toolkits available that allow the sophisticated fitting of elaborate models to suspension bushes. Measured elastomer data is characterised by a distinctive ‘scimitar-like’ shape, as shown in Figure 3.53, in which the loading characteristic continues in a fairly intuitive way until the maximum is reached; upon unloading, load drops immediately and significantly more steeply before returning to a parallel line some way below the initial loading line.

To move on from the simple hysteretic model and get closer to the measured form rapidly becomes quite arcane in its detail but similar principles can be employed. It will be noted that the described scimitar shape has a level of rotational symmetry when a fully developed loop (i.e. one not including a start-up/shutdown) is used. This means first that the mathematical expressions must discern the difference between loading and unloading:

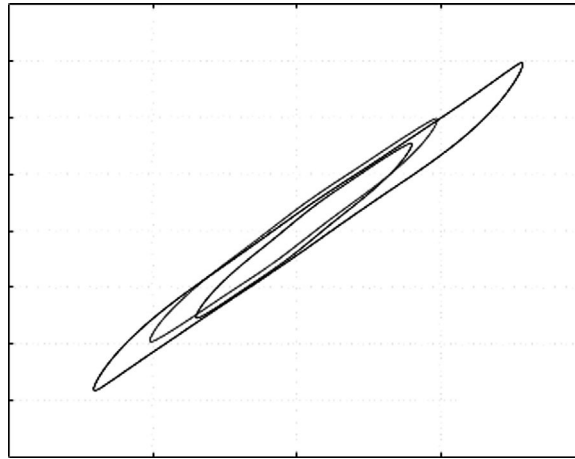


**FIGURE 3.52**

A simple hysteretic damping response at 0.5 Hz (top) and 2 Hz (bottom). Viscous traces are included as a comparison.

$$\text{SIGN}(\text{DZ} \cdot \text{VZ}) = \text{LOADFLAG}$$

Note that in many multibody codes a Fortran-style **SIGN** function is used in which sign returns 1 if the number is greater than or equal to 0 and  $-1$  if the number is less than 0; the Excel/Star Office-style **SIGN** function returns 0 if the expression evaluates to 0. Looking for a computationally imprecise real number to actually equal 0 is like looking for a needle in a haystack, but it does mean that the presence of zeroes can trip up test case models in Excel when they would compute in a completely satisfactory manner inside a general-purpose package.

**FIGURE 3.53**

A Typical 'Scimitar' Shape Hysteresis Loop from an Elastomer. (Karlsson and Persson, 2003.)

The variable **LOADFLAG** can be used to switch between two different computational schemes to give a rotationally symmetric shape to the resulting trace:

```
LOADFLAG > 0
F = k•DZ + Fh•SIGN(VZ)
LOADFLAG < 0
F = k•DZ + Fh•[2•tanh( VZ/(Vref•ABS(DZ)) ) - SIGN(VZ) ]
```

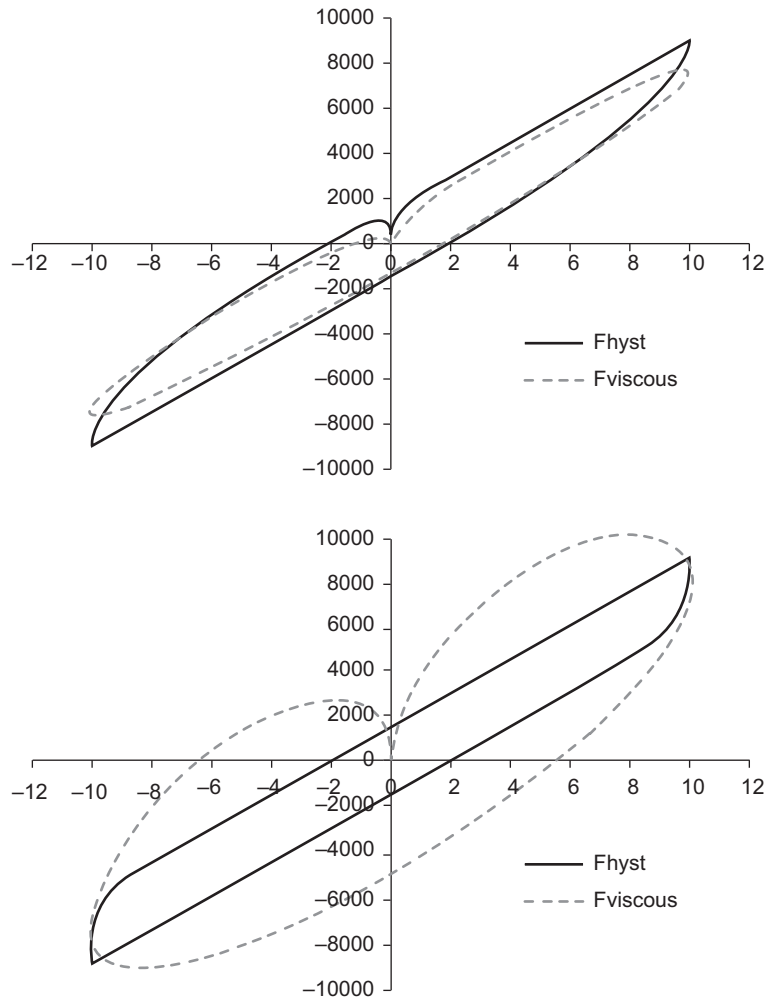
In this form the model appears suitable for use as is but detailed investigation shows that when starting from rest a discontinuity appears. To remedy this, a further flag can be calculated and used:

```
SIGN(DZ•AZ) = STARTSTOPFLAG
STARTSTOPFLAG < 0
SCALE = 1
STARTSTOPFLAG > 0
SCALE = tanh(VZ•AccFactor)
```

The hysteretic force model is then given in its entirety by:

$$F_{\text{hyst}} = F \cdot \text{SCALE}$$

This model is somewhat more complex than the simple model but does capture the rotationally symmetric aspect of the behaviour as shown in [Figure 3.54](#). Whether or not this is important to the final outcome is for the user to decide; it should be noted that, as the frequency of excitation rises they become very similar to each other, so the more complex model only has value at frequencies under about 10 Hz with the parameters shown.



**FIGURE 3.54**

A rotationally symmetric hysteresis model at 0.5 Hz (left) and 2 Hz (right) with the reference viscous model for comparison. Values for  $k$ ,  $F_h$ ,  $V_{ref}$  and  $AccFactor$  were 750 N/mm, 1500 N, 4 mm/s and 0.1 s/mm, respectively.

Both these models are in essence a spring in parallel with a Coulomb damping term, a variation on the Kelvin–Voigt model. They are just about the simplest representation of elastomer behaviour that captures the essential frequency independence. Other representations for elastomer behaviour are available (Maxwell, Zener and many others), and the interested reader is referred to Karlsson and Persson (2003) for an informed comparison between them.

If a yet more elaborate representation of elastomer behaviour is required, the use of simple function expressions can become somewhat limiting and it becomes more convenient to move into some kind of user subroutine. At this juncture many nonexpert users of the code will wish to pick up an existing library of models and some vendors of general-purpose software will supply a toolkit aimed specifically at this aspect of a system model, crucially including tools to fit measured data to the parameterised modelling scheme — which can be somewhat specialist in its own right.

This discussion of hysteretic elastomer modelling leads nicely on to some more general observations about modelling force generation mechanisms within a multi-body model. In essence, there are two approaches.

One is to take some known science and to use it directly; we might call this ‘causal’ modelling, where the cause of the force is captured directly. An example of this might be the force from a spring, which can be shown to be given by:

$$F_{\text{coil}} = (Gd^4 DZ) / (8D^3 n) \quad (3.89)$$

where:

- G = shear modulus;
- d = wire diameter;
- DZ = deflection;
- D = mean coil diameter;
- n = number of coils.

A detailed derivation of this expression is beyond the scope of this book but the interested reader is referred to, for example, ‘Roark’s Formulas for Stress and Strain’ (Young et al., 2011) for background in the theory of elasticity. In this example, we can measure the physical dimensions of the spring and look up the standard material properties, and predict the force a spring might produce without having to actually have a spring in our hands.

The second approach is to take some experimental observations and represent them with some form of mathematics that describes the resulting characteristic but makes no attempt to describe the underlying mechanism. We might call this ‘empirical’. It is important not to confuse empirical modelling with causal modelling, and it is also important to note that empirical models can often be more accurate since they are not based on an understanding of detailed physical phenomena — which history shows us is an ever-evolving and sometimes strangely shifting — but rather on robust and repeatable experimental observations.

A good example of useful empirical modelling is in the force generated by aerofoils (airfoils in the US). Typically they are represented below stall by some empirical form similar to:

$$F_{\text{aero}} = 0.5 \rho (C_{L0} + C_{L\alpha} \alpha_A) V X^2 \quad (3.90)$$

where:

$\rho$  = local density of air;

$C_{L0}$  = coefficient of lift at zero angle of attack;

$C_{L\alpha}$  = variation in coefficient of lift with angle of attack;

$\alpha_A$  = angle of attack.

A great deal of high quality experimentation was carried out by the National Advisory Committee for Aeronautics (later to morph into NASA) and the results published. With data reduction techniques it is possible to relate a certain description of a wing profile with coefficients  $C_{L0}$  and  $C_{L\alpha}$  but it should be noted there is no description of aspect ratio, camber or detailed geometry of the aerofoil in these coefficients.

The aerofoil case is a good example, where causal models are remarkably awkward to use but empirical models are very simple. The prospect of modelling the full dynamic behaviour of an aircraft manoeuvring by solving the causal computational fluid dynamics (CFD) problem is heroic in scale, but a comprehensive empirical flight dynamics model can easily be managed on a laptop; such a model is currently provided with every copy of the excellent textbook 'Flight Dynamics' by Stengel (2004).

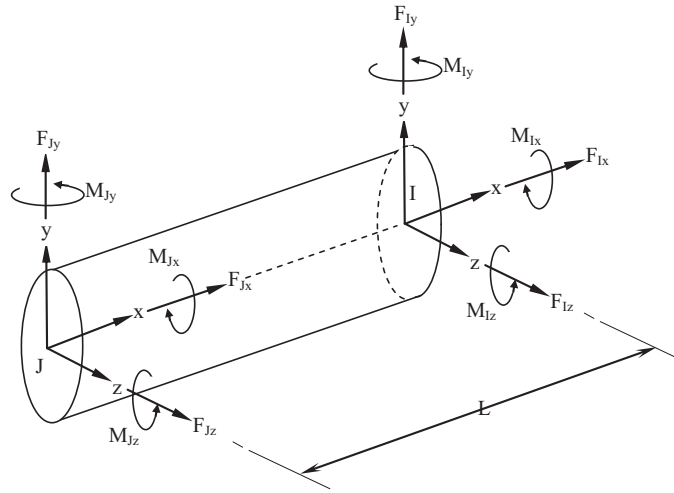
As a brief aside, there is popular belief that the principle of 'equal transit times' describes the lift generated by an aerofoil. A moment with a pocket calculator shows that the difference in path length on a Boeing 747 wing (about 2%) generates just about enough lift to suspend a European hatchback and nothing like enough to lift 400 tonnes of jumbo off the runway. Some argue that the wing simply deflects the air downwards a bit like a tea tray, but again some time with a pocket calculator shows that it helps but is still nothing like the right amount of lift.<sup>8</sup>

The exact details are somewhat arcane but in principle the special shape of the aerofoil steers a very large amount of air downward and the rate of change of momentum gives a force in the opposite direction, exactly as Newton predicted. The special shape of the aerofoil is necessary to keep the airflow attached and not allow it to separate; if such a separation occurs then a much smaller volume air is induced to change direction and the lift falls away dramatically. This special shape is what makes an aerofoil an aerofoil and why a tea tray does not work as a wing. There appears a great deal of controversy around these relatively simple (and apparently uncontentious) ideas, which the authors find somewhat inexplicable. However, it is not within the remit of this book to address such matters; the student is encouraged to critically consult reputable textbooks, preferably written by people who use the know-how to actually design wings, and not be too credulous of less reliable sources.

Tyres are another subject where causal models are remarkably difficult to formulate, even using sophisticated explicit finite element methods, but a number of

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<sup>8</sup>Stengel notes that a simple representation such as this actually works quite well for hypersonic flight but grossly underpredicts lift for subsonic flight.

**FIGURE 3.55**

Massless beam element.

empirical descriptions exist, which are entirely adequate for the subject of researching the whole vehicle behaviour. These are described in detail in Chapter 5.

The most advanced examples of the modelling of force elements extend to the incorporation of finite element type representations of beams and flexible bodies into the MBS model. In modelling a vehicle the most likely use of a beam-type element is going to be in modelling the rollbars or possibly if considered relevant an appropriate suspension member such as a tie rod. So-called 'semi-independent' suspensions, also described as 'twistbeams' also require a beam-type representation, as do leaf springs.

The beam element in general-purpose programs requires the definition of an I frame on one body and a J frame on another body to represent the ends of the beam with length  $L$ , as shown in Figure 3.55. The beam element transmits forces and moments between the two markers, has a constant cross section and obeys Timoshenko beam theory.

The beam centroidal axis is defined by the  $x$ -axis of the J frame and when the beam is in an undeflected state, the I frame lies on the  $x$ -axis of the J frame and has the same orientation. The forces and moments shown in Figure 3.55 are:

Axial Forces  $F_{Ix}$  and  $F_{Jx}$

Shear Forces  $F_{Iy}$ ,  $F_{Iz}$ ,  $F_{Jy}$  and  $F_{Jz}$

Twisting Moments  $M_{Ix}$  and  $M_{Jx}$

Bending Moments  $M_{Iy}$ ,  $M_{Iz}$ ,  $M_{Jy}$  and  $M_{Jz}$

The forces and moments applied to the I frame are related to the displacements and velocities in the beam using the beam theory equations presented in matrix form as

$$\begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{iz} \\ M_{Ix} \\ M_{Iy} \\ M_{Iz} \end{bmatrix} = - \begin{bmatrix} k_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 & 0 & k_{26} \\ 0 & 0 & k_{33} & 0 & k_{35} & 0 \\ 0 & 0 & 0 & k_{44} & 0 & 0 \\ 0 & 0 & k_{35} & 0 & k_{55} & 0 \\ 0 & k_{26} & 0 & 0 & 0 & k_{66} \end{bmatrix} \begin{bmatrix} d_x - L \\ d_y \\ d_z \\ r_x \\ r_y \\ r_z \end{bmatrix} - \begin{bmatrix} c_{11} & c_{21} & c_{31} & c_{41} & c_{51} & c_{61} \\ c_{21} & c_{22} & c_{32} & c_{42} & c_{52} & c_{62} \\ c_{31} & c_{32} & c_{33} & c_{43} & c_{53} & c_{63} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{54} & c_{64} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{65} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad (3.91)$$

The terms  $d_x$ ,  $d_y$  and  $d_z$  in Eqn (3.91) are the x-, y- and z-displacements of the I frame relative to the J frame measured in the J reference frame. The terms  $r_x$ ,  $r_y$  and  $r_z$  are the relative rotations of the I frame with respect to the J frame measured about the x-axis, y-axis and z-axis of the J frame. It should be noted here that the rotations in the beam are assumed to be small and that large angular deflections are not commutative. In these cases, typically when deflections in the beam approach 10% of the undeformed length, the theory does not correctly define the behaviour of the beam. The terms  $V_x$ ,  $V_y$ ,  $V_z$ ,  $\omega_x$ ,  $\omega_y$  and  $\omega_z$  are the velocities in the beam obtained as time derivatives of the translational and rotational displacements.

The stiffness and damping matrices are symmetric. The terms in the stiffness matrix are given by:

$$\begin{aligned}
 K_{11} &= \frac{EA}{L} & K_{22} &= \frac{12EI_{zz}}{L^3(1+P_y)} & K_{26} &= \frac{-6EI_{zz}}{L^2(1+P_y)} & K_{33} &= \frac{12EI_{yy}}{L^3(1+P_z)} \\
 K_{35} &= \frac{6EI_{yy}}{L^2(1+P_z)} & K_{44} &= \frac{GI_{xx}}{L} & K_{55} &= \frac{4+P_zEI_{yy}}{L(1+P_z)} & K_{66} &= \frac{4+P_yEI_{zz}}{L(1+P_y)}
 \end{aligned}$$

where  $P_y = 12 EI_{zz}A_{SY}/(GAL^2)$  and  $P_z = 12 EI_{yy}A_{SZ}/(GAL^2)$ . Young's modulus of elasticity for the beam is given by  $E$  and the shear modulus is given by  $G$ . The cross-sectional area of the beam is given by  $A$ . The terms  $I_{yy}$  and  $I_{zz}$  are the second moments of area about the neutral axes of the beam cross section. For a solid circular section with diameter  $D$  these would, for example, be given by  $I_{yy} = I_{zz} = \frac{\pi D^4}{64}$ .  $I_{xx}$  is the second moment of area about the longitudinal axis of

the beam. Again for a solid circular section this is given by  $I_{xx} = \frac{\pi D^4}{32}$ .

The final part of the definition of the terms in the stiffness matrix is the correction factors for shear deflection in the y and z directions for Timoshenko beams. These

are given in the y direction by  $A_{SY} = \frac{A}{I_{yy}^2} \int_A \left( \frac{Q_y}{I_z} \right)^2 dA$  and in the z direction by



$A_{sz} = \frac{A}{I_{zz}^2} \int_A \left( \frac{Q_z}{l_y} \right)^2 dA$ . The terms  $Q_y$  and  $Q_z$  are the first moments of area about the beam section y and z-axes respectively. The terms  $l_y$  and  $l_z$  are the dimensions of the beam cross section in the y and z directions of the cross-section axes.

The structural damping terms  $c_{11}$  through to  $c_{66}$  in Eqn (3.91) may be input directly or by using a ratio to factor the terms in the stiffness matrix. In a similar manner to the dampers and bushes discussed earlier it is possible to define a beam with nonlinear properties using more advanced elements that allow the definition of general force fields.

As with the bush elements the beam will produce an equilibrating force and moment acting on the J frame using

$$\{F_{ji}\}_j = -\{F_{ij}\}_j \quad (3.92)$$

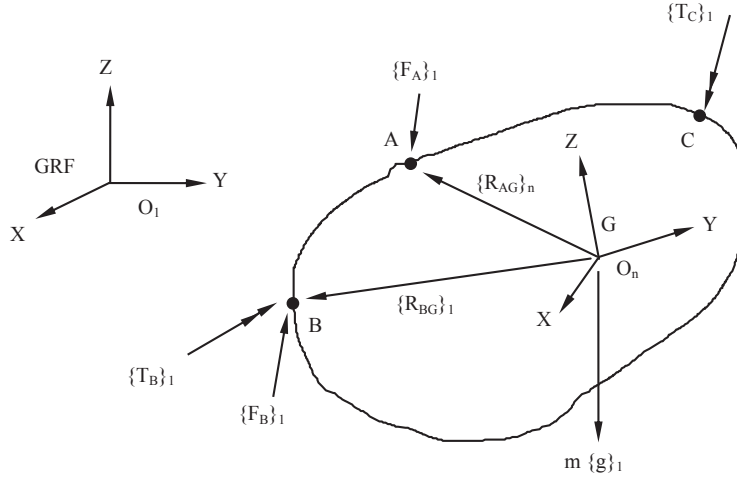
$$\{M_{ji}\}_j = -\{M_{ij}\}_j - \{d_{ij}\}_j \times \{F_{ij}\}_j \quad (3.93)$$

where  $\{d_{ij}\}_j$  is the position vector of the I frame relative to the J frame, as measured in the J reference frame.

Beam elements are generally massless and so they are often used in groups with small masses stationed at their interfaces. It can be readily imagined that a large number of beams and masses is functionally indistinguishable from a one-dimensional finite element model; in judging how many beams to use, the practitioner must decide principally on how well the deformed shape of the beam needs to be described; in the case of a twistbeam suspension, for example, the camber compliance mechanism produces a distinctive S-shape when viewed from ahead or behind; a single beam element will not capture this adequately, nor will two. The authors' habits have been to use 8–10 elements along a beam where such a deflection is occurring. If the deflections are substantially more complex than this, the use of full finite element representation of the component could be considered — the practitioner is encouraged to experiment with representative test cases before resting business-critical decisions on matters of modelling convenience.

### 3.2.10 Summation of forces and moments

Having considered the definition of force elements in terms of model definition we may conclude by considering the formulation of the equations for the forces and moments acting on a body. An applied force or moment can be defined using an equation to specify the magnitude, which may be functionally dependent on displacements, velocities, other applied forces and time. Using the example in Figure 3.56, there is an applied force  $\{F_A\}_1$  acting at point A, the weight of the body  $m\{g\}_1$  acting at the centre of mass G, a force  $\{F_B\}_1$  and a torque  $\{T_B\}_1$  due to an element, such as a bush or beam connection to another part. In addition there is an applied torque  $\{T_C\}_1$  acting at point C. Note that at this stage all the force and torque vectors are assumed to be resolved parallel to the GRF and that  $\{g\}_1$  is the vector of acceleration due to gravity and is again measured in the GRF.

**FIGURE 3.56**

Applied Forces and Torques on a Body. GRF, ground reference frame.

The summation of applied forces resolved in the GRF as required in Eqn (3.94) is obtained in this example by:

$$\sum \{F_{nA}\}_1 = \{F_A\}_1 + \{F_B\}_1 + m\{g\}_1 \quad (3.94)$$

The summation of moments about G is not so straightforward. MSC ADAMS performs the moment calculations about the axes of the Euler-axis frame. It is therefore necessary to use the transformation matrix  $[A_{n1}]$  to transform forces and torques to the part frame  $O_n$  and to use  $[B_n]^T$  to transform from the part frame to the Euler-axis frame.

$$\{F_A\}_n = [A_{n1}] \{F_A\}_1 \quad (3.95)$$

$$\{F_B\}_n = [A_{n1}] \{F_B\}_1 \quad (3.96)$$

$$\{T_B\}_n = [A_{n1}] \{T_B\}_1 \quad (3.97)$$

$$\{T_C\}_n = [A_{n1}] \{T_C\}_1 \quad (3.98)$$

It is now possible to calculate the moments at G due to the forces at A and B working in the part frame.

$$\{M_A\}_n = \{R_{AG}\}_n \times \{F_A\}_n \quad (3.99)$$

$$\{M_B\}_n = \{R_{BG}\}_n \times \{F_B\}_n \quad (3.100)$$

The next step is to transform the moments and torques to the Euler-axis frame and to summate as required in Eqn (3.101).

$$\sum \{M_n\}_e = [B_n]^T \{M_A\}_n + [B_n]^T \{M_B\}_n + [B_n]^T \{T_B\}_n + [B_n]^T \{T_C\}_n \quad (3.101)$$

Other general-purpose packages work similarly in principle but for package-specific detail the documentation should be consulted.

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### 3.3 Analysis capabilities

#### 3.3.1 Overview

Once the model has been assembled the main code may be used to carry out kinematic, static, quasi-static or dynamic analyses. Kinematic analysis is applicable to systems possessing zero rigid body DOF. Any movement in this type of system will be due to prescribed motions at joints — note this is joints in the more general sense rather than the Simpack-style usage of the word. The program uncouples the equations of motion and force and then solves separately and algebraically for displacements, velocities accelerations and forces.

For static analysis the velocities and accelerations are set to zero and the applied loads are balanced against the reaction forces until an equilibrium position is found. This may involve the system moving through large displacements between the initial definition and the equilibrium position and therefore requires a number of iterations before convergence on the solution closest to the initial configuration. Static analysis is often performed as a precursor to a dynamic analysis. An example would be to perform a static analysis on a full vehicle model before a dynamic handling simulation. This establishes the configuration of the vehicle at ‘kerb height’ before the vehicle moves forward during the dynamic phase of the simulation. Quasi-static analysis is a series of static equilibrium solutions at selected time steps. Although the system can be considered to be moving the dynamic response is not required. An example would be to perform a quasi-static analysis on a vehicle mounted on a tilting surface. As the surface rotates to different angles with time the static equilibrium of the vehicle can be calculated at selected angles.

Dynamic analysis is performed on systems with one or more DOF. The differential equations representing the system are automatically formulated and then numerically integrated to provide the position, velocities, accelerations and forces at successively later times. Although the user will select output at various points in time the program will often compute solutions at many intermediate points in time. The interval between each of these solution points is known as an integration time step. The size of the integration time step is constantly adjusted using internal logic, although the user may override the system defaults if so desired.

While often recommended as good practice, the authors' experience can be that finding static equilibrium prior to a dynamic analysis might well fail when the model contains errors. The failures are often difficult to diagnose by attempting static equilibrium but often lay themselves open quite clearly if a finely resolved dynamic solution is attempted. For this reason an initial 'drop onto the road' dynamic solution is helpful to debug vehicle models and to assess their plausibility before commencing the manoeuvre of interest. If the model is good it should have no problem solving for this event — including spinning the wheels up from rest if the velocity is nonzero — and should converge to running in a more or less straight line with no driver control input. (A 'more or less' straight line because there are often small force offsets with real measured tyre data at zero slip angle.)

A final possibility for solution of models is sometimes referred to as 'dynamic equilibrium'. This apparently oxymoronic description is essentially a series of dynamic solutions to converged equilibrium positions with no interim reporting.

Before considering the implementation of these analysis methods in more detail we must consider the fundamental methods used to solve linear and nonlinear equations. An additional modal analysis capability may be used to linearise the nonlinear equations of motion about a selected operating point and then find the natural frequencies and mode shapes associated with the system. It is also possible to extract the linearised state-space plant model in a format suitable for input to a control system design package.

### 3.3.2 Solving linear equations

MBS programs solve both linear and nonlinear equations during the analysis phase. Linear equations can be assembled in matrix form as shown in [Eqn \(3.102\)](#).

$$[A] [x] = [b] \quad (3.102)$$

where

- [A] is a square matrix of constants;
- [x] is a column matrix of unknowns;
- [b] is a column matrix of constants.

The formulations in a MBS program generally lead to a matrix [A], where most of the elements are zero. As such the matrix is referred to as sparse and the ratio of nonzero terms to the total number of matrix elements is referred to as the sparsity of the matrix. In a general-purpose program the sparsity of the matrix is typically in the range of 2–5% (Wielenga, 1987). The computer solvers that are developed in MBS to solve linear equations can be designed to exploit the sparsity of the [A] matrix leading to relatively fast solution times. This is one of the reasons, apart from improvements in computer hardware, that MBS programs can appear to solve quite complex engineering problems in seconds or minutes. These solution speeds can be quite notable when compared with other computer software such as finite elements or CFD programs.

The solution of Eqn (3.102) follows an established approach that initially involves decomposing or factorising the [A] matrix into the product of a lower triangular and an upper triangular matrix.

$$[A] = [L] [U] \quad (3.103)$$

For the lower triangular matrix [L], all the elements above the diagonal are set to zero. For the upper triangular matrix [U], all the elements on and below the diagonal are set to zero. In the following step a new set of unknowns [y] is substituted into Eqn (3.102) leading to

$$[L] [U] [x] = [b] \quad (3.104)$$

$$[U] [x] = [y] \quad (3.105)$$

$$[L] [y] = [b] \quad (3.106)$$

The terms in the [L] and [U] matrices may be obtained by progressive operations on the [A] matrix, where the terms in one row are all factored and then added or subtracted to the terms in another row. This process can be demonstrated by considering the expanded [A], [L] and [U] matrices as shown in Eqn (3.103), where to demonstrate the influence of sparsity some of the terms in the [A] matrix are initially set to zero.

$$\begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & 1 & U_{23} & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & 0 & 0 & A_{24} \\ A_{31} & 0 & A_{33} & 0 \\ A_{41} & 0 & A_{43} & A_{44} \end{bmatrix} \quad (3.107)$$

The fact that the [A] matrix is sparse leads to the [L] and [U] matrices also being sparse with subsequent savings in computer simulation time and memory storage. If we start by multiplying the rows in [L] into the first column of [U] we can begin to obtain the constants in the [L] and [U] matrices from those in the [A] matrix:

$$L_{11} = A_{11}$$

$$L_{21} = A_{21}$$

$$L_{31} = A_{31}$$

$$L_{41} = A_{41}$$

In a similar manner for the second column of [U] we get:

$$\begin{aligned}
 L_{11} U_{12} &= A_{12} & \text{therefore} & U_{12} = A_{12} / A_{11} \\
 L_{21} U_{12} + L_{22} &= 0 & \text{therefore} & L_{22} = -A_{21} A_{12} / A_{11} \\
 L_{31} U_{12} + L_{32} &= 0 & \text{therefore} & L_{32} = -A_{31} A_{12} / A_{11} \\
 L_{41} U_{12} + L_{42} &= 0 & \text{therefore} & L_{42} = -A_{41} A_{12} / A_{11}
 \end{aligned}$$

Moving on to the third column of [U] gives:

$$\begin{aligned}
 L_{11} U_{13} &= 0 & \text{therefore} & U_{13} = 0 \\
 L_{21} U_{13} + L_{22} U_{23} &= 0 & \text{therefore} & U_{23} = 0 \\
 L_{31} U_{13} + L_{32} U_{23} + L_{33} &= A_{33} & \text{therefore} & L_{33} = A_{33} \\
 L_{41} U_{13} + L_{42} U_{23} + L_{43} &= A_{43} & \text{therefore} & L_{43} = A_{43}
 \end{aligned}$$

Finishing with the multiplication of the rows in [L] into the fourth column of [U] gives:

$$\begin{aligned}
 L_{11} U_{14} &= A_{14} \\
 L_{21} U_{14} + L_{22} U_{24} &= A_{24} \\
 L_{31} U_{14} + L_{32} U_{24} + L_{33} U_{34} &= 0 \\
 L_{41} U_{14} + L_{42} U_{24} + L_{43} U_{34} + L_{44} &= A_{44}
 \end{aligned}$$

therefore

$$U_{14} = A_{14} / A_{11}$$

$$U_{24} = -(A_{24} - A_{21} A_{14} / A_{11}) / (A_{21} A_{12} / A_{11})$$

$$U_{34} = (- (A_{31} A_{14} / A_{11}) - (A_{31} A_{12} / A_{11}) (A_{24} - A_{21} A_{14} / A_{11}) / (A_{21} A_{12} / A_{11})) / A_{33}$$

$$\begin{aligned}
 L_{44} = & A_{44} - A_{41} A_{14} / A_{11} - (A_{41} A_{12} / A_{11}) (A_{24} - A_{21} A_{14} / A_{11}) / (A_{21} A_{12} / A_{11}) \\
 & - A_{43} (- (A_{31} A_{14} / A_{11}) - (A_{31} A_{12} / A_{11}) (A_{24} - A_{21} A_{14} / A_{11}) / (A_{21} A_{12} / A_{11})) / A_{13}
 \end{aligned}$$

From the preceding manipulations we can see that, for this example some of the terms in the [L] and [U] matrices come to zero. Using this we can update [Eqn \(3.107\)](#) to give

$$\begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & 0 & U_{14} \\ 0 & 1 & 0 & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & 0 & 0 & A_{24} \\ A_{31} & 0 & A_{33} & 0 \\ A_{41} & 0 & A_{43} & A_{44} \end{bmatrix} \quad (3.108)$$

Consideration of Eqn (3.108) reveals that some elements in the factors [L] and [U] are nonzero where the corresponding elements in [A] are zero. Such elements, for example  $L_{22}$ ,  $L_{32}$ , and  $L_{42}$  here, are referred to as ‘fills’. Having completed the decomposition process of factorising the [A] matrix into the [L] and [U] matrices the next step, forward—backward substitution, can commence.

The first step is a forward substitution, utilising the now known terms in the [L] matrix to find the terms in [y]. For this example if we expand [L] [y] = [b] as given in Eqn (3.106), we get

$$\begin{bmatrix} L_{11} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & 0 \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad (3.109)$$

therefore

$$\begin{aligned} y_1 &= b_1 / L_{11} \\ y_2 &= (b_2 - L_{21}y_1) / L_{22} \\ y_3 &= (b_3 - L_{31}y_1 - L_{32}y_2) / L_{33} \\ y_4 &= (b_4 - L_{41}y_1 - L_{42}y_2 - L_{43}y_3) / L_{44} \end{aligned}$$

The next step is a back substitution, utilising the terms found in the [U] matrix to find the overall solution for the terms in [x]. For this example if we expand [U] [x] = [y] as given in Eqn (3.105), we get

$$\begin{bmatrix} 1 & U_{12} & 0 & U_{14} \\ 0 & 1 & 0 & U_{24} \\ 0 & 0 & 1 & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \quad (3.110)$$

therefore

$$\begin{aligned} x_4 &= y_4 \\ x_3 &= (y_3 - U_{34}x_4) \\ x_2 &= (y_2 - U_{24}x_4) \\ x_1 &= (y_1 - U_{12}x_2 - U_{14}x_4) \end{aligned}$$

It can be seen that in the solution of Eqn (3.109), it is necessary to divide by the diagonal terms in [L]. These are referred to as ‘pivots’ and must be nonzero to avoid a singular matrix and failure in solution. There will also be problems if the pivots are so small that they approach a zero value. In these circumstances the condition of the matrices is said to be poor. The answer to this is to rearrange the order of the

equations, referred to as pivot selection, until the best set of pivots is available. This process is also called refactorisation. The mathematics has been developed so that the process will also attempt to minimise the number of fills to assist with a faster solution. The sequence of operations can be stored to speed solution as the simulation progresses unless the physical configuration of the system changes to a point where the matrix changes sufficiently to justify the reselection of a set of pivots. It should be noted that, in general, solution of these equations will involve much larger matrices than the four by four examples shown here and the sparsity of the matrix will be more apparent for these larger problems.

### 3.3.3 Nonlinear equations

In the case of nonlinear equations an iterative approach must be undertaken in order to obtain a solution. A set of nonlinear equations may be described in matrix form using

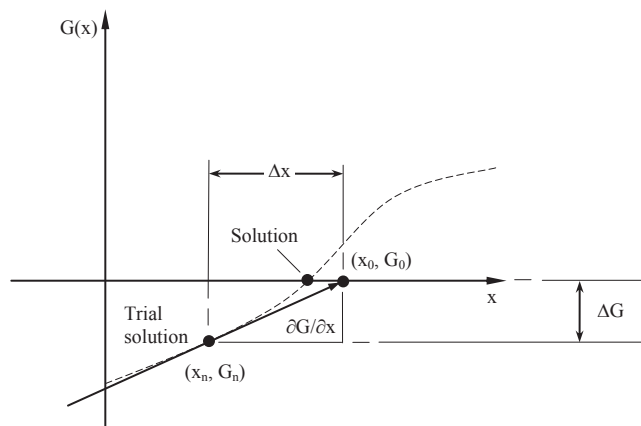
$$[G][x] = 0 \quad (3.111)$$

where

- [x] is a set of unknown variables;
- [G] is a set of implicit functions dependent on [x].

Consider the solution of a single nonlinear equation  $G(x) = 0$ , where  $G$  is an implicit function dependent on  $x$ . If we plot a graph of  $G(x)$  against  $x$  the solution is the value of  $x$  where the curve intersects the  $x$ -axis, as shown in Figure 3.57.

The solution may be obtained using Newton–Raphson iteration based on the assumption that very close to the solution the curve may be approximated to a straight line where it crosses the  $x$ -axis. This is illustrated in Figure 3.57, where



**FIGURE 3.57**

Application of Newton–Raphson Iteration.



the line is determined by a trial solution, for say the  $n$ th iteration, located at  $(x_n, G_n)$  and the slope or derivative of the curve  $\frac{\partial G}{\partial x}$  at this point.

If we take the point  $(x_o, G_o)$  to be that at which the straight line crosses the  $x$ -axis then the gradient  $\partial G/\partial x$  is given by

$$\frac{\partial G}{\partial x} = \frac{G_o - G_n}{x_o - x_n} = \frac{\Delta G}{\Delta x} \quad (3.112)$$

Rearranging this we can write

$$(\partial G/\partial x) \Delta x = \Delta G \quad (3.113)$$

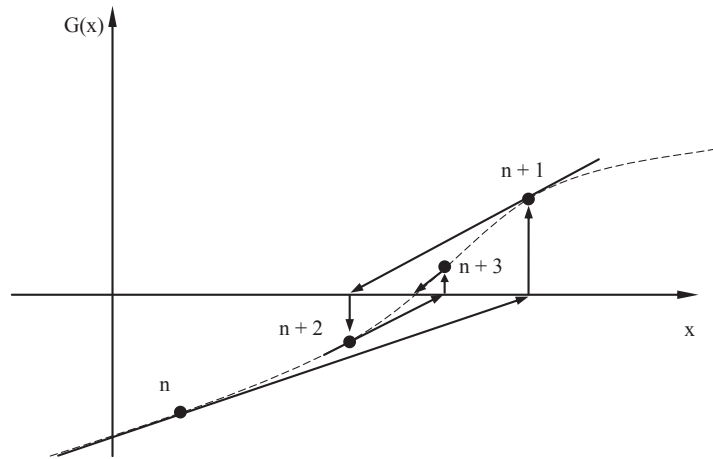
We can now extend this to demonstrate how the method can be used to iterate and close in on the solution. This is shown graphically in [Figure 3.58](#).

Extending the single equation  $G(x)$  to a set of simultaneous nonlinear equations, we can write

$$\left[ \frac{\partial G}{\partial x} \right] [\Delta x] = [\Delta G] \quad (3.114)$$

where  $\left[ \frac{\partial G}{\partial x} \right]$  is a matrix of partial derivatives representing the slopes of straight lines used in the solution. This is usually referred to as the Jacobian matrix.  $[\Delta x]$  is a column matrix of updates to the values of  $x$  for each equation and  $[\Delta G]$  is a column matrix of terms representing the current error in each equation.

Once all the terms in  $[\Delta G]$  approach zero a solution has been achieved. In practice, since this method involves the programming of a numerical approximation, a small error tolerance is used to determine that the terms in  $[\Delta G]$  have converged.



**FIGURE 3.58**

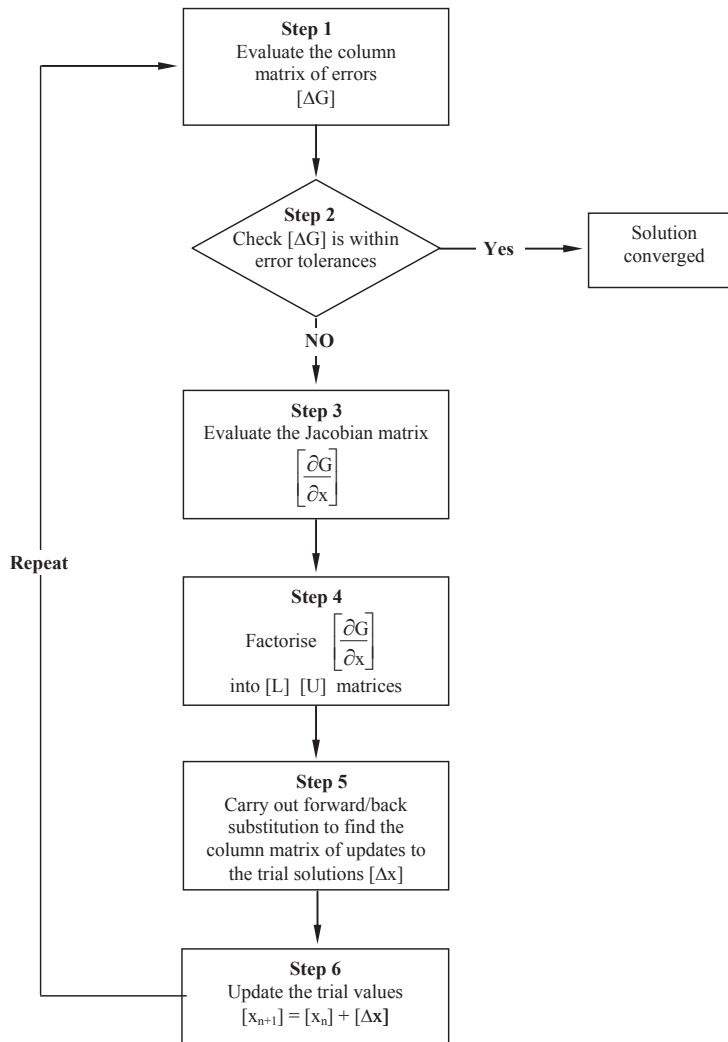
Convergence of Newton–Raphson Iteration.

For example, if we were to use an error tolerance equal to  $10^{-4}$  then we could say that convergence has been achieved when for each equation in the system.

$$-10^{-4} \leq \Delta G \leq 10^{-4} \quad (3.115)$$

The iterative process used here can be illustrated using the flow chart shown in Figure 3.59.

The Newton–Raphson method can be modified to speed up the solution process by not performing Step 3 and Step 4 in Figure 3.59. These two steps are the most computationally intensive and a more rapid solution may be achieved by calculating



**FIGURE 3.59**

Flow chart illustrating solution of nonlinear equations.

the Jacobian matrix, for example, on every second or third iteration. The graphical illustration of the process in Figure 3.58 shows how the use of the slope  $\frac{\partial G}{\partial x}$  from the previous iteration may often be an acceptable modification to the solution process.

### 3.3.4 Integration methods

The solution of nonlinear equations for dynamic systems with one or more DOF is a fundamental component of the application of MBS analysis to engineering problems. A number of methods have been developed and the process is commonly referred to as integration. The example here is based on the description given by (Wielenga, 1987) of a backwards differentiation formula method. The method described here can be considered to have two phases. The first of these is the use of a polynomial fit through past values of a given equation to predict a value at the next integration time step (solution point). The second step is to use the Newton-Raphson method described in the previous section to correct the prediction and achieve convergence. As such this method may also be described as a predictor-corrector approach. The equations being solved are first order differential equations and are also referred to as state equations. The state equations are implicit and have the general formulation  $G(x, \dot{x}, t) = 0$ .

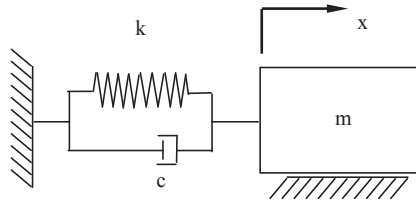
Clearly most dynamic problems such as the unforced vibration of the simple 1 DOF system shown in Figure 3.60 involve acceleration and have an equation of motion that is second order as shown in Eqn (3.116).

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = 0 \quad (3.116)$$

The implementation of this as first order differential equations is simply a matter of introducing a new variable, for example  $z$ , for the velocity and writing Eqn (3.116) as two implicit first order differential equations:

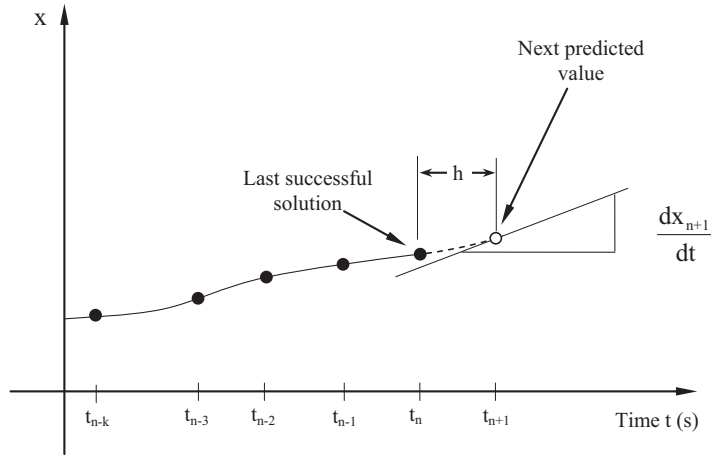
$$z - \frac{dx}{dt} = 0 \quad (3.117)$$

$$m \frac{dz}{dt} + cz + kx = 0 \quad (3.118)$$



**FIGURE 3.60**

Simple one degree of freedom mass, spring, damper system.

**FIGURE 3.61**

Use of predictor to fit a polynomial through past values.

The predictor phase of the solution can be explained with the help of the plot of the value of the state variable  $x$  as a function of time  $t$  as shown in Figure 3.61. The current solution point, or  $n$ th integration time step, is shown to be occurring at time  $t_n$ . Previous successful solutions have been computed at times  $t_{n-1}$ ,  $t_{n-2}$ ,  $t_{n-3}$ , ...  $t_{n-k}$ . The next value to be predicted,  $x_{n+1}$  will occur at time  $t_{n+1}$ . The time between each solution point is the integration time step  $h$ . Note that this should not be confused with output steps that are generally defined before the analysis and are used to fix the time interval at which results will be calculated for printing and plotting. The integration time step must be at least as small as the output time step to compute solutions but will generally be smaller in order to obtain a solution. In some programs — and notably real-time applications — the integration time steps may be fixed but usually with general-purpose programs they will be variable and will use programmed logic to determine the optimum step size for the problem in hand.

For the solution at the next time step to lie on the polynomial both the value of the state variable  $x_{n+1}$  and the derivative  $\frac{dx_{n+1}}{dt}$  must satisfy the polynomial.

The derivative  $\frac{dx_{n+1}}{dt}$  at the next time step  $t_{n+1}$  can be related to the unknown future value  $x_{n+1}$  and the past computed values using

$$\frac{dx_{n+1}}{dt} = P_n(x_{n+1}, x_n, x_{n-1}, x_{n-2}, x_{n-3}, \dots, x_{n-k}) \quad (3.119)$$

For the solution at the next time step to lie on the polynomial both the value of the state variable  $x_{n+1}$  and the derivative must satisfy the polynomial. For successful computation of the next solution at time  $t_{n+1}$  there are therefore two unknowns,

$x_{n+1}$  and  $\frac{dx_{n+1}}{dt}$ , that must be found. This requires two equations: the first being the polynomial in Eqn (3.119) and the second being the state equation  $G(x, \dot{x}, t)=0$ . Using the Newton–Raphson approach described in the previous section, the solution takes the form in Eqn (3.120)

$$\frac{\partial G}{\partial \dot{x}} \Delta \dot{x}_{n+1} + \frac{\partial G}{\partial x} \Delta x_{n+1} = -G(\dot{x}_{n+1}, x_{n+1}, t) \quad (3.120)$$

If we substitute a term  $\beta$  that represents the ratio  $\Delta \dot{x}_{n+1} = \beta \Delta x_{n+1}$  we end up with the following two equations on which the solution is based

$$\left[ \beta \frac{\partial G}{\partial \dot{x}} + \frac{\partial G}{\partial x} \right] \Delta x_{n+1} = -G(\dot{x}_{n+1}, x_{n+1}, t) \quad (3.121)$$

$$\Delta \dot{x}_{n+1} = \beta \Delta x_{n+1} \quad (3.122)$$

The integration process can be thought of as having two distinct phases. The first of these is the predictor phase that results in values of  $x_{n+1}$  and  $\dot{x}_{n+1}$  that satisfy a polynomial fit through past values. The second is the corrector phase that iterates using the process shown in Figure 3.58, until the error is within tolerance and the solution can progress to the next time step. Parameters can be set that control the solution process. Examples of these are the initial, maximum and minimum integration step sizes to be used and the order of the polynomial fit. Parameters used during the corrector phase include the acceptable error tolerance, the maximum number of iterations and the pattern or sequence to be used in updating the Jacobian matrix during the iterations. In general these will default to values programmed into the software but with experience users will find the most suitable settings for the analysis in hand.

The integration step size is variable and if a solution is not achieved at the next prediction point the solver can reduce the integration time step and alter the order of the polynomial to attempt another solution. This typically occurs when there is a sudden change in an equation associated with a physical event, such as an impact or clash of parts. A general rule in modelling is to never program an equation that instantly changes value at a certain time. Problems will be avoided by using, for example, a function that allows a smooth transition from one value to another over a physically small time interval.

Consideration of the predictor corrector approach will indicate that at the start of a transient solution the solver may step forward and backward initially as it establishes a suitable scheme to progress the solution. Experienced users will again be aware of this and avoid programming important events to occur immediately after the start. For example, the simulation of a 5 s lane change manoeuvre may be best accomplished by an initial static analysis followed by the transient simulation for say 1 s of straight line driving before any steering inputs are made, notwithstanding the debugging difficulties mentioned in Section 3.3.1.

Advanced applications may also involve the incorporation of control algorithms based on a discrete time step. A possible solution here is to fix the integration minimum and maximum step size to ensure the solver computes solutions at the fixed time steps of the ABS algorithm. This may of course result in inefficient computation at some stages of the simulation. On the other hand the fixed step scheme must be refined enough to deal with any sudden nonlinear events during the analysis.

Commercial MBS codes often have a range of integrators available that will adopt variations on the solution process discussed here. A commonly used method is referred to as the Gstiff integrator (Gear, 1971). Rather than storing past values of the polynomial a Taylor expansion (3.82) is used to store the polynomial in a form using the current value of the state variable  $x$  and the integration step size  $h$ . This is known as a Nordsieck vector. At a current time  $t$  the Nordsieck vector  $[N_t]$  has the form.

$$[N_t]^T = \left[ x \quad h \frac{dx}{dt} \quad \frac{h^2}{2} \frac{d^2x}{dt^2} \quad \frac{h^3}{3!} \frac{d^3x}{dt^3} \quad \dots \quad \frac{h^k}{k!} \frac{d^kx}{dt^k} \right] \quad (3.123)$$

The components of  $[N_t]$  are added together to predict the next value of  $x$  at a time step  $h$  forward to a new time  $t+h$ . As the simulation progresses the Nordsieck vector is updated by premultiplying by the Pascal triangle matrix to give a new vector  $[N_{t+h}]$ . An example of this is shown for a polynomial with an order  $k$  equal to five in Eqn (3.124).

$$[N_{t+h}] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} [N_t] \quad (3.124)$$

The predicted values in  $[N_{t+h}]$  lie on the polynomial described by  $[N_t]$  but are subject to further change as the state equations have not yet been corrected using the process shown in Figure 3.58. During the corrector phase the starting values of  $x$  and  $\frac{dx}{dt}$  are taken from  $[N_{t+h}]$ . As the value of  $x$  changes during the Newton-Raphson iteration process the components of the Nordsieck vector are updated by  $[\Delta N]$  where

$$[\Delta N] = [c] [\Delta x] \quad (3.125)$$

and

$$[c]^T = [c_0 \ c_1 \ c_2 \ c_3 \ \dots \ c_k] \quad (3.126)$$

The matrix  $[c]$  contains constants with values dependent on the polynomial order  $k$  (Orlandea, 1973).

Many other integration schemes are possible and the reader is encouraged to consult the software documentation for details. It is the author's general observation that a well-conditioned model will solve well with the default integrator; if the model will only solve with one particular integrator, this may be a clue that something is poorly conditioned within it. In principle the system should not 'notice' with which integrator it is being solved. However, experimentation may show that some integrators are substantially faster than others for certain classes of problems.

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### 3.4 Eigensolutions

As well as the time domain solution, it is extremely instructional to interrogate a model for its eigensolutions as part of the commissioning process, even if their solution is not part of the immediate analytical requirement. It serves as a rather unimpeachable witness to the actual content of the model (as distinct from the expected or intended content of the model, which is rarely in question).

Eigenvalues have many interesting properties to a mathematician but their primary interest to an engineer is that they allow some familiarity with the character of the time domain solution without actually having to compute it.

If the system has resonances, eigensolutions for the system equations describe them. They are always inherently present even if they are not being excited; the propensity of a passenger car to oscillate on its suspension springs is always there whether or not it is actually doing so.

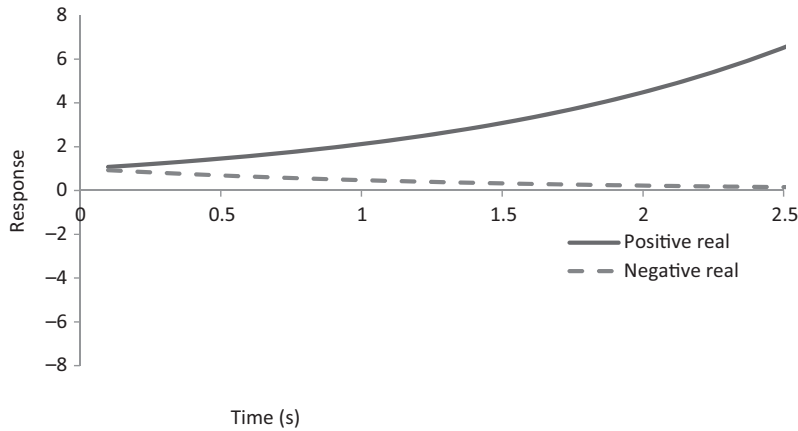
The equations used to describe the system are differential equations, which have been the subject of study for over 300 years. It is reasonable to investigate the solutions of the differential equations for a form  $Ae^{\lambda t}$ .

The presumption of a form of solution is common practice in the solution of differential equations but it sometimes mystifies students — it seems as if it is simply not cricket to know the answer before one starts. Be that as it may, if a form for the solution is presumed but turns out to have been inappropriately chosen, it usually emerges in short order. While such a statement hardly constitutes a rigorous mathematical proof, it will have to suffice since unfortunately such a proof is beyond the scope of this book.

In the suggested form of solution, the value  $\lambda$  is also the eigenvalue for the solution being considered. If  $\lambda$  is a positive real number then the response of the system is unbounded and grows with time. If  $\lambda$  is a negative real number then the response of the system is bounded and converges to a constant value. [Figure 3.62](#) illustrates this.

If  $\lambda$  is an imaginary number then the response of the system is oscillatory since

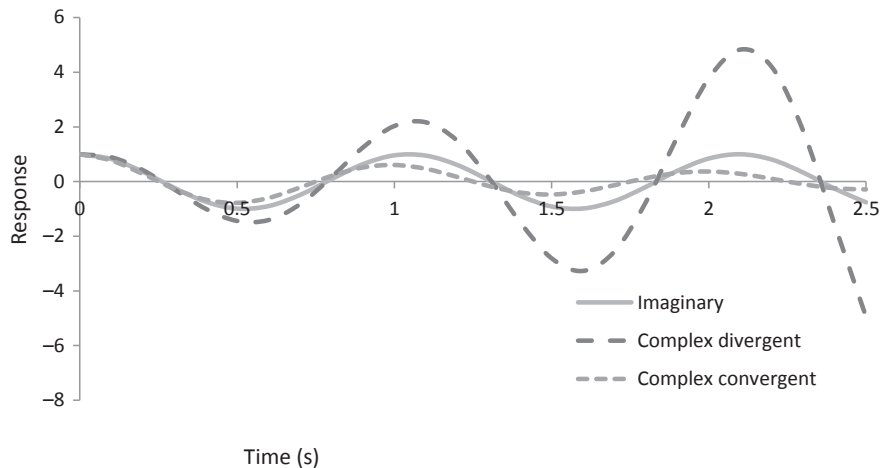
$$e^{-i\omega t} = \cos(\omega t) + i \sin(\omega t) \quad (3.127)$$

**FIGURE 3.62**

The expected form of time domain solution for real eigenvalues.

which is oscillatory in time. If  $\lambda$  is a complex number and the real part of  $\lambda$  is positive the response is oscillatory and unbounded. If  $\lambda$  is a complex number and the real part of  $\lambda$  is negative the response is oscillatory and convergent. Figure 3.63 illustrates this.

Systems with multiple DOF have multiple eigenvalues. Eigenvalues can tell us whether the system will converge or not without having to perform repeated calculations to determine the time domain response and are thus appealing in their own right, as well for the audit purposes described earlier. They are a logical first step to examining system behaviour.

**FIGURE 3.63**

The expected form of time domain solutions with complex roots.



Considering the general form of the solution, we can immediately write two derivative functions:

$$x(t) = Ae^{\lambda t} \quad (3.128)$$

$$\frac{dx}{dt} = \lambda Ae^{\lambda t} = \lambda x \quad (3.129)$$

$$\frac{d^2x}{dt^2} = \lambda^2 Ae^{\lambda t} = \lambda^2 x \quad (3.130)$$

To find eigensolutions, external inputs to the system are set to zero. Considering the system described in Eqn (3.116), there are no external inputs and so the equation can be used as is. Neglecting damping, the second derivative can be substituted using Eqn (3.130) above

$$-\lambda^2 mx + kx = 0 \quad (3.131)$$

If  $x$  is nonzero, which is the case for any nontrivial solutions, this can be seen to lead directly to

$$\lambda^2 = \frac{-k}{m} \quad (3.132)$$

$$\lambda = \sqrt{\frac{-k}{m}} = i\sqrt{\frac{k}{m}} \quad (3.133)$$

This result is intimately familiar to many engineering students — an oscillatory response with a frequency in radians/second given by the magnitude of  $\lambda$ .

We can consider other systems too, to illustrate the power of the eigensolution in granting insight to the behaviour. For example, consider a simple pendulum as shown in Figure 3.64.

The mass of the pendulum is considered concentrated at the tip. The weight of the pendulum,  $mg$ , acts vertically downward. The lateral offset between the weight and the reaction force gives a couple restoring the pendulum to the middle position. Using the small angle approximation, we can write:

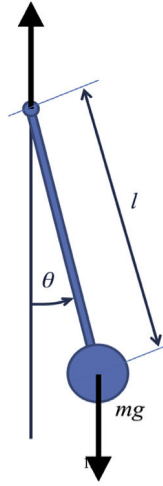
$$M_y = -mg \sin \theta \cdot l \approx -mg \theta \cdot l \quad (3.134)$$

Noting there are no moving frames of reference, we also can see by inspection

$$M_y = I_y \frac{d^2\theta}{dt^2} \quad (3.135)$$

Having presumed the pendulum is a particle at its tip, its inertia about the pivot is given by

$$I_y = ml^2 \quad (3.136)$$

**FIGURE 3.64**

Pendulum example.

Hence

$$-mg\theta \cdot l = \frac{d^2\theta}{dt^2} ml^2 \quad (3.137)$$

$$-g\theta = \frac{d^2\theta}{dt^2} l \quad (3.138)$$

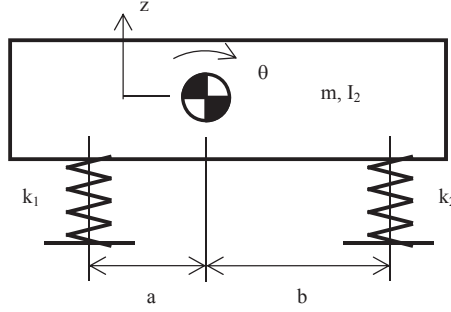
As before, we can presume a solution of the form  $Ae^{\lambda t}$ , which allows us via the same process as previously to arrive at

$$-g\theta = -\lambda^2 \theta l \quad (3.139)$$

$$\lambda = \sqrt{\frac{-g}{l}} = i\sqrt{\frac{g}{l}} \quad (3.140)$$

The significance of this is that it also predicts an oscillatory response – but note the oscillation occurs despite the absence of any elastic element in the system. Students who have learned the resonant frequency of a mass on a spring by rote have missed the generality of system behaviour and the possibility for responses other than resonance; it is the authors' experience that this can cause considerable difficulties when trying to understand the behaviour of a vehicle on pneumatic tyres, as described in Chapter 7.

Coupled systems may also be considered using this technique. Consider the vehicle in side view, suspended by elastic elements at each axle, as shown in [Figure 3.65](#).

**FIGURE 3.65**

Vehicle heave and pitch model.

The DOF are  $z$  and  $\theta$  – we might call them ‘heave’ and ‘pitch’, respectively. To derive the equations of motion, consider each DOF in turn:

1. Imagine lifting the block a unit displacement in  $z$ 
  - a. Both springs exert a negative heave force
  - b. Spring 1 exerts a negative pitch moment
  - c. Spring 2 exerts a *positive* pitch moment
2. Imagine rotating the block a unit displacement in  $\theta$ 
  - a. Spring 1 exerts a negative heave force
  - b. Spring 2 exerts a *positive* heave force
  - c. Both springs exert a negative pitch moment

By writing all the terms individually and arranging them together, this allows to write the equations for the system in matrix form:

$$\begin{bmatrix} \frac{-(k_1 + k_2)}{m} & \frac{-ak_1 + bk_2}{m} \\ \frac{-ak_1 + bk_2}{I} & \frac{-(a^2k_1 + b^2k_2)}{I} \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} \ddot{z} \\ \ddot{\theta} \end{bmatrix} \quad (3.141)$$

As before, we presume the solution is of the form  $Ae^{\lambda t}$  to give:

$$\begin{bmatrix} \frac{-(k_1 + k_2)}{m} & \frac{-ak_1 + bk_2}{m} \\ \frac{-ak_1 + bk_2}{I} & \frac{-(a^2k_1 + b^2k_2)}{I} \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} \quad (3.142)$$

$$\begin{bmatrix} \frac{-(k_1 + k_2)}{m} - \lambda^2 & \frac{-ak_1 + bk_2}{m} \\ \frac{-ak_1 + bk_2}{I} & \frac{-(a^2k_1 + b^2k_2)}{I} - \lambda^2 \end{bmatrix} \begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.143)$$

As before, setting the determinant of the matrix equal to zero will return the eigenvalues:

$$\begin{vmatrix} \frac{-(k_1 + k_2)}{m} - \lambda^2 & \frac{-ak_1 + bk_2}{m} \\ \frac{-ak_1 + bk_2}{I} & \frac{-(a^2k_1 + b^2k_2)}{I} - \lambda^2 \end{vmatrix} = 0 \quad (3.144)$$

We can write the individual components of the matrix in a more manageable form:

$$a_{11} = \frac{-(k_1 + k_2)}{m} \quad (3.145)$$

$$a_{21} = \frac{-ak_1 + bk_2}{m} \quad (3.146)$$

$$a_{12} = \frac{-ak_1 + bk_2}{I} \quad (3.147)$$

$$a_{22} = \frac{-(a^2k_1 + b^2k_2)}{I} \quad (3.148)$$

$$\begin{vmatrix} a_{11} - \lambda^2 & a_{12} \\ a_{21} & a_{22} - \lambda^2 \end{vmatrix} = 0 \quad (3.149)$$

This yields a quadratic equation in  $\lambda^2$ , which can be solved in the normal way:

$$(a_{11} - \lambda^2)(a_{22} - \lambda^2) - a_{12} \cdot a_{21} = 0 \quad (3.150)$$

$$\lambda^4 - \lambda^2(a_{11} + a_{22}) - a_{12} \cdot a_{21} + a_{11} \cdot a_{22} = 0 \quad (3.151)$$

$$\lambda^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (3.152)$$

$$A = 1 \quad (3.153)$$

$$B = -(a_{11} + a_{22}) \quad (3.154)$$

$$C = -a_{12} \cdot a_{21} \quad (3.155)$$

Even with this relatively simple system, the equations are starting to become unwieldy. The numerical solution of the pitch-heave problem is not unduly onerous, though and can be easily attempted in MS-Excel, Octave or Matlab:

```

m = 1200; % mass kg
I = 1541; % pitch inertia kg m^2
a = 1.22; % CG to front axle m
b = 1.37; % CG to rear axle m
kf = 43882; % front axle ride rate N/m
kr = 44583; % rear axle ride rate N/m

a11 = -(kf + kr)/m
a21 = (-a*kf + b*kr)/m
a12 = (-a*kf + b*kr)/I
a22 = -(a^2*kf + b^2*kr)/I

A = 1;
B = -(a11 + a22);
C = -a12*a21 + a11*a22;
Bsq4AC = B^2 - 4*A*C;
lambda1 = ((-B + Bsq4AC^0.5)/(2*A))^0.5;
lambda2 = ((-B - Bsq4AC^0.5)/(2*A))^0.5;

f1 = imag(lambda1)/(2*pi)
f2 = imag(lambda2)/(2*pi)

```

The following results were obtained by the authors using this code and serve as a reference check when implementing it:

```

a11 = -73.721
a21 = 6.2856
a12 = 4.8947
a22 = -96.685
Bsq4AC = 650.42
f1 = 1.3547
f2 = 1.5752

```

Considering the system described by Eqn (3.116) once more, we shall now include the damping, since this is a more general case. For this second order system it is not so obvious how to retrieve the eigenvalues. Recall that for reasons of numerical convenience, the second order system was represented as the combination of two first order systems by introducing the auxiliary system state variable,  $z$ . We can write

$$\dot{x} = z \quad (3.156)$$

$$\dot{z} = \left( \frac{-k}{m} \right) x + \left( \frac{-c}{m} \right) z \quad (3.157)$$

We may still presume solutions of the form  $Ae^{\lambda t}$ , as before. This allows us, as previously

$$\dot{x} = \lambda x \quad (3.158)$$

$$\dot{z} = \lambda z \quad (3.159)$$

Taking these expressions and rearranging things into matrix form allows us to write

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (3.160)$$

Note that nothing new has been added here — this is simply a restatement of the expressions in [Eqns \(3.156\) and \(3.157\)](#). This can be quite trivially rearranged to give

$$\begin{bmatrix} -\lambda & 1 \\ -k/m & (-c/m) - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad (3.161)$$

The eigenvalues for the system can be found by setting the determinant of the left hand matrix to zero. This process results in the search for the roots of a polynomial — in this case a quadratic — and for this reason, eigenvalues are sometimes referred to as ‘roots’ or ‘roots of the characteristic equation’. For a  $2 \times 2$  matrix such as this, it can be seen by inspection that

$$\begin{vmatrix} -\lambda & 1 \\ -k/m & (-c/m) - \lambda \end{vmatrix} = 0 \quad (3.162)$$

$$\Rightarrow \lambda = \frac{-c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}} \quad (3.163)$$

Note that the eigenvalues have the same real part with equal and opposite imaginary components — they are known as *complex conjugates*, mirrored about the real axis. Note also that if  $c = 0$  then the solution is identical to that of the undamped system.

This approach is very powerful for more complex systems. The matrix immediately to the right of the equal sign in [Eqn \(3.160\)](#) is the Jacobian as mentioned in [Section 3.3.3](#) and can be formed by taking partial derivatives:

$$\begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}(\dot{x}) & \frac{\partial}{\partial z}(\dot{x}) \\ \frac{\partial}{\partial x}(\dot{z}) & \frac{\partial}{\partial z}(\dot{z}) \end{bmatrix} \quad (3.164)$$

Thus, more generally we can find eigenvalues for systems with or without springs and with coupled, complex behaviour of  $n$  system states  $x_i = x_1 \dots x_n$  by solving for the determinant of the Jacobian being equal to zero:

$$\begin{vmatrix} \frac{\partial}{\partial x_1}(\dot{x}_1) - \lambda & \frac{\partial}{\partial x_2}(\dot{x}_1) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_1) \\ \frac{\partial}{\partial x_1}(\dot{x}_2) & \frac{\partial}{\partial x_2}(\dot{x}_2) - \lambda & & \frac{\partial}{\partial x_n}(\dot{x}_2) \\ \dots & & \dots & \dots \\ \frac{\partial}{\partial x_1}(\dot{x}_n) & \frac{\partial}{\partial x_2}(\dot{x}_n) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_n) - \lambda \end{vmatrix} = 0 \quad (3.165)$$

With complex systems, the solution of the determinant becomes nontrivial when the matrix gets large. While MS-Excel offers commands such as MDETERM, notionally allowing its built-in solver to hunt for eigenvalues, it typically only solves for a single eigenvalue, and even then will often fail if the matrix becomes poorly conditioned during its numerical hunt for the solution. The impeccable 'Numerical Recipes' (Press et al, 1992) has this to say:

*You have probably gathered by now that the solution of eigensystems is a fairly complicated business. It is. It is one of the few subjects covered in this book for which we do not recommend that you avoid canned routines.*

The reader who is determined to assemble their own solution to the problem is urged to consult *Recipes*, where the routines provided are entirely useful.

### 3.5 Systems of units

As with all engineering analysis it is important that consistent units are used throughout any calculation or simulation exercise. For static analysis the choice of a system of units can be quite forgiving as long as consistency is observed. For dynamic analysis more care is required. At a basic level if we consider the application of Newton's Second Law to a body  $n$  with mass  $m_n$  and acceleration  $\{A_{Gn}\}_1$  at the mass centre  $G_n$  we have with the vector convention used here

$$\sum \{F_n\}_1 = m_n \{A_{Gn}\}_1 \quad (3.166)$$

It is important to note that Eqn (3.166) is only valid for certain systems of units. For a metric system of SI units (force = N, mass = kg, acceleration = m/s<sup>2</sup>) Eqn (3.86) can be used as it stands. The system of units used mainly in this text, and popular in Europe, is to use millimetres as the unit of choice for length. Clearly Eqn (3.166) would produce incorrect forces unless the right hand side of Eqn (3.166) was divided by a constant, in this case 1000, to ensure consistency.

Early users of general-purpose programs needed to define such a constant when defining gravitational forces, even for dynamic models operating in a zero gravity

<b>Table 3.7</b> Units Consistency							
Measurement	SI	Metric	Metric	Metric	FPS	FPS	IPS
Length	m	mm	cm	m	ft	ft	in
Velocity	m/s	mm/s	cm/s	m/s	ft/s	ft/s	in/s
Acceleration	m/s <sup>2</sup>	mm/s <sup>2</sup>	cm/s <sup>2</sup>	m/s <sup>2</sup>	ft/s <sup>2</sup>	ft/s <sup>2</sup>	in/s <sup>2</sup>
Mass	kg	kg	kg	kg	slug	lbm	lbm
Force	N	N	kgf	kgf	lbf	lbf	lbf
Inertia	kgm <sup>2</sup>	kgmm <sup>2</sup>	kgcm <sup>2</sup>	kgm <sup>2</sup>	slug ft <sup>2</sup>	lbm ft <sup>2</sup>	lbm in <sup>2</sup>
Gravity	9.81 m/s <sup>2</sup>	9807 mm/s <sup>2</sup>	981 cm/s <sup>2</sup>	9.81 m/s <sup>2</sup>	32.2 ft/s <sup>2</sup>	32.2 ft/s <sup>2</sup>	386.1 in/s <sup>2</sup>
UCF	1.0	1000	981	9.81	1.0	32.2	386.1

*UCF, units consistency factor; FPS, foot-pound-second system; IPS, inch-pound-second system.*

environment. As such users were reminded of some of the basic fundamentals of dynamics when using such software. More modern programs require the users to define only a set of units and apply the correction internally. Table 3.7 is provided to show the required corrections to various systems of units. Although for most users of MBS this will be for reference only, advanced users developing subroutines that link with MBS may still be required to implement a constant to ensure consistency. An early term used to define the constant was the ‘gravitational constant’ although a more recent and applicable definition is the ‘units consistency factor’ (UCF) where for any system of units the following equation is valid for the given UCF in Table 3.7.

$$\sum \{F_n\}_1 = \frac{m_n \{A_{Gn}\}_1}{UCF} \quad (3.167)$$

### 3.6 Further comments on pre- and postprocessing

Several general-purpose programs have rather specialist layers over the top of them to facilitate modelling vehicles in particular. In this sense they become a single-purpose general-purpose program, which appears a very complex way of going about things. Nevertheless, they allow the detailed complexity of a general-purpose program and a highly capable numerical engine to be brought to bear on the vehicle analysis problem, which more specialist programs are often unable to achieve — ironically because they lack the generality of the general-purpose programs.

An early customisation of MSC ADAMS for automotive applications was the ADAMS/Vehicle program originally developed as a commercially available product, which has been used by engineers from the Newman/Hass Indy Car racing



team (Trungle, 1991). The program allowed a suspension model to be created, carry out an analysis and postprocess the results without specialist knowledge of MSC ADAMS. The program could also be used to automatically generate a full vehicle model, hence the title. The preprocessor included a number of established suspension configurations where the data were input via screen templates using familiar suspension terminology.

Simpack provides a detailed vehicle ‘wizard’ with a large amount of detail in the models and templates available to allow more or less any conceivable level of complexity to be approached from something significantly less intimidating than a clean sheet of paper. Other packages, such as Dymola, also include vehicle libraries for the same reason.

ADAMS/Car was developed working with a consortium of major vehicle manufacturers including Audi, BMW, Renault and Volvo. The vehicle manufacturers’ involvement included developing the specification for the functionality of the software. This included, for example, determining the suspension systems that would be included, the manoeuvres to be simulated and the outputs and their manner of presentation. For this reason, the ADAMS/Car library content and workflow seems to be regarded as something of a benchmark for other software. An example of the ADAMS/Car graphical interface is shown in [Figure 3.66](#).

In practice the different software systems work in different ways but there are two basic categories of user:

1. The Expert User who has access to the fundamental software modelling elements. As such they are able, for example, to create or modify system model templates of suspensions and steering systems. They would also be able to modify or create test procedures to be simulated.
2. The Standard User who is not necessarily a multibody system expert but is able to use the existing templates to enter data and create models using familiar terminology and to produce reliable results according to a robust protocol. They include design, test and development engineers in addition to analysts.

In parallel to ADAMS/Car the ADAMS/Chassis system is also used for vehicle work and offers similar capability. The ADAMS/Chassis program was originally developed in house by Ford in the late 1980s. The program was originally called ADAMS/Pre and as the name suggests the early implementation was a preprocessor that automatically formatted ADAMS data sets. In its current form it has additional capability to run customised simulations and has its own postprocessor. Ford allowed the program to be taken on and developed by another company before the product was acquired by the developers of MSC ADAMS. Due to its origin ADAMS/Chassis appears on the surface unlike any of the other customised MSC ADAMS programs. The program uses a graphical interface based more on data forms to enter the data. An example is shown in [Figure 3.67](#).

Any software requires a high level of programming skill on the part of the Expert User who is going to customise or develop the way models are generated, simulations are run or results are plotted and reported. In addition to vehicle dynamics

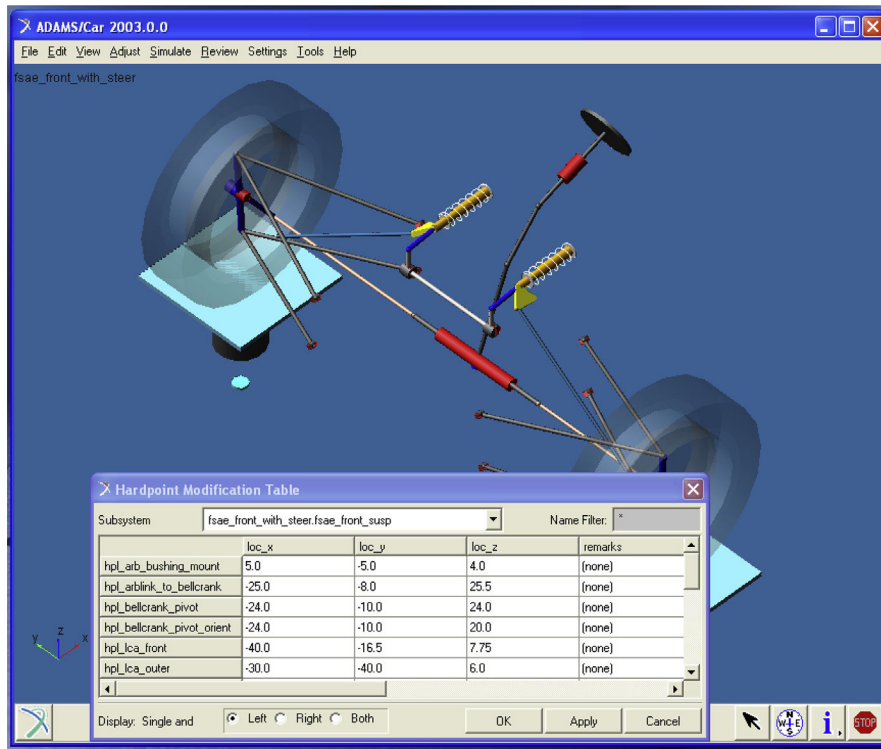


FIGURE 3.66

ADAMS/Car graphical user interface.

(Provided courtesy of MSC.Software.)

knowledge the expert would need a working knowledge of the core solver language, C++ and FORTRAN in order to perform any meaningful customisation. For this reason, such experts are few and far between and tend to be well looked after by organisations that employ them.

All the integrated 'single-purpose general-purpose' programs include a substantial list of preprogrammed vehicle manoeuvres and subsystem test procedures, with both inputs and outputs defined.

It is perhaps in the calculation of outputs such as these that the customised software offers a significant benefit to the analyst but also a risk to the culture of the organisation. With a general-purpose program, every model starts with the proverbial 'blank sheet'. Everything is uniformly difficult, even the easy things. This generates a steep learning curve, which often surprises the authors at the rapidity of its scaling by students — so perhaps it is not as steep as it looks. Once it has been scaled, nothing looks particularly difficult any more and everything becomes possible.

In contrast, with a single-purpose general-purpose program even quite difficult things are very easy *providing someone else has set up the model and procedure.*

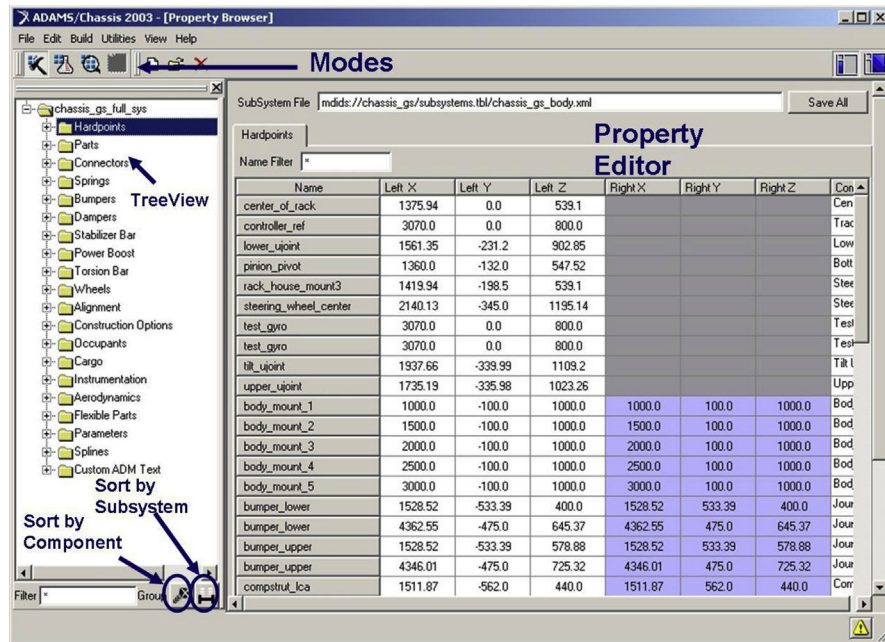


FIGURE 3.67

ADAMS/Chassis user interface.

(Provided courtesy of MSC Software.)

This leads to a somewhat institutionalised frame of mind in which anything that is not already defined looks impossible and it becomes much more difficult to implement changes in process or procedure; the organisation becomes a slave to what is possible in the currently released version of the software and stops pushing the boundaries. More importantly, things get done that *can be done* instead of that *need to be done*.

It is with this note of caution that the student is encouraged to set their aspiration towards becoming the Expert User, with the labour-saving automation provided by the software but the free-ranging ability to tackle any problem that comes from the clean sheet mentality.

# Modelling and Analysis of Suspension Systems

# 4

*Any suspension will work, if you don't let it*

**Colin Chapman**

In this chapter, the basic role of the suspension system is discussed from a functional, analytical perspective. Several types of suspension are introduced and methods of their analysis are described in a typical commercial multibody systems (MBS) analysis package. Immediately familiar to students is the system shown in [Figure 4.1](#), this being a suspension system referred to in Europe as a double wish-bone system. In the USA the practice is to call the same system a short-long arm suspension system. We will be considering the modelling and analysis of suspension systems as separate units, a practice sometimes referred to as quarter vehicle modelling. The discussion in this chapter will be restricted to passive systems and will not address active suspension systems.

It is also intended in this chapter to demonstrate the manner in which the vector based methods described in Chapter 2 can be applied to carry out three-dimensional kinematic, static and dynamic force analyses of a double wish-bone suspension system. The model geometry has also been analysed using an equivalent MBS model allowing a comparison of results from theory and MBS and providing readers with an insight into the computational processes involved.

The traditional treatment of suspension systems, dealing usually with a projection in one plane, is used here only where useful to explain standard terms that the student or developing practitioner should be aware of.

The concept of a roll centre is one that has been used extensively by vehicle designers in order to relate suspension layout to vehicle handling performance, particularly when considering understeer or oversteer. The roll centre is introduced in this chapter using a classical treatment together with a description of the typical process used in MBS for computation. Some of the mystery surrounding it is also debunked.

Before progressing to a detailed treatment of the modelling and analysis issues it is important to fully describe the function of a suspension system and the needs that must be addressed. The initial sections of this chapter intend to identify these needs in a systematic manner providing a checklist against which suspension performance may be evaluated during the vehicle design process. The follow-on treatment of modelling and simulation can then be related back to the various needs now described.

**FIGURE 4.1**

Aston Martin Vanquish double wishbone front suspension system.

---

## 4.1 The need for suspension

The term suspension seems at first hearing an odd one when considering the function in a modern vehicle as the vehicle body appears to sit on rather than be suspended from the mechanism. With growing commercial activities, coach bodies hung from so-called ‘C-Springs’ allowed faster travel over the poor roads of the late eighteenth century and so gave an advantage. Despite the change from hanging to standing, the term ‘suspension’ has stuck in the English-speaking world.

In its simplest form, a modern road vehicle suspension may be thought of as a linkage to allow the wheel to move relative to the body and some elastic element to support loads while allowing that motion. As suspensions become more complex, the need for well-controlled damping forces and multidirectional compliance emerges. MBS analysis can help quantify an existing design in terms of these parameters or help to synthesise a new design from a set of target parameters.

Most practical vehicles have some form of suspension, particularly when there are four or more wheels. The suspension system addresses two basic needs:

- Reduction of vertical wheel load variations
- Isolation of road inputs from body

However, the introduction of a suspension system introduces some tasks of its own; each additional interface and component brings with it some structural compliance. This may lead to a delay in transmitting loads to the body, with a possible degradation to the vehicle-handling task. It also offers the opportunity (or risk) of modifying the

**Table 4.1** Suspension Design Process Activities

	Wheel Load Variation	Body Isolation	Handling Load Control	Compliant Wheel Plane Control	Kinematic Wheel Plane Control	Component Loading Environment
Investigate design strategies	✓	✓	✓	✓	✓	✓
Set design targets	✓	✓	✓	✓	✓	✓
Verify proposed designs	✓	✓	✓	✓	✓	✓

way the wheel is presented to the road. The new components must themselves be capable of surviving the design life when the vehicle is used as intended. Therefore, as a consequence of its existence the suspension generates four more needs:

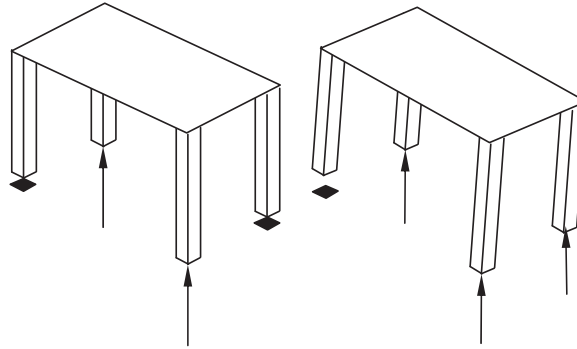
- Control of transmission of handling loads to body
- Control of wheel plane geometry due to compliant effects
- Control of wheel plane geometry due to kinematic effects
- Comprehension of component load environment

Modern MBS analysis tools offer the opportunity to evaluate each of the six needs listed, either to investigate design strategies for whole systems, set design targets for components or to verify the performance of proposed designs. Any rigorous design process must therefore be able to quantify matters in these 18 different ways as shown in [Table 4.1](#).

It is easy to focus on one small subset of the task above but to do so fails to deliver the full range of benefits possible from MBS analysis software. The software is generally expensive and the skilled personnel to operate it are hard to come by and so such a focus represents a lost opportunity for the organisation.

#### 4.1.1 Wheel load variation

Since a vehicle with four wheels is statically indeterminate with respect to the calculation of reaction forces, only with some elastic behaviour ('compliance') in the suspension can a determinate solution be formulated unless some presumption of symmetry is made; otherwise, the 'wobbly table' case exists where there are at least three possible load distributions ([Figure 4.2](#)). This leads to abrupt fluctuations in load of a type barely tolerable in restaurants and unconscionable for road vehicle behaviour. For the problem shown we have one system and three possible reaction load solutions. These include being balanced on the two longest legs (left in [Figure 4.2](#)) — unlikely in practice but a solution in theory — or being balanced on

**FIGURE 4.2**

A classic case of static indeterminacy.

three legs (right in Figure 4.2) in one of two stable states. This system represents a classic case of static indeterminacy.

Consideration of a shopping trolley, a notionally rigid vehicle with four wheels that displays many irksome traits, shows that even on the most well-prepared supermarket floor the absence of suspension causes problems with load distribution between the four wheels due to inconsistent manufacture. The least loaded wheel is prone to shimmy and the uneven wheel-load distribution can emphasise frictional asymmetry in the castors with even the most fastidious grocery-packing practices, giving the familiar ‘mind of its own’ sensation.

Less infuriating ‘rigid’ vehicles include some agricultural equipment that uses compliant behaviour in the tyre sidewalls to accommodate uneven terrain and racing karts, which use a significant amount of compliance in the frame structure of the vehicle to modify vertical wheel loads. An example of this is shown in Figure 4.3 where the inclusion of the racing kart frame flexibility clearly influences the yaw rate response for a simulated manoeuvre.

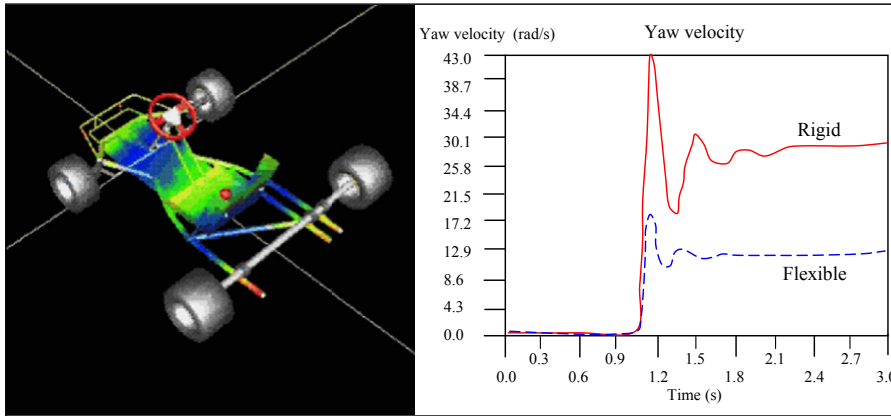
More formally, the problem may be posed as a rigid platform, Body 2, of total weight  $m_2\{g\}_1$ , with reactions acting at wheel locations A, B, C and D as shown in Figure 4.4.

Consider first the equations that would be needed for equilibrium of forces. Note that for rigour we are continuing with the vector notation established in Chapter 2.

$$\sum \{F_2\}_1 = \{0\}_1 \quad (4.1)$$

$$\{F_A\}_1 + \{F_B\}_1 + \{F_C\}_1 + \{F_D\}_1 - m_2\{g\}_1 = \{0\}_1 \quad (4.2)$$

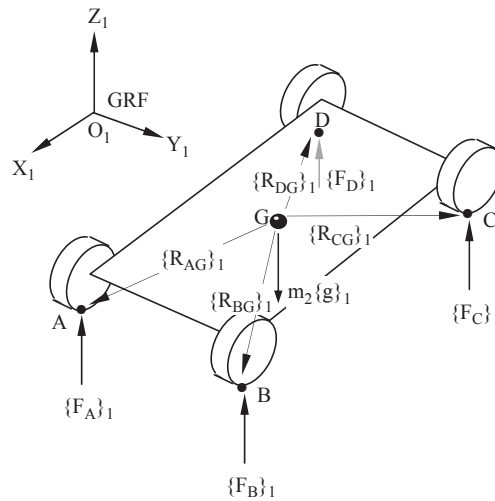
$$\begin{bmatrix} 0 \\ 0 \\ F_{Az} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_{Bz} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_{Cz} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F_{Dz} \end{bmatrix} - m_2 \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (4.3)$$



**FIGURE 4.3**

The influence of racing kart frame flexibility.

*Courtesy of MSC software.*



**FIGURE 4.4**

The wheel load reaction problem. GRF, ground reference frame.

From Eqn (4.3) we now get the expression in Eqn (4.4) that we would normally get quickly by inspection.

$$F_{Az} + F_{Bz} + F_{Cz} + F_{Dz} - m_2 g = 0 \quad (4.4)$$

If we continue now to take moments about the mass centre G of Body 2 we have

$$\sum \{M_G\}_1 = \{0\}_1 \quad (4.5)$$



$$\{R_{AG}\}_1 \{F_A\}_1 + \{R_{BG}\}_1 \{F_B\}_1 + \{R_{CG}\}_1 \{F_C\}_1 + \{R_{DG}\}_1 \{F_D\}_1 = \{0\}_1 \quad (4.6)$$

If we expand the vector moment terms for the force acting at A only we get

$$\begin{bmatrix} 0 & -AG_z & AG_y \\ AG_z & 0 & -AG_x \\ -AG_y & AG_x & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ F_{Az} \end{bmatrix} = \begin{bmatrix} AG_y F_{Az} \\ -AG_x F_{Az} \\ 0 \end{bmatrix} \quad (4.7)$$

From Eqn (4.7) it is clear that we can now write Eqn (4.6) as

$$AG_y F_{Az} + BG_y F_{Bz} + CG_y F_{Cz} + DG_y F_{Dz} = 0 \quad (4.8)$$

$$-AG_x F_{Az} - BG_x F_{Bz} - CG_x F_{Cz} - DG_x F_{Dz} = 0 \quad (4.9)$$

It can now be seen that we have the classical case of an indeterminate problem where there are insufficient equations available to solve for the unknowns. The four unknowns here are  $F_{Az}$ ,  $F_{Bz}$ ,  $F_{Cz}$  and  $F_{Dz}$ . There are however only the three Eqns (4.4), (4.8) and (4.9) available to effect a solution. Trying to arrange the equations in matrix form for solution demonstrates the futility of the problem.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ AG_y & BG_y & CG_y & DG_y \\ -AG_x & -BG_x & -CG_x & -DG_x \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} F_{Az} \\ F_{Bz} \\ F_{Cz} \\ F_{Dz} \end{bmatrix} = \begin{bmatrix} mg \\ 0 \\ 0 \\ ? \end{bmatrix} \quad (4.10)$$

If a modified formulation is adopted, the problem can be posed more completely. If the rigid platform is presumed to be sprung in the manner of a normal road vehicle with an effective wheel rate,  $k_A$ ,  $k_B$ ,  $k_C$  and  $k_D$ , at A, B, C and D then a new formulation is possible. Writing  $A_{2z}$ ,  $B_{2z}$ ,  $C_{2z}$  and  $D_{2z}$  for the height of corners A, B, C and D on Body 2 and  $A_{1z}$ ,  $B_{1z}$ ,  $C_{1z}$  and  $D_{1z}$  for the ground height corners at A, B, C and D we can then define a preload  $F_{pA}$ ,  $F_{pB}$ ,  $F_{pC}$  and  $F_{pD}$  on each spring such that if the z-coordinates of Body 2 at the corner are equal to the z-coordinates of the ground, Body 1, at each corner then the spring load is equal to the preload. This leads to the following equations for the spring force at each corner:

$$F_{Az} = k_A (A_{2z} - A_{1z}) + F_{pA} \quad (4.11)$$

$$F_{Bz} = k_B (B_{2z} - B_{1z}) + F_{pB} \quad (4.12)$$

$$F_{Cz} = k_C (C_{2z} - C_{1z}) + F_{pC} \quad (4.13)$$

$$F_{Dz} = k_D (D_{2z} - D_{1z}) + F_{pD} \quad (4.14)$$

At first sight it would appear this is simply a more elaborate formulation of the previous problem; there are still four unknown quantities (the corner heights of Body 2) and three Eqns (4.4), (4.8) and (4.9). However, consideration of the rigid platform yields a fourth relationship

$$D_{2z} = C_{2z} + (B_{2z} - A_{2z}) \quad (4.15)$$

This leads to, after some manipulation of Eqn (4.10), substituting in the spring Eqns (4.11)–(4.13) for  $F_{Az}$ ,  $F_{Bz}$ ,  $F_{Cz}$  and relying on Eqn (4.15) to solve for  $F_{Dz}$ , the equations in Eqn (4.16)

$$\begin{bmatrix} k_A - k_D & k_B + k_D & k_C + k_D \\ k_A A G_y - k_D D G_y & k_B B G_y + k_D D G_y & k_C C G_y + k_D D G_y \\ k_A A G_x - k_D D G_x & k_B B G_x + k_D D G_x & k_C C G_x + k_D D G_x \end{bmatrix} \begin{bmatrix} A_{2z} \\ B_{2z} \\ C_{2z} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \quad (4.16)$$

where

$$\lambda_1 = m_2 g + k_A A_{1z} - F_{pA} + k_B B_{1z} - F_{pB} + k_C C_{1z} - F_{pC} + k_D D_{1z} - F_{pD} \quad (4.17)$$

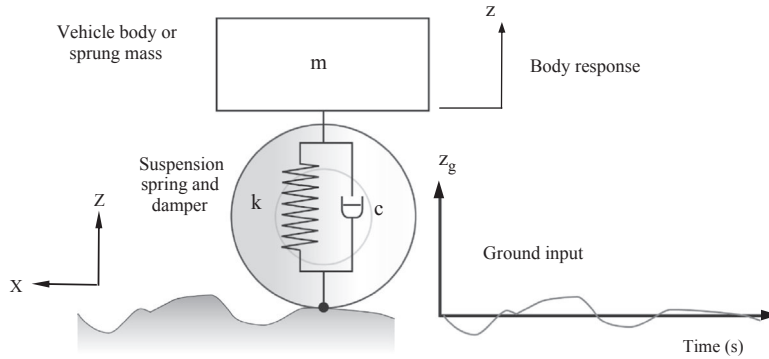
$$\lambda_2 = (k_A A_{1z} - F_{pA}) A G_y + (k_B B_{1z} - F_{pB}) B G_y + (k_C C_{1z} - F_{pC}) C G_y + (k_D D_{1z} - F_{pD}) D G_y \quad (4.18)$$

$$\lambda_3 = (k_A A_{1z} - F_{pA}) A G_x + (k_B B_{1z} - F_{pB}) B G_x + (k_C C_{1z} - F_{pC}) C G_x + (k_D D_{1z} - F_{pD}) D G_x \quad (4.19)$$

The equations in Eqn (4.16) may be solved by Gaussian elimination and substitution. It can be seen that preloads ( $F_{pA}$ ,  $F_{pB}$ ,  $F_{pC}$  and  $F_{pD}$ ), the ground heights ( $A_{1z}$ ,  $B_{1z}$ ,  $C_{1z}$  and  $D_{1z}$ ) and the corner stiffness ( $k_A$ ,  $k_B$ ,  $k_C$  and  $k_D$ ) and wheel locations are the inputs to the calculations. The unknowns that are found from Eqn (4.16) are the body heights at three corners ( $A_{2z}$ ,  $B_{2z}$  and  $C_{2z}$ ). These are used to find  $D_{2z}$  from Eqn (4.15) leading to the force solution in each corner from Eqns (4.11)–(4.14). Thus, the presence of an elastic suspension for the road wheels allows a solution to the ‘wobbly table’ problem for vehicles with four or more wheels even when traversing terrain that is not smooth.

### 4.1.2 Body isolation

The interaction of a single wheel with terrain of varying height is frequently idealised as shown in Figure 4.5. This is a so-called ‘quarter vehicle’ model and is widely used to illustrate the behaviour of suspension systems. It may be thought of as a stationary system under which a ground profile passes to give a time-varying ground input,  $z_g$ . Note that at this point we are assuming the tyre to be rigid. Whether classical models or ‘literal’ MBS models are used, the methods used to comprehend body isolation are the same.

**FIGURE 4.5**

A classical quarter vehicle ride model.

Classically, the system may be formulated as a single second-order differential equation

$$m\ddot{z} + c(\dot{z} - \dot{z}_g) + k_s(z - z_g) = 0 \quad (4.20)$$

In order to more fully understand the isolation behaviour of the suspension, a fully developed (steady state) harmonic solution of the equation above may be presumed to apply. 'Harmonic' simply means that both input and output may be described with a sine function, which may be conveniently expressed in the form

$$z = A.e^{(i\omega t + \phi)} \quad (4.21)$$

where  $i$  is the imaginary square root of  $-1$ . Thus for the two derivatives of  $z$  it may be written

$$\dot{z} = i\omega A e^{(i\omega t + \phi)} = i\omega z \quad (4.22)$$

and

$$\ddot{z} = -\omega^2 A e^{(i\omega t + \phi)} = -\omega^2 z \quad (4.23)$$

This is a more problem-specific version of the general Jacobian formulation advanced in Chapter 3. The reason for the problem-specific formulation is to illustrate isolation behaviour specifically. The original Eqn (4.20) can then be written

$$-m\omega^2 z + i\omega c(z - z_g) + k(z - z_g) = 0 \quad (4.24)$$

Rearranging this in the form of a transfer function gives

$$H(\omega) = \frac{\text{output}}{\text{input}}(\omega) = \frac{z}{z_g} = \frac{(k) + i(c\omega)}{(k - m\omega^2) + i(c\omega)} \quad (4.25)$$

This expression relates the amplitude and phase of ground movements to the amplitude and phase of the body movements and is commonly reproduced in many vibration theory books and courses.

Considering the transfer function, several behaviours may be observed. At frequencies of zero and close to zero, the transfer function is unity. This may reasonably be expected; if the ground is moved very slowly the entire system translates with it, substantially undistorted. The behaviour of the system may be described as 'static' where its position is governed only by the preload in the spring and the mass carried by the spring; dynamic effects are absent.

At one particular frequency where  $\omega = \sqrt{k/m}$ , the real part of the denominator becomes zero and the transfer function is given by

$$H(\omega) = \frac{(k) + i(c\omega)}{i(c\omega)} = \frac{-ik}{i(c\omega)} + 1 \quad (4.26)$$

At this frequency, the behaviour of the system is called 'resonant'. Examination of the transfer function shows it is at its maximum value. Since  $k$ ,  $c$  and  $\omega$  are all positive real numbers, the transfer function shows that ground inputs are amplified at this frequency. If  $c$  is zero, i.e. there is no damping present in the system, then the transfer function is infinite. If  $c$  is very large, the transfer function has an amplitude of unity. For typical values of  $c$ , the amplitude of the transfer function is greater than unity.

At substantially higher frequencies where  $\omega \gg \sqrt{k/m}$ , the transfer function is dominated by the term  $-m\omega^2$  in the denominator and tends towards

$$H(\omega) = \frac{1}{-m\omega^2} \quad (4.27)$$

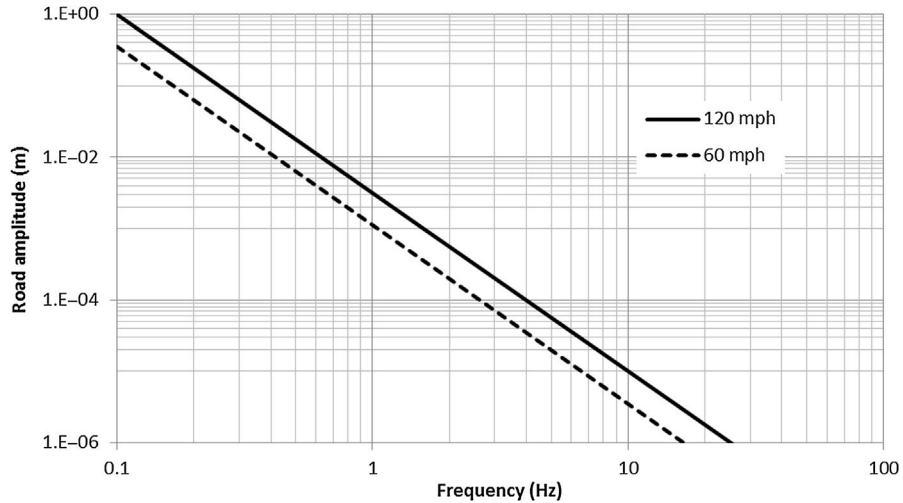
At these higher frequencies, the amplitude of the transfer function falls away rapidly.

It may therefore be noted that if it is desired to have the body isolated from the road inputs, the system must operate in the latter region where  $\omega \gg \sqrt{k/m}$ . In fact this is a general conclusion for any dynamic system; for isolation to occur it must operate above its resonant frequency. Given that the mass of the vehicle body is a function of things beyond the suspension designer's control, it is generally true that the only variables available to the designer are spring and damper calibration.

The preceding analysis would suggest that the softest springs possible, to position the resonant frequency as low as possible, are preferred. In a purely technical sense this is indeed true; however, our vehicles are to have as their primary purpose the transport of people over roads, which constrains our choices further.

Road surfaces are, to a first approximation, a random process passing under the car. They can be described by the expression

$$z_g(\omega) = \frac{K(2\pi V)^{R-1}}{\omega^R} \quad (4.28)$$

**FIGURE 4.6**

Road surface frequency content.

where  $K$  and  $R$  are constants. Figure 4.6 illustrates a road amplitude spectrum with  $K$  and  $R$  set to  $10^{-5}$  and 2.5 respectively. Values of both  $K$  and  $R$  vary with the nature of the road being studied.  $K$  is a straightforward amplitude measure. For novel applications or for previously unknown markets, some measurements of road profiles is desirable; for the developed Western world the values given are generally appropriate. For military applications, a value of 1 for  $R$  may be appropriate. The very smoothest roads are unlikely to have  $R$  higher than 4.

It can be seen that the spectral content as seen by the car varies with vehicle speed. Gillespie (1992) gives a ‘rational’ argument to suggest that spectral content rises with speed squared. However, Hales (1989) gives a different view, reproduced in essence here. The road surfaces is a process that may be approximated by a straight line on a log–log plot and represented by

$$\log[z_g(n)] = \log[K] - R \cdot \log[n] \quad (4.29)$$

implying that

$$z_g(n) = \frac{K}{n^R} \quad (4.30)$$

This is a spatial representation of the road profile, with  $n$  defined in cycles per metre. It appears under the car at frequency of  $V \cdot n$  Hz, where  $V$  is the forward speed of the car, thus

$$V \cdot n = \frac{\omega}{2\pi} \Rightarrow n = \frac{\omega}{2\pi V} \quad (4.31)$$

We can write

$$z_g(n) = \frac{g(n)}{n} = g(n) \cdot \frac{2\pi V}{\omega} = z_g(\omega) \cdot 2\pi V \quad (4.32)$$

if we acknowledge that  $g(n) = g(\omega)$  – i.e. that the process is unchanged whatever we choose to express it as a function of. Hence:

$$z_g(\omega) = \frac{1}{2\pi V} \cdot \frac{K}{n^R} = \frac{1}{2\pi V} \cdot \frac{K}{(\omega/2\pi V)^R} = \frac{K(2\pi V)^{R-1}}{\omega^R} \quad (4.33)$$

For the purposes of positioning the ride mode, then the frequency domain relationship between ground input,  $z_g$  and displacement of the vehicle body,  $z$ , is given by

$$\frac{z(\omega)}{z_g(\omega)} = \frac{k^2 + \omega^2 c^2}{(k - \omega^2 m)^2 + \omega^2 c^2} \quad (4.34)$$

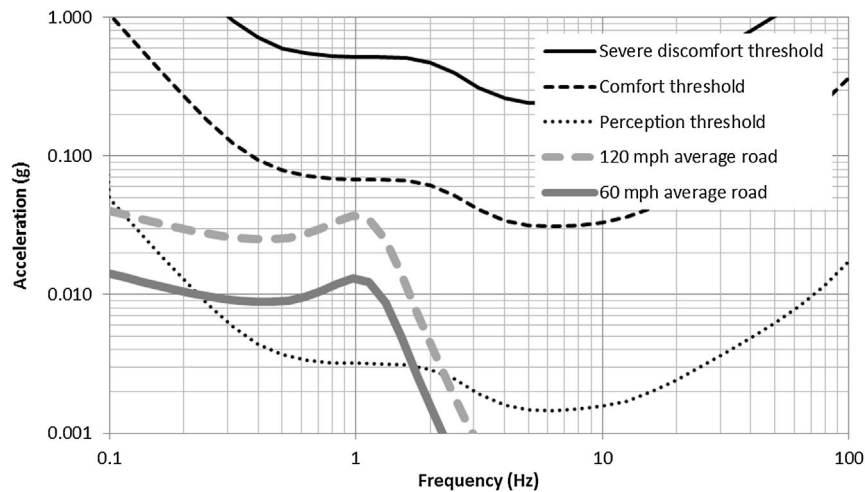
The acceleration environment is the prime concern in ride studies. Assuming harmonic solutions (i.e. made up of sine waves), we may write

$$\ddot{z}(\omega) = \frac{\ddot{z}(\omega)}{\ddot{z}_g(\omega)} \cdot \ddot{z}_g(\omega) = -\omega^2 \frac{\ddot{z}(\omega)}{\ddot{z}_g(\omega)} \cdot \ddot{z}_g(\omega) \quad (4.35)$$

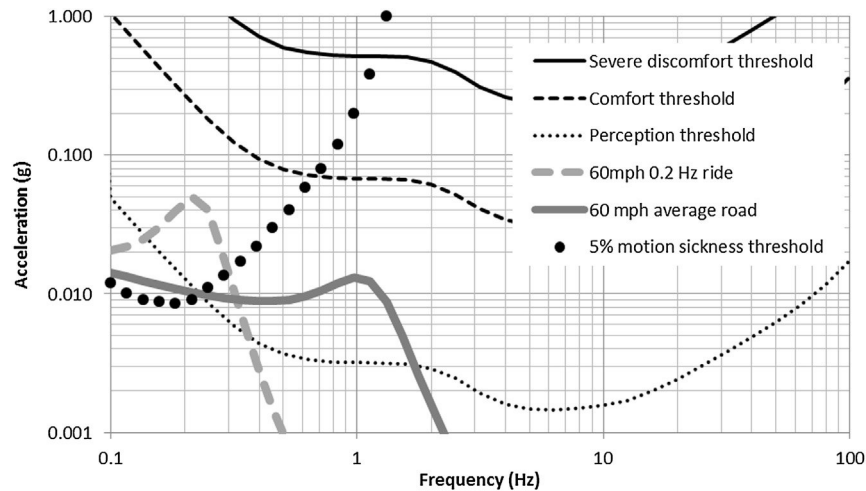
Thus we have all that is needed to calculate a harmonic acceleration response of the car over a typical road profile. If the acceleration response is compared to the curves in ISO2631-1:1997 (1997), then some direct measure of ride comfort can be made.

Figure 4.7 shows a prediction of the same vehicle travelling at 60 and 120 mph on the same road, with threshold figures based on 1 h exposure to the vibration environment. Considering Figure 4.7, it may be supposed that a better riding vehicle could be made by positioning the primary ride resonance at a lower frequency in order to better match the shape of the threshold curves and thus improve the aggregate ride over the frequency range of interest. Notwithstanding the difficulty in general that lower ride frequencies mean larger suspension motions, if we presume these difficulties can be overcome, another difficulty remains.

Figure 4.8 shows the acceleration response for a 0.2 Hz primary ride car, which has a much lower exceedance of the perception threshold. However, superimposed on the graph is a motion sickness threshold for 5% of the population at 1 h exposure. It can be seen that in the region substantially below resonance for the 1 Hz car, the motion sickness threshold exceedance is entirely in the region in which the vehicle does not amplify road inputs; in other words, motion sickness is induced by irregularities in the road and not exacerbated in the vehicle. For the 0.2 Hz car, however, it can be seen that the vehicle resonance on its suspension contributes to a significant exceedance of the threshold motion sickness – the suspension will make people ill. Considering Figure 4.8 in some detail, it would seem that a practical limiting

**FIGURE 4.7**

A prediction of the harmonic response of a vehicle body in comparison with the acceleration curves laid down in ISO2631-1:1997.

**FIGURE 4.8**

A prediction of the harmonic response of a vehicle body in comparison with the acceleration curves laid down in ISO2631-1:1997 with the inclusion of motion sickness threshold data, comparing a 0.2 Hz vehicle with a 1 Hz vehicle.

suspension frequency is where the 5% motion sickness line crosses the comfort threshold. This is around 0.65 Hz.

Given that the human frame is primarily engineered for walking, a more typical value for the resonant frequency of the body mass on the road springs is just over

1 Hz, this being the frequency at which we walk for the majority of the time. This is one of several reasons why babies may be readily nursed to sleep in cars; the motion of the vehicle gives an acceleration environment not unlike being held in the arms of a walking adult.

For motorsport or military vehicles, it may be acceptable to use the severe discomfort threshold; in this case the practical lower limit for ride frequencies is around 1.25 Hz.

It should be noted that the threshold values for accelerations drop off significantly above 2 Hz. For this reason, motor vehicles rarely have primary ride modes much above 2.5 Hz, even quite aggressively suspended ones. There is thus a 'window' in which we position the primary ride behaviour of our vehicles to make them compatible with the operators. For autonomous vehicles that never carry humans, these restrictions are relaxed and the suspension primary ride frequency can be chosen on the basis of some other functional aspect.

Our real vehicle systems are rarely so straightforward to address as this, however. Real vehicle suspensions have vertical and longitudinal compliance behaviour. Suspension components rotate as well as translate and the sprung mass has rotational as well as translational freedoms. Modern MBS analysis software allows us to describe the individual components of the system and will automatically calculate the contribution of, for example, sprung mass pitch inertia to the acceleration solution for the body. The software also allows the linearisation of the system about an operating point and the calculation of modes of vibration of the system. For example, the longitudinal compliance typically snubs out under hard braking and the fore-aft isolation suffers; proprietary software will allow the calculation of fore-aft resonant frequencies under cruise and braking conditions and their comparison by an intelligent user will explain (and perhaps generate solutions for) harsh behaviour over small obstacles while braking.

At no stage with added complexity do the basic rules for dynamic systems break down; there remains a subresonant, stiffness-dominated regime, a resonant, damping-dominated regime and a post-resonant, mass-dominated regime. The software user who keeps a grasp of these basic concepts, while allowing the software to undertake the task of assembling the equations of motion and solving them, is productive within an engineering organisation.

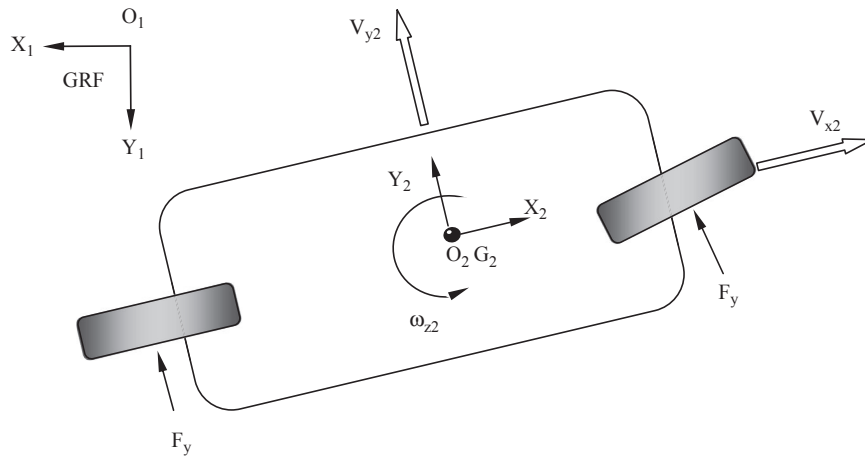
### 4.1.3 Handling load control

The simplest possible representation for a vehicle manoeuvring in the ground plane is shown in [Figure 4.9](#). If we break from the full 3D vector notation the following pair of differential equations describes it fully:

$$\sum M_{2z} = I_{zz} \dot{\omega}_{2z} \quad (4.36)$$

$$\sum F_{2y} = m_2 (\dot{V}_{2y} + V_{2x} \omega_{2z}) \quad (4.37)$$



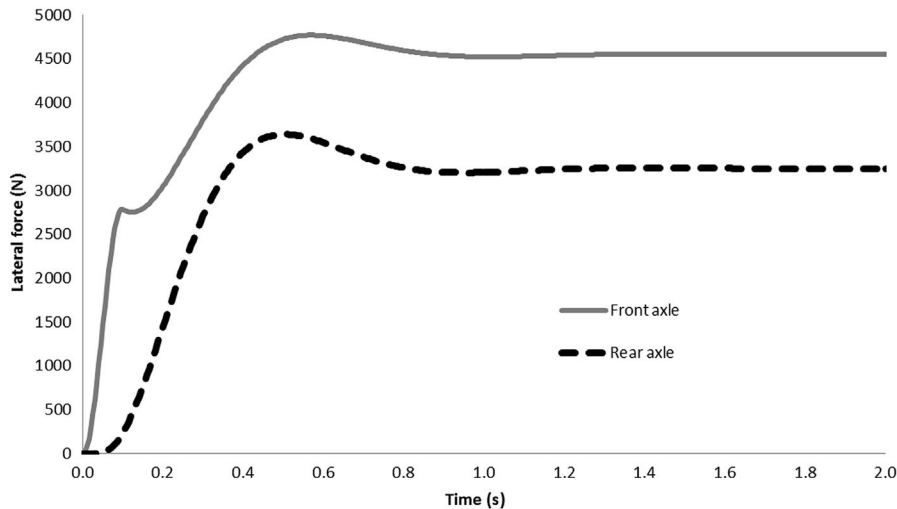
**FIGURE 4.9**

The simplest possible representation of a vehicle manoeuvring in the ground plane. GRF, ground reference frame.

These equations, their significance and a more formal derivation are well described in the seminal IME paper (Segel, 1956). The formulation states that yaw acceleration is the applied yaw moments divided by the yaw inertia of the vehicle, and that lateral acceleration is the applied lateral forces divided by the mass of the vehicle. The additional term in the lateral force expression reflects a body-centred formulation, which is more convenient when the model is expanded to more than the two degrees of freedom shown.

The two equations are correctly referred to as a two-degree-of-freedom ('2 DoF') model; they are sometimes referred to as a 'bicycle' model but the authors dislike this description since it implies that the description may be suitable for two-wheeled vehicles, which it most certainly is not. Readers interested in two-wheeled vehicles are recommended to study Chapter 10 of 'Tire and Vehicle Dynamics' (Pacejka, 2012).

Even with this simplest possible representation, it can be seen that the vehicle may be thought of as a free-floating 'puck' (as used in ice hockey), to which forces are applied by the tyres in order to manipulate its heading and direction. To many casual observers it appears that the vehicle runs on little 'rails' provided by the tyres and that the function of the tyres is simply to provide a cushion of air beneath the steel wheel rims. This is simply not so, and examination of the behaviour of rally cars in the hands of skilled drivers reveals behaviour which visually resembles that of a hovercraft. All vehicles on pneumatic tyres behave as the rally cars behave, adopting a sideslip angle to negotiate even the slightest curve. Since this angle is typically less than a degree it is not always apparent to the untrained observer; it may however be seen on high speed, steady corners such as motorway interchanges if vehicles are observed attentively.



**FIGURE 4.10**

Side forces to a 0.02 rad input applied between  $t = 0$  and  $t = 0.1$  with a cosine ramp function — front axle characteristic is common in simulation but rarely observed in practice since such rapid steer inputs are uncommon.

The generation of the forces necessary to initiate the turn, to constrain the vehicle at the correct sideslip angle and to return it to the straight-running condition is the role of the tyres. In order to successfully control the vehicle, however, those loads must be transmitted to the sprung mass. This is a key role for the vehicle suspension system.

Close examination of the vehicle behaviour described by the equations above can demonstrate the slight phasing of the forces necessary to allow the vehicle to accelerate in yaw and be constrained to the desired yaw velocity (Figure 4.10). Errors in that phasing generated by flexibility (compliance) on one axle or the other can lead to an error in the vehicle behaviour as perceived by the driver. This error may take the form of a disconcerting delay in response to the steering if the front axle has more compliance than the rear, or a rather more serious delay in the action of the rear axle in constraining the body slip angle to its required value. In the latter case, particularly for aggressive transient manoeuvres (i.e. accident avoidance) the rear tyre slip angle may exceed its critical value, leading to a divergent behaviour of the vehicle — that is to say a spin.

Since for modern vehicles the isolation of road inputs is a high priority, there is always a desire to introduce some elastomeric elements into the vehicle between the suspension elements and the vehicle body. MBS analysis tools allow the study of handling degradation due to the introduction of such elastomers and allow the ride/refinement compromise to be quantified before excessive experimentation is carried out on the vehicle. Detailed comments on the

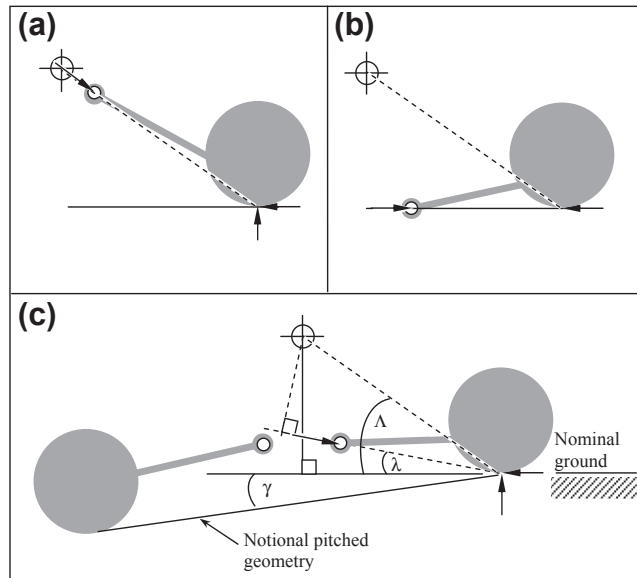
representation of elastomers are given in Chapter 3. MBS analysis in the right hands allows an understanding of those design parameters that dominate refinement performance and those that dominate handling performance. The understanding so gained allows better conceptual design of suspension systems in order to separate clearly the refinement and handling functionality for elastomeric elements. Modern multilink rear suspensions are a good example of such well-separated systems and are a significant part of the simultaneous improvement in both ride and handling, traditionally areas of mutual exclusivity, that has befallen modern road cars.

Even for a suspension with no elastomers, such as may be used on a competition vehicle, some structural compliance is always present. MBS analysis allows this compliance to be optimised and matched front to rear in order to avoid onerous design constraints using conservative stiffness targets, which would almost certainly incur some sort of weight disadvantage.

A final, important element of the study of handling load transfer is one that allows a single, unified treatment but rarely receives it. The notions of anti-dive/squat and 'roll centre' are rarely discussed together and yet their influence on vehicle handling loads is via the same mechanism; they lend themselves to the consistent rational treatment given here. The so-called 'roll centre' concept has to be one of the most nonintuitive and misunderstood modifiers of vehicle behaviour since empirically observed in racing circles in the 1920s. The understanding developed then is most applicable to beam axles and adds more difficulty than clarity for independent suspensions. The authors prefer the concept of 'sprung' and 'unsprung' loadpaths to the vehicle body.

Figure 4.11 shows a single wheel of an independently suspended vehicle viewed from the side. A braking load is applied at the contact patch, reflecting the fitment of outboard brakes, as is a common practice. Were the brakes inboard or were this a tractive case, the load would be applied at the wheel hub height and the diagrams would be redrawn appropriately. In this case the suspension is of a leading arm type. Since all independent suspensions may be represented as some form of equivalent length virtual swinging arm (although the length and pivot location vary with some types more than others) this remains a useful notion.

Three different orientations for the swinging arm are shown. The first has the swinging arm pivot on a direct line between the contact patch and the mass centre. Considering the wheel and arm together as a single entity and noting the ability of the pivot to support no moments, we may draw the reaction force at the pivot as being on a line between the contact patch and the pivot. The horizontal magnitude is the same as the applied lateral force at the wheel, giving a full solution for the force at the inboard pivot. The reaction on the sprung mass is equal and opposite to the force on the pivot, with a line of action passing directly through the mass centre. This is widely recognised as a 'no-dive' (no-pitch) type of suspension. Although there is no body pitch, this does not mean there is no load transfer between rear and front wheels. We may therefore conclude that the braking load is carried to the vehicle mass centre entirely through the suspension linkage

**FIGURE 4.11**

(a) No-dive suspension — pitch moment transfer solely via an unsprung loadpath. (b) Pitch moment transfer solely via a sprung loadpath. (c) Typical arrangement; pitch moment carried by a combination of sprung and unsprung loadpaths.

components and that none is carried in the suspension springs — i.e. via an ‘unsprung’ loadpath.

The second diagram has the swinging arm pivot at ground level. Using similar logic as before, the force at the pivot may be drawn as purely lateral, equal and opposite to that at the wheel. This in turn means the horizontal force is applied to the body at ground level, giving a pitch moment. That pitch moment cannot be reacted until the suspension has deformed sufficiently to give an equal and opposite moment on the sprung mass. In this case, the load transfer between rear and front axles is performed entirely by the suspension springing and none is carried in the suspension linkage components — i.e. via a ‘sprung’ loadpath.

The third diagram shows a more typical situation with some of the pitch moment carried by an unsprung loadpath and most carried by a sprung loadpath. Some fraction that is a function of the two angles  $\lambda$  and  $\Lambda$  may be calculated and expressed as an ‘anti-dive’ fraction or percentage, or alternatively the anti-pitch angle  $\lambda$  may be quoted separately. The authors prefer:

$$\text{Anti-dive \%} = 100 (\lambda + \gamma) / (\Lambda + \gamma) \quad (4.38)$$

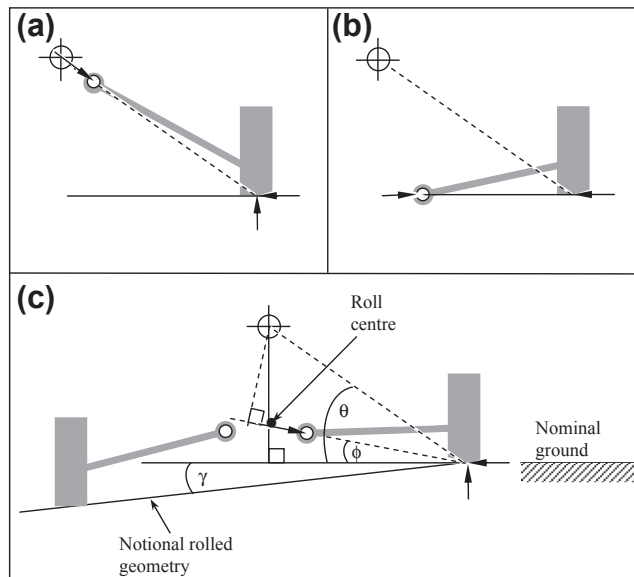
Other texts give differing descriptions and definitions. What matters is not the definition, although it is important to be certain how the quantities in use are defined

if they are to be compared one with another, but the significance of the sprung and unsprung load transfers themselves:

- Unsprung load transfer occurs via the stiff metallic elements in the system and is thus very rapid. It is limited in speed by the frequency of the wheel hop mode, a mode of vibration in which the unsprung mass oscillates on the tyre stiffness somewhere of the order of 15 Hz.
- Sprung load transfer occurs via the elastic elements of the system and is limited in speed by the frequency of the primary suspension mode. This is of the order of 1.5 Hz.

It may be seen then that unsprung load transfer is some 10 times faster than sprung load transfer. Herein lies the key to understanding some of the most important effects of the so-called ‘roll centre’. Figure 4.12 is very similar to Figure 4.11 except that it shows the vehicle from the front instead of the side. Otherwise, the diagrams are identical. Figure 4.12(a) shows a ‘no-roll’ suspension with load transfer entirely by an unsprung loadpath whereas Figure 4.12(b) shows a suspension that transmits load entirely via a sprung loadpath.

The point frequently but ambiguously referred to as the ‘roll centre’ is where the line of action of the unsprung loadpath crosses the vehicle centre line. As with the



**FIGURE 4.12**

(a) No-roll suspension — roll moment transfer solely via an unsprung loadpath. (b) Roll moment transfer solely via a sprung loadpath. (c) Typical arrangement; roll moment carried by a combination of sprung and unsprung loadpaths.

anti-pitch behaviour, the absolute height is of less importance than the distribution of loads between sprung and unsprung loadpaths. It is not any kind of centre of motion.

Again as with the anti-pitch behaviour, some fraction that is a function of the two angles  $\phi$  and  $\theta$  may be calculated and expressed as an 'anti-roll' fraction or percentage, or alternatively the anti-roll angle,  $\phi$ , may be quoted separately. Alternatively, an anti-roll fraction or percentage could be quoted based on the fraction of the 'roll centre height' compared to the mass centre height. The authors prefer to use a ratio of the two angles  $\phi$  and  $\theta$  to express the anti-roll fraction similarly to before:

$$\text{Anti-roll \%} = 100 (\phi + \gamma) / (\theta + \gamma) \quad (4.39)$$

For lateral handling loads, the same ideas of relative speed between unsprung and sprung loadpaths apply. This has a particular importance when considered in the light of the phasing of front and rear axle forces in order to manipulate the yaw moments on the body. For a vehicle in yaw, the rate of load transfer may thus be set differently at different ends of the vehicle in order to modify the transient behaviour as compared to the steady state behaviour. For example, it is typical for vehicles to run around 20% rear anti-roll and only around 6% front anti-roll. This means that as a manoeuvre develops, load transfer from the outside rear tyre may briefly outpace load transfer from the front tyre. The resulting yaw moment acts to stabilise the vehicle and mitigates sudden, aggressive steer inputs. Vehicles that do not have this type of geometry, notably those with trailing arm rear suspensions, are unable to benefit from these effects.

For both longitudinal and lateral load transfers there is no conceptual reason why either the anti-pitch angle or anti-roll angle may not be negative. Motorcycles, for example, have a negative anti-pitch angle equal to the steer axis rake when they are fitted with conventional telescopic forks. This has the disadvantage of requiring extra performance from the suspension springs since they must carry more than the straightforward load transfer one might instinctively expect. The performance cost of this is discussed in [Sections 4.9.2 and 4.9.3](#).

MBS analysis allows both an understanding of the load transfers in a rig-based environment, such as may be measured on the MIRA Kinematics & Compliance rig (Whitehead, 1995) and also during real driving manoeuvres. In both situations, the ability of an MBS model to retrieve forces in each suspension member in convenient frames of reference while working with quarter, half or full vehicle models is a powerful tool to unscramble some of these less-than-intuitive effects with vehicle designs.

#### 4.1.4 Compliant wheel plane control

Hand-in-glove with an understanding of the load transmission paths and time delays associated with the activity manoeuvring the vehicle in the ground plane comes an understanding of the resulting motion of the wheel plane with respect to the ground. From the treatment of tyres that follows in Chapter 5, and as described briefly in the introductory chapter, it is clear that the angles at which the tyres are presented to the

road are of crucial importance in modifying the forces generated by the tyres and hence the resulting motion of the vehicle.

There are some subtle and intricate effects present in real vehicle systems that defy simplistic comprehension and evaluation. For example, the deformation of anti-roll bars re-orientates the links with which they are connected to the moving suspension members and may introduce forces that ‘steer’ the wheel plane with respect to the vehicle. This may not have been considered at the time the suspension was schemed conceptually but yet may modify the behaviour of the vehicle in practice.

MBS analysis allows the construction of rig-like models for such wheel plane compliant behaviour before prototype vehicles exist. It also allows a systematic and well-controlled comparison of behaviour with different levels of compliance in order to establish the influences of the different aspects of wheel plane compliance on vehicle behaviour. Typically such studies carry across several revisions of a model within a market segment and are thus of some strategic benefit.

#### 4.1.5 Kinematic wheel plane control

Suspension arrangements typically consist of some connection of linkages to a device for holding the bearings in which the wheel actually turns. The interaction of those individual elements comprising the linkage means that the wheel plane typically undergoes some sort of translation and rotation as the suspension articulates.

This motion of the wheel plane is traditionally defined with respect to the vehicle body, although for a moving vehicle it is the angles and velocities with respect to the road that are of import. However, using the vehicle body gives a convenient frame of reference and so the descriptions following will do so:

1. Toe change (steer) with suspension articulation gives a lateral force and yaw moment by directly adding to or subtracting from the tyre slip angle and is readily understood. Some care is needed when discussing this characteristic since there are a number of possibilities for definition; steer may be defined as a right hand positive rotation about the vehicle vertical axis or it may be defined as a handed rotation, different on left and right sides of the vehicle (‘toe-out’) or even as a term relating the behaviour to the yaw moment it induces on the vehicle (‘roll understeer’).
2. Camber change<sup>1</sup> acts to mitigate the angle with which the tyre is presented to the road due to the roll of the body with respect to road surface. Typical passenger vehicles roll ‘out’ of turns, with the inside edge of the vehicle platform being further away from the road than the outside. Were the wheels to remain perpendicular to the vehicle platform at all times, they would be presented to the road with a camber/inclination angle equal to the vehicle roll angle. For most independent suspension types, the wheels camber somewhat with respect to the

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<sup>1</sup>It might be more sensibly named ‘inclination change’ but the term camber change is nevertheless in popular use.

body such that they are presented to the road at something less than the vehicle roll angle. This is referred to as ‘camber compensation’. Were the wheels to remain upright with respect to the road, this would be 100% or ‘full’ camber compensation. As the wheel is presented to the road progressively more and more upright, the tyre is loaded more and more evenly across its width and lasts longer. This is of primary concern for competition vehicles. Camber/inclination angles also generate forces in their own right and the tendency of a tyre leaned ‘out’ of the turn is to reduce the cornering forces generated by tyre slip angles; therefore by balancing camber compensations front to rear some influence may be exerted on the overall handling balance of the vehicle. Such treatment is very evident on, for example, the Bugatti Type 35.

3. It is typical for independent suspensions to move the contact patch of the tyre laterally as they articulate (Figure 4.14). Although less intuitively direct than the toe change mechanism, half-track change (lateral displacement) influences the lateral velocity of the tyre contact patch via roll rate. The slip angle of the tyre is affected; since the angle is defined as the arctangent of lateral and longitudinal velocities, an increase in lateral velocity directly increases slip angle. If the track change is plotted against bump travel, with both measured at the contact patch, a direct indication of the anti-roll angle is obtained with no need for knowledge of construction methods for any particular type of suspension.

Some practitioners attempt to calculate combined measures for both suspensions on the same axle or indeed all the suspensions on the vehicle. For beam axles there is some logic in combining the characteristics since the wheels are physically joined but for independent suspensions, calculating some combined metric is of questionable value. For example, attempting to combine anti-roll angles across one axle in a

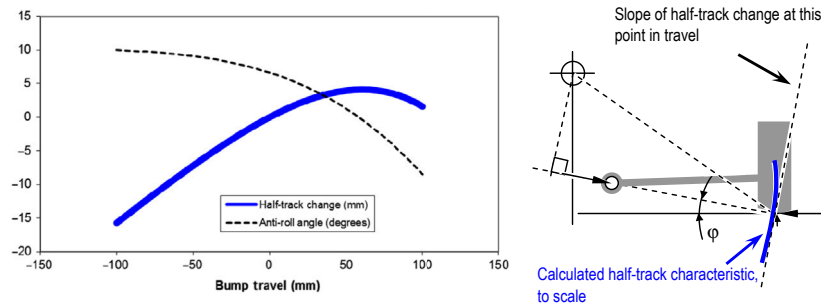


**FIGURE 4.13**

Vehicle cornering pose.

*Courtesy of Prodrive.*



**FIGURE 4.14**

Relationship between calculated half-track change and anti-roll angle.

purely geometric manner, when their relative importance is determined by wheel loading is clearly nonsensical, as shown in Figure 4.13. If the wheel is in the air, how can it contribute the motion of the vehicle?

Equally, there is often a desire to calculate some of these measures, particularly camber values and anti-roll elements, with respect to the ground. Unfortunately the location and orientation of the ground cannot reliably be determined using only a quarter vehicle model. Nonlinear force–deflection characteristics are typical for the suspension elements and so the frequently used ‘symmetric roll’ presumption is often flawed. The amount of roll generated for a particular lateral loading varies with suspension calibration and the amount of roll moment carried on a particular axle varies with suspension calibration. Thus the boundary conditions cannot be known for a quarter vehicle model with any useful degree of certainty except for symmetric events. It is strongly suggested that the simplest interpretations of the vehicle kinematic measures be used and comparisons drawn between these simple measures. For comprehending the effects on a full vehicle, a full vehicle model is recommended.

#### 4.1.6 Component loading environment

MBS models of the quarter vehicle type are often used to distribute design loads through the different suspension members with a view to sizing them intelligently in the first instance. A distinction needs to be drawn between design loads and service loads when discussing MBS analyses for this purpose. Design loads are calculated by postulating notional extremes for possible loads that may be induced in reality. An example of this might be the case where a car has been parked between two kerbs, between which it is a snug fit, and the driver then applies maximum effort to the handwheel. The exact philosophy for the selection and implementation of design loads is typically historical; they have been empirically defined and honed over many years of development experience. In practice they often correspond to events described as ‘extreme service loads’ or ‘abuse

loads'. They are events that the vehicle must survive but may need some attention immediately following it; in the steering example given, the steering alignment may be distorted by events but the vehicle would probably be required to remain capable of being driven. In industry many companies have standard cases such as the 3G bump case where the static wheel load is factored up to represent dynamic abuse situations such as striking a road hump at speed. The specification of abuse loads can be traced back to a publication in the *Automobile Engineer* by Garrett (1953) where a range of recommended wheel loads was proposed for the design of vehicles of that period. Design loads are frequently viewed as 'one-off' events in the life of the vehicle.

In contrast to design loads, there is another category of loads to which the vehicle is exposed. These are the service loads. This is the loading environment to which the vehicle is subjected during its durability sign-off testing. Durability sign-off criteria vary widely but they typically consist of a specified number of repetitions of different events at prescribed speeds and loading conditions. They induce a large number of repetitions of events that are probably more commonplace in the life of a vehicle — driving up kerbs, for example. In contrast to the design events, the vehicle is expected to emerge largely undamaged from the durability sign-off procedure — although no expectation of remaining service life is usually associated with durability sign-off. The relationship between durability sign-off criteria and actual usage of the vehicle is another question entirely; durability sign-off procedures themselves are often compiled in the light of historical warranty costs and other such influences.

There is, then, a key difference between design and service events. The design events are slightly fictitious and are usually analysed using a static or quasi-static procedure. While of questionable 'accuracy' they are of tremendous value. They may be calculated with the scarcest data about the vehicle and allow the early intelligent sizing of many different components, once applied to the wheel of a quarter vehicle model and distributed about a conceptual suspension layout. The resulting loading environment for each component may be used by designers to sketch a meaningful first-sight design. This can avoid packaging difficulties induced by ill-informed sketching of components and the subsequent 'stealing' of space needed for section modulus in key components.

Service loads are altogether more detailed. They are typically well-defined events where road profiles and speeds are well known. Once a reasonably representative geometry and suspension setting data set is available, the description of events may be combined with some form of enveloping tyre model in order to predict load-time histories for each component in the suspension. This in turn may be passed to finite element (FE)-based fatigue calculation methods in order to evaluate the number of times a component may be subject to this event before failure. Hypotheses such as the Linear Damage Accumulation Hypothesis (Miner's Rule) may be used to sum up the contribution of different events to the life of the component. Interested readers are referred to the *SAE Fatigue Design Handbook* (Rice, 1997) for more information on component fatigue analysis.

With the calculation of service loads, instead of a single set of peak loads as produced from the design loads there is a vast amount of time-domain data. Typical data rates for service load events are around 200 points per second and service load events last of the order of 10 s. With a repertoire of up to 20 events describing a typical durability sign-off, it may be seen that the calculation of service loads results in an increase in the quantity of results data of the order of five orders of magnitude. Calculation and subsequent processing times rise too, though not quite by the same amount, since linear static FE calculation times are a significant proportion of the overall elapsed time following design load calculations. However, as an overall process time amplification the expectation should be between two and three orders of magnitude for the calculation and use of service loads as compared to design loads. For some simple components, the use of a previously verified and correlated rig test may prove less onerous to the organisation.

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## 4.2 Types of suspension system

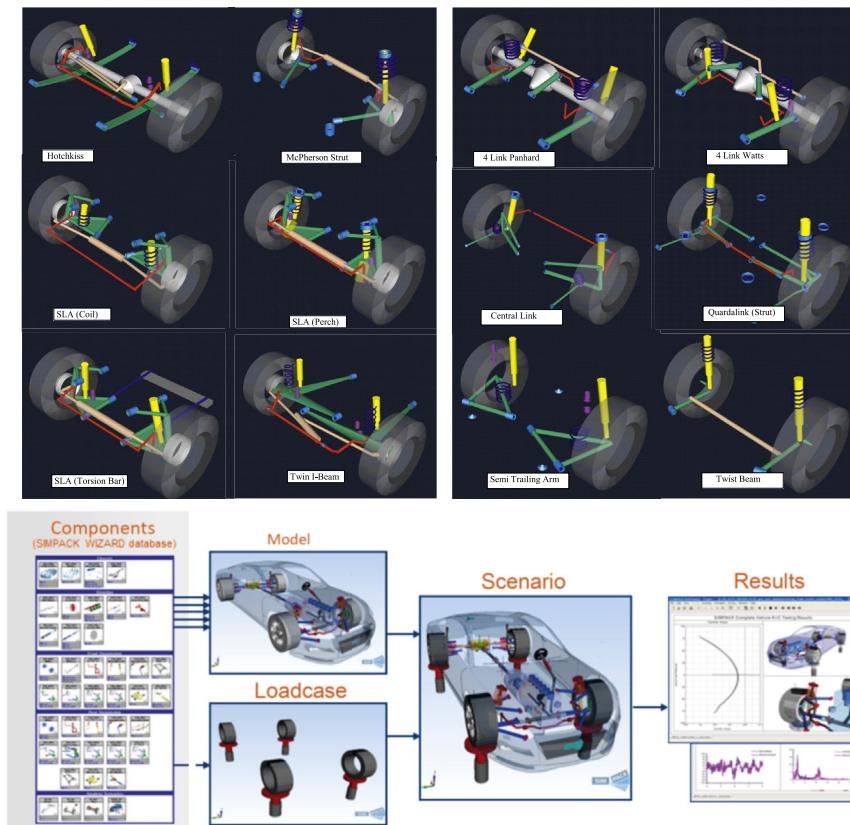
There are various suspension systems used on cars and trucks and as described in Chapter 3, single-purpose versions of general purpose programs provide templates with preprogrammed configurations of suspension systems commonly used by automotive manufacturers. Many established textbooks on vehicle dynamics and some that focus on suspensions provide a detailed treatment of the various types of suspension system and their function. The thorough treatment and discussion contained in 'Road Vehicle Suspensions' (Matchinsky, 1998) is particularly commended. The coverage here will be to briefly mention some of the most common systems and to then direct our attention to the MBS modelling and simulation environment. The double wishbone and McPherson strut systems are very common and will provide the basis for the ensuing discussion.

Most of the 'single-purpose general-purpose' program libraries contain a number of different suspension types and are typically open enough to allow the user to add new types (Figure 4.15).

As stated the double wishbone suspension system will form the basis of much of the following discussion in this chapter. For readers new to the subject area and unfamiliar with the system the main components are indicated in Figure 4.16.

Figure 4.16 shows the main components the modelling of which will be dependent on the type of analysis to be performed and the outputs that must be produced. If the model is to be used only for the prediction of suspension characteristics such as camber angle or half-track change (HTC) with bump movement (BM) then an accurate definition of the mass and inertial properties of the rigid bodies will not be required. This information would be required however if the model is to be used for a dynamic analysis predicting the response of the suspension to inputs at the tyre contact patch.

The modelling of the connections between the suspension links will also depend on the type of vehicle and whether the suspension is for the front or rear of the vehicle. On

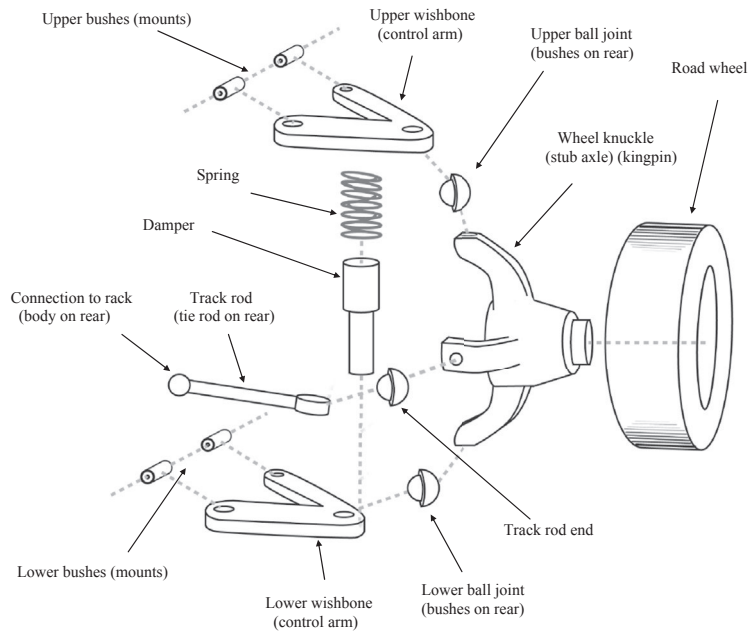
**FIGURE 4.15**

Graphical representation of suspension configurations in ADAMS/Car and the SIMPACK vehicle wizard.

*Provided courtesy of MSC software and Simpack.*

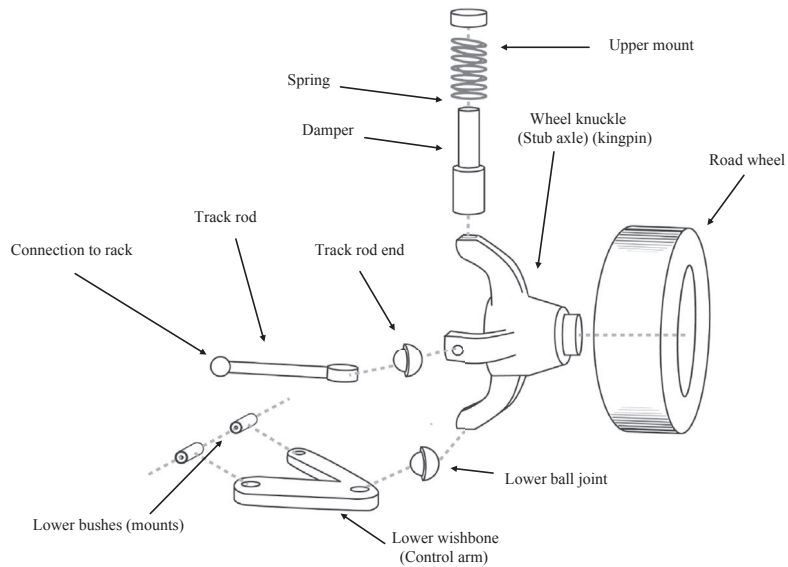
the front of the vehicle the connections between the control arms and the wheel knuckle would be modelled using spherical joints to represent the ball joints used here. On the rear these connections are more likely to include the compliance effects of rubber bushes. On a racing car where ride comfort is not an issue the suspension model is likely to be rigidly jointed throughout. Not shown in Figure 4.16 are the bump and rebound stops that would need to be included when considering situations where the wheel loads are severe enough to generate contact with these force elements.

The other type of suspension system that is very common on road vehicles is the McPherson strut system as illustrated in Figure 4.17. The main difference between this system and the double wishbone system is the absence of an upper control arm and the combination of the spring and the damper into a single main strut the body of which is the major component in the system.



**FIGURE 4.16**

Double wishbone suspension system.



**FIGURE 4.17**

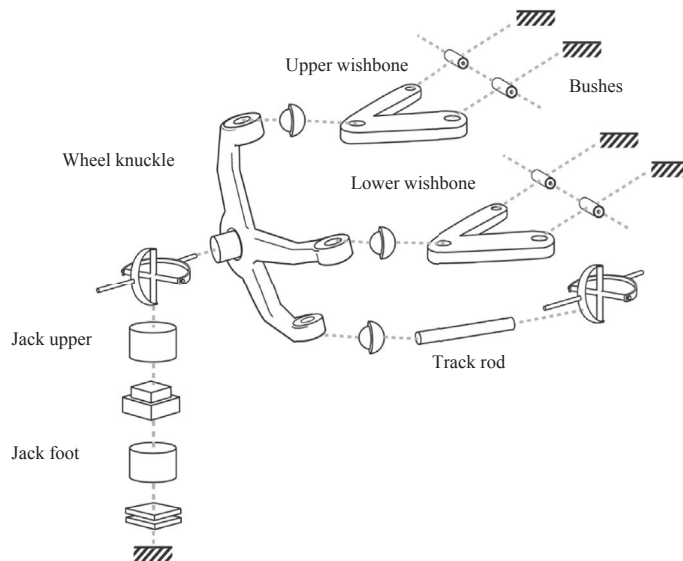
McPherson Strut suspension system.

### 4.3 Quarter vehicle modelling approaches

One of the earliest documented applications of the MSC ADAMS program by the automotive industry (Orlandea and Chase, 1977) was the use of the software to analyse suspension geometry. This approach is now well established and will be discussed further in the next section of this chapter. The output from this type of analysis is mainly geometric and allows results such as camber angle or roll centre position to be plotted graphically against vertical wheel movement.

The inclusion of bush compliance in the model at this stage will depend on whether the bushes have significant influence on geometric changes in the suspension and road wheel as the wheel moves vertically relative to the vehicle body. With the development of multilink type suspensions, such as the rear suspension on the Mercedes Model W201 (von der Ohe, 1983), it would appear difficult to develop a model of the linkages that did not include the compliance in the bushes. This type of suspension was used as a benchmark during the IAVSD exercise (Kortum and Sharp, 1991) mentioned in Chapter 1 comparing the application of MBS analysis programs in vehicle dynamics.

This modelling issue is best explained by an example using the established double wishbone suspension system. The modelling of the suspension using bushes to connect the upper and lower arms to the vehicle body is shown in Figure 4.18. Vertical motion is imparted to the suspension using a jack upper connected to the jack lower by a translational joint. A translational motion is applied at this joint



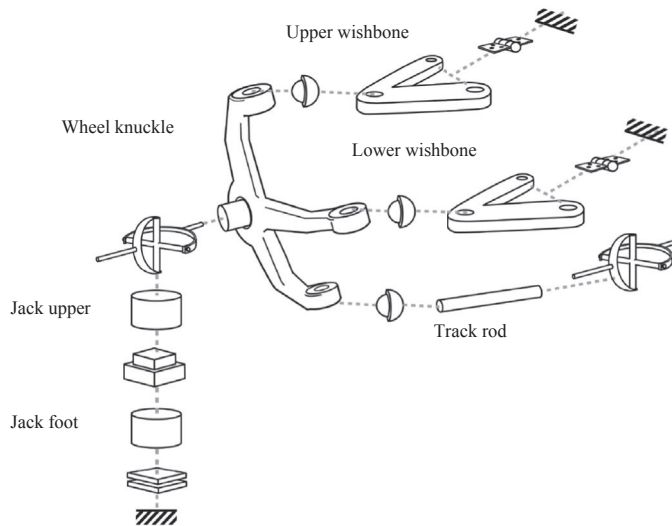
**FIGURE 4.18**

Double wishbone suspension modelled with bushes.

to move the jack over a range of vertical movement equivalent to moving between the full bump and full rebound positions. Although the jack upper is shown below the upright in [Figure 4.18](#), it is connected to the wheel using a universal joint at the wheel centre (WC); this also illustrates the difficulty in the software being unable to use exploded diagrams referred to previously. Note the orientation of the universal joint is entirely theoretical since no practicably realisable universal joint will work with an angle at or close to  $90^\circ$ . This constraint is necessary to prevent a spin degree of freedom for the jack.

The planar joint constrains the wheel centre or wheel footprint to remain in the plane at the foot of the jack while allowing it to move in the lateral or longitudinal directions while the presence of the universal joint does not constrain the wheel to change orientation. The wheel itself is not explicitly represented since its spin degree of freedom is of no consequence to this model. For the suspension modelled in this manner it is possible to calculate the degrees of freedom for the system as shown in [Table 4.2](#).

<b>Table 4.2</b> Degree of Freedom (DOF) Calculation for Suspension System with Bushes			
<b>ADAMS-Style</b>			
<b>Component</b>	<b>Number</b>	<b>DOF</b>	<b><math>\Sigma</math> DOF</b>
Parts	6	6	36
Translationals	1	-5	-5
Universals	2	-4	-8
Sphericals	3	-3	-9
Planars	1	-3	-3
Motions	1	-1	-1
$\Sigma$ DOF for system = 10			
<b>SIMPACK Style</b>			
<b>Joint/Constraint</b>		<b>DOF</b>	
Universal – steering arm to ground		2	
Spherical – upright from steering arm		3	
Spherical – lower wishbone from upright		3	
Spherical – upper wishbone from upright		3	
Universal – jack upper from upright		2	
Translational – jack foot from jack upper		1	
Planar constraint		-3	
Applied motion		-1	
Total		10	

**FIGURE 4.19**

Double wishbone suspension modelled with joints.

The double wishbone suspension model shown in [Figure 4.18](#) can be simplified to represent the bushes connecting the upper arm and the lower arm to the vehicle body by revolute joints as shown in [Figure 4.19](#).

It can be seen that this system is identical to the previous one, save for the deduction of 10 degrees of freedom, five for each revolute constraint, leaving the system with zero degrees of freedom — a kinematic model — and allows a kinematic analysis to be performed. The fact that at least one of the degrees of freedom constrained in this model is due to a time-dependent motion, input at a joint, means that the model will move and operate as a mechanism rather than ‘lock’ as a structure.

This model is a slightly different scheme to that presented in the first edition but there are a number of advantages to the slightly more complex layout. When attached at each corner of a full vehicle model, the motion can be removed and replaced with a description of the tyre stiffness and damping to allow an eigensolution of the model to check for expected behaviour before adding a running tyre model. An eigensolution is always recommended to check that the practitioner has modelled what they *think* they have modelled, which with restrictive graphical interfaces is by no means certain. Any worthwhile model audit will contain a stationary eigensolution of the full vehicle less the running tyre model; the addition of compliance is plainly seen to add a large number of high-frequency modes of vibration and thus the penalty in run time is made clear to the user when including compliance. The highest frequency of vibration provides valuable guidance as to the minimum time step that may need to be employed.



**Table 4.3** ADAMS Solver-Level Statements for a Joint, Linear Bush and Nonlinear Bush

Nonlinear Bush	Linear Bush	Joint
GFORCE/16,I=1216 ,FLOAT=011600,RM=1216 ,FX=CUBSPL(DX(1216,0116,1216),0,161)\ ,FY=CUBSPL(DY(1216,0116,1216),0,161)\ ,FZ=CUBSPL(DZ(1216,0116,1216),0,162)\ ,TX=CUBSPL(AX(1216,0116),0,163)\ ,TY=CUBSPL(AY(1216,0116),0,163)\ ,TZ=0.0 SPLINE/161 ,X=-1.8, -1.5, -1.4, -1.22, -1.123, -1.0, -0.75, -0.5, -.25, 0, 0.25, 0.5, 0.75, 1.0, 1.123, 1.22, 1.4, 1.5, 1.8 ,Y=15350, 10850, 9840, 6716, 5910, 5059, 3761, 2507, 1253, 0,-1253, -2507, -3761, -5059, -5910, -6716, -9840, -10850, -15350 SPLINE/162 ,X=-5, -4, -3, -2.91, -2.75, -2.5, -2, -1.5, -1, -0.5, 0, 0.5, 1, 1.5, 2, 2.5, 2.75, 2.91, 3, 4, 5 ,Y=7925, 3925, 1925, 1790, 1626, 1450, 1136, 830, 552, 276, 0, -276, -552, -830, -1136, -1450, -1626, -1790, -1925, -3925, -7925 SPLINE/163, ,X=-0.22682, -0.20939, -0.19196, -0.17453, -0.1571, -0.13963, -0.10472, -0.06981, -0.03491, 0, 0.03491, 0.06981, 0.10472, 0.13963, 0.1571, 0.17453, 0.19196, 0.20939, 0.22682 ,Y=241940, 198364, 160018, 125158, 93387, 75415, 52951, 35702, 18453, 0, -18453, -35702, -52951, -75415, -93387, -125158, -160018, -198364, -241940	BUSH/ 16,I=1216,J=0116 ,K=7825,7825,944	JO/ 16,REV, I=1216, J=0116

With a fixed constraint applied to the vehicle body, the motion can be reinstated and static equilibrium runs performed to retrieve wheel force versus displacement characteristics including bump stops and the influence of bush compliance, which can sometimes be unexpected with, for example, large lateral loads on the lower wishbone when the bump stop is heavily engaged, leading to unexpected steering behaviour.

Remaining with the fixed constraint to the body and removing the applied motion, design loads can be applied to the vehicle corners for load distribution analysis directly, with no need to modify the model further.

The decision to model the bushes will have a significant impact on the collation of data and the effort required to input and check the values. This is illustrated in [Table 4.3](#) where the typical solver-level inputs required to model a connection as a rigid joint, linear bush and nonlinear bush are compared. The fact that modern software presents the user with graphical dialogue boxes to enter the information does not change the fundamental escalation of the required amount of data for the model including compliance.

There are three main types of analysis for which individual suspension models are likely to be used during the design and development of a vehicle. The quantity and type of data will vary from vehicle to vehicle and the type of analysis to be performed. A summary of the typical modelling data is provided for guidance in [Table 4.4](#). Note that for some suspensions, such as a twist beam type, the modelling of structural compliance will also need to be included.

Before leaving the subject of quarter vehicle suspension models it is worth considering the implementation of suspension connections as used in some race cars. The advent of inter-university competitions where students design, build and race a vehicle has become popular on automotive courses in recent years. An example of such a vehicle is a Formula Student car built by students at Coventry University shown in [Figure 4.20](#).

The modelling of part of the suspension system is shown in [Figure 4.21](#). The pushrod connects the suspension arm to the bell crank and uses a spherical joint at one end and a universal joint at the other end to constrain unwanted spin of the pushrod about its own axis. The bell crank is attached to the chassis with a revolute joint, the rotation about which is resisted by the springdamper unit. It can be noted that it is not really necessary to model the spring damper as rigid bodies. The definition of the forces generated will be sufficient to simulate the handling of the full vehicle.

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## 4.4 Determination of suspension system characteristics

The suspension design process discussed at the beginning of this chapter has been summarised in [Table 4.1](#) by six areas in which the suspension performance can be assessed. Ultimately the quality of the design will be judged on the performance of the full vehicle, but an early assessment of the suspension design as an individual unit is essential. In order to quantify the performance of the suspension system a

<b>Table 4.4</b> Indicative Data Requirements for Individual Suspension Analyses
<b>Kinematic Vertical Rebound to Bump Analysis</b>
Coordinates of suspension linkage connections
<b>Quasi-static Vertical Rebound to Bump Analysis</b>
Coordinates of suspension linkage connections Bush stiffnesses (if this affects the movement) Spring and bump/rebound rubber characteristics
<b>Static or Quasi-static Durability Analysis</b>
Coordinates of suspension linkage connections Bush stiffnesses Spring and bump/rebound rubber characteristics Component flexibility (some suspensions)
<b>Dynamic Durability or Vibration Analysis</b>
Coordinates of suspension linkage connections Mass and inertial properties Bush stiffnesses Bush damping data Spring and bump/rebound rubber characteristics Damper properties Component flexibility (some suspensions)

range of characteristics may be determined through simulation of a single suspension system or quarter vehicle model. During this chapter it will be shown that a single suspension system can be analysed in a number of ways that will provide information to support the six suspension design activities that have been identified.

It should also be noted, however, that while the emphasis in this chapter is to explain the function and modelling of suspension systems using quarter models, so-called ‘single-purpose general-purpose’ programs such as ADAMS/Car and the SIMPACK Vehicle Wizard extend the modelling as stated to a half vehicle model analysed using a virtual test rig, an example of which is shown in [Figure 4.22](#).

Such a system also allows the additional incorporation of rollbars and a steering system to investigate the cross coupling of left and right suspension systems through rollbar compliance and the effects of steering inputs in isolation or in combination with suspension movement.

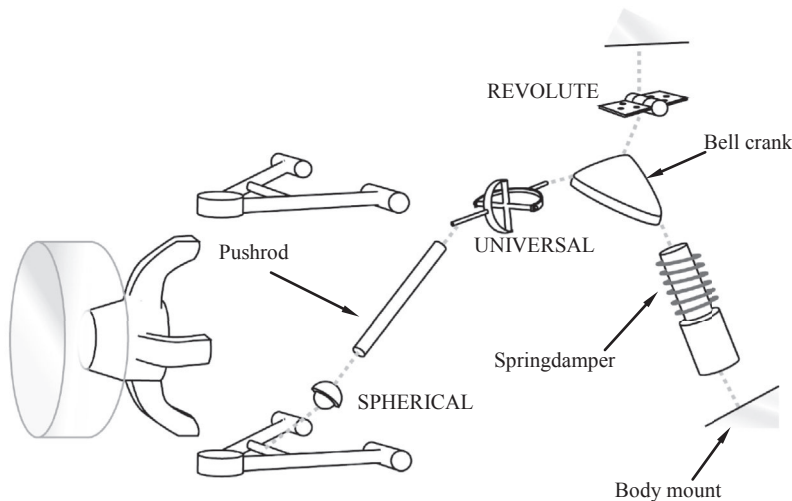
A more detailed discussion of suspension analysis methods, such as those used to investigate body isolation issues, will follow later in this chapter but using a virtual rig such as this the following are typical of some of the analyses performed:

1. The wheels may be moved vertically relative to the vehicle body through a defined bump-rebound travel distance. For the half model shown in [Figure 4.20](#)



**FIGURE 4.20**

Coventry University Formula Student Car.



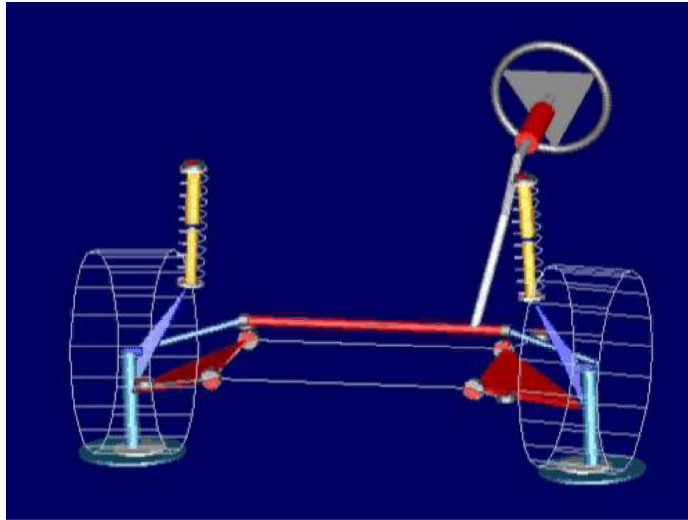
**FIGURE 4.21**

Modelling of PushRod and Bell Crank mechanism in student race car.

the vertical movement may involve single, opposite or parallel wheel travel representing ride or roll motions for the vehicle. The measured outputs allow the analyst to consider, depending on the model used, aspects of kinematic and compliant wheel plane control.

2. Lateral force and aligning torque may be applied at the tyre contact path allowing measurement for example of the resulting toe angle change and lateral deflection of the wheel (compliant wheel plane control).

In addition to the above basic types of analysis it is also possible to use an MBS suspension model to consider wheel envelopes where under the full range of

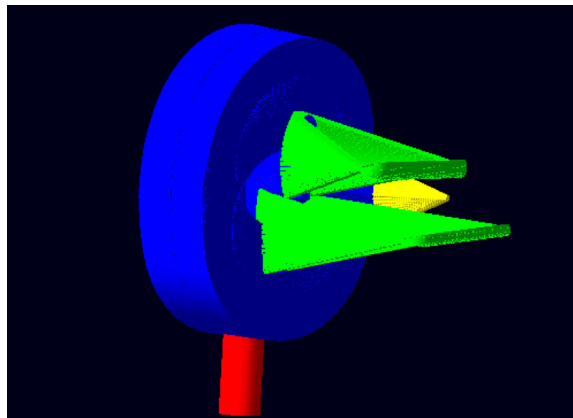
**FIGURE 4.22**

Use of virtual test rig to analyse a half vehicle suspension model.

*Provided courtesy of MSC software.*

suspension travel and steering inputs an envelope mapped by the outer surface of the tyre can be developed allowing the clearance with surrounding vehicle structure to be checked.

In practice this has been achieved by using the wheel centre position, orientation and tyre geometry from the MBS simulation as input to a CAD system where the

**FIGURE 4.23**

Superimposed animation frames giving visual indication of wheel envelope.

clearances can be checked. An example of the graphic visualization of a wheel envelope, for vertical wheel travel only, using superimposed animation frames is shown in Figure 4.23. Some CAD packages have an embedded kinematic solver for this purpose and generate the resulting surface within the CAD package directly.

## 4.5 Suspension calculations

### 4.5.1 Measured outputs

A glossary of terms providing a formal specification of various suspension characteristics has been provided by the Society of Automotive Engineers (SAE Publication, 1976). In the past, variations in formulations and terminology have been provided by researchers, authors and also practicing engineers following corporate methodologies. The concept of a roll centre has also been subject to a number of definitions (Dixon, 1987).

As discussed in Chapter 3, vehicle-specific programs offer a range of precomputed outputs for suspension characteristics. The user documentation provided with those software systems includes an extensive description of each output and need not be repeated here. For completeness those outputs considered to be most common in their usage and most relevant only to the following discussion in this textbook will be described in this chapter.

As discussed in the previous sections, one of the main uses of an MBS model of a suspension system is to establish during the design process geometric position and orientation as a function of vertical movement between the rebound and bump positions. As the output required does not include dynamic response it is suitable to use a kinematic or quasi-static analysis to simulate the motion. It should be noted that this information could also be obtained using a CAD package or a program developed solely for this purpose. Some CAD packages include an implementation of a spherical kinematic solver, which may come directly from a multibody software provider, and the authors have also seen dedicated Excel and Open Office spreadsheets used with a spherical kinematic formulation to the same end. The fact is that an MBS program used is often associated with the stages of model development described in Chapter 1 that lead through from the individual suspension model to a model of the full vehicle.

A large number of parameters can be measured on an existing suspension system and laboratory rigs such as the Kinematics and Compliance measurement facility (or K&C Rig) described by Whitehead (1995) have been developed specifically for this purpose. The descriptions provided here will be limited to the most commonly calculated outputs, which are:

- Bump motion (spindle rise)
- Wheel recession
- Half track change
- Steer (toe) angle

- Camber angle
- Castor angle
- Steer axis inclination
- Suspension trail
- Ground-level offset
- Wheel rate
- Roll centre height

In each case the parameters are presented as XY plots with the bump motion as the independent variable usually on the x-axis. The calculation of these outputs can be programmed to create a variable that is derived directly from measured system variables or is based on a trigonometric or algebraic derivation. It should be noted that parameters such as the camber angle determined here are measured relative to the vehicle body which is assumed to be fixed and should not be confused with the inclination or presentation angle measured between the tyre and the road surface discussed later in Chapter 5.

The vertical motion is imparted to the wheel using a body to represent a jack with the wheel centre constrained to remain in the plane of the jack using an inplane joint primitive as described in Chapter 3 and shown in [Figure 4.24](#). The motion applied to the translational joint between the jack and the ground will be that needed to move the wheel between the rebound and bump positions. If the model is defined in a position midway between these it is convenient to define the motion as sinusoidal with a cycle of 1 s. This would provide results in the bump position at 0.25 s, the rebound position at 0.75 s and allow for presentation purposes smooth animation of continuous cycles. Running the analysis for 1 s with, say, 80 output steps would ensure that output is calculated in the full bump and rebound positions.

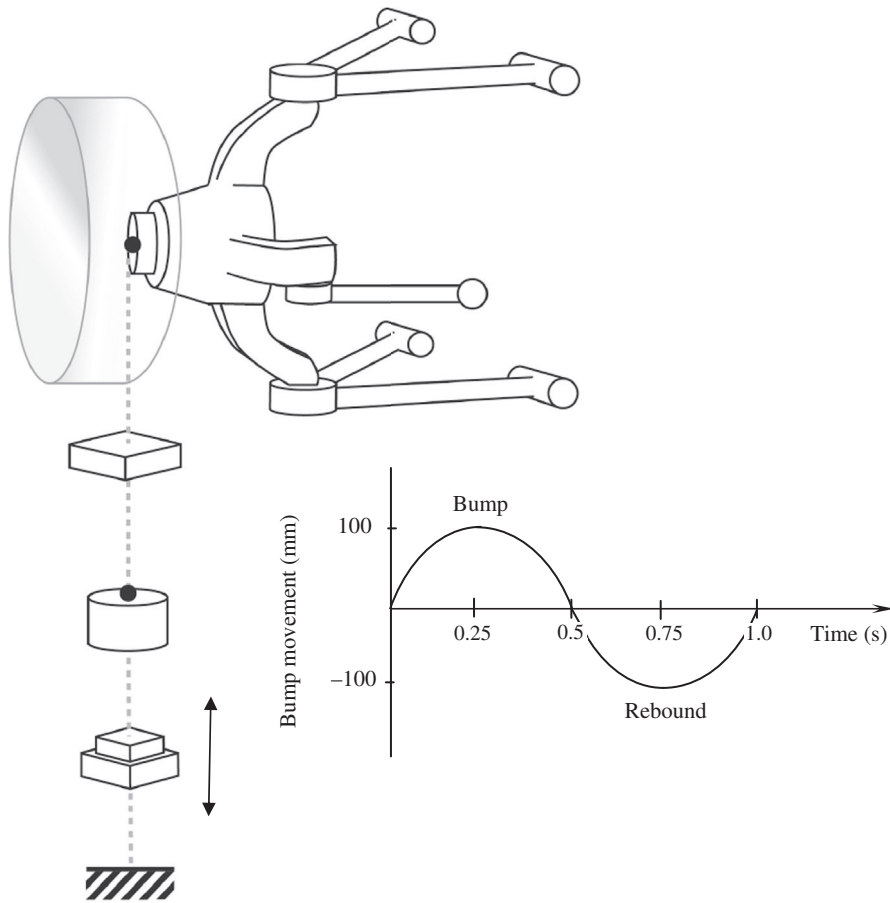
The motion statement applied to the translational joint that would accomplish this for a suspension where the total movement between the rebound and bump is 200 mm would be.

```
FUNCTION = 100*SIN(TIME*2*PI)
```

Note that the TIME variable is in seconds and is converted to radians within the function to represent one cycle over 1 s of simulation time. Some software allows the use of degrees directly within functions with the addition of the letter 'D' after the number but in general computer software works in radians with trigonometric functions.

It should also be noted that if the movement is not symmetric, that is to say that the distance moved in bump is different to that moved in rebound, a more complicated function will be needed for the motion input. In the following example the suspension is required to move 110 mm into the bump position and 90 mm into the rebound position. It is still desirable to have an overall sinusoidal motion for animation purposes and so an Arithmetic If is used in the function to switch the amplitude at the half cycle position of 0.5 s as follows:

```
FUNCTION = IF(TIME-0.5: 110*SIN(TIME*2*PI), 0.0, 90*SIN(TIME*2*PI))
```

**FIGURE 4.24**

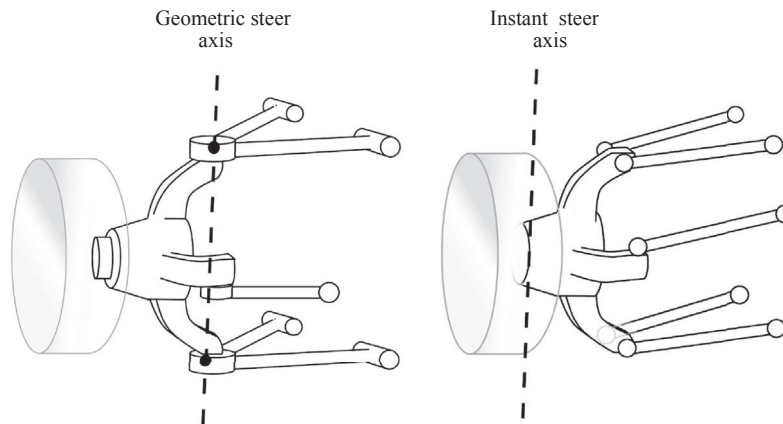
Input of vertical motion at the wheel centre.

### 4.5.2 Suspension steer axes

Suspension characteristics, such as castor angle, trail and the steering axis inclination, require an initial computation of the suspension system steer axis. Generally the concept of a steer axis is straightforward when considering, for example, the double wishbone system described earlier. In such a case it is easy to see that the wheel will steer about an axis passing through the lower and upper ball joints.

For a McPherson strut suspension system, the steer axis may be defined to pass through a point located at the lower ball joint and a point located where the upper part of the strut is mounted to the vehicle body. Note that this line is not necessarily parallel to the sliding axis of the upper part of the strut.



**FIGURE 4.25**

Geometric and Instant steer axes of a suspension system.

For some suspensions, such as a multilink system, the location of the steer axis is not immediately evident from the suspension geometry. In these cases it is necessary that the software be programmed to calculate the steer axis as the instant axis of rotation of the wheel carrier parts (Figure 4.25).

Using the instant axes method the left and right steer axes can be computed from the suspension's compliance matrix. The process involves locking the spring to prevent wheel rise and applying an incremental steering torque or force against an elastic restoring force temporarily added to, for example, the steering rack. The resulting translation and rotation of the wheel carrier parts can then be used to compute the instant axis, and hence steer axis, of rotation for each wheel carrier.

The authors have often encountered discussions of a steer axis for an unsteered axle but these appear something of a *non sequitur*. It is generally true that a lateral force applied at some point will generate a compound lateral, camber and steer (toe) motion and it is perfectly reasonable to combine the lateral and toe motion to calculate some elastic axis in the same manner as described previously, save for the fact that there is no need to introduce an artificial aligning force. However, some practitioners erroneously advocate arbitrarily joining some suspension hard-points and declaring the resulting line a steering axis; it appears to escape their notice that more than one arbitrary connection of points is generally possible and that in any case the true elastic axis is not generally aligned with suspension hard-points.

Note that the formulations of suspension output that follow are for a quarter vehicle suspension model located on the right side of the vehicle using the general vehicle coordinate system in this text with the x-axis pointing to the rear, the y-axis to the side and the z-axis upwards. Needless to say users must ensure the formulations are consistent with the vehicle coordinate system and the side of the vehicle

being considered to ensure the correct sign for the calculated outputs. For each of the suspension characteristics discussed a typical system variable calculation is provided. This will assist users of MBS programs who need to develop their own calculations without access to the automated outputs of more elaborate software.

### 4.5.3 Bump movement, wheel recession and half track change

As stated earlier it can be the practice to impart vertical motion to a suspension system at either the wheel centre or base of wheel. In the following example the displacements at the wheel centre are used to determine the suspension movement. The displacements at the base of wheel would be corrected for camber, steer and castor angle changes and dependent on the suspension geometry. On the real vehicle the displacements of the tyre contact patch relative to the road wheel would also result due to the effects of tyre distortion. This is discussed later in Chapter 5.

Bump motion is the independent variable and is taken as positive as the wheel moves upwards in the positive  $z$ -direction relative to the vehicle body. Similarly wheel recession and half track change are taken as positive as the wheel moves back and outwards in the positive  $x$ - and  $y$ -directions respectively. The displacements are obtained simply by comparing the movement of a frame at the wheel centre relative to an initially coincident fixed frame on the ground. The displacements are shown in [Figure 4.26](#).

### 4.5.4 Camber and steer angle

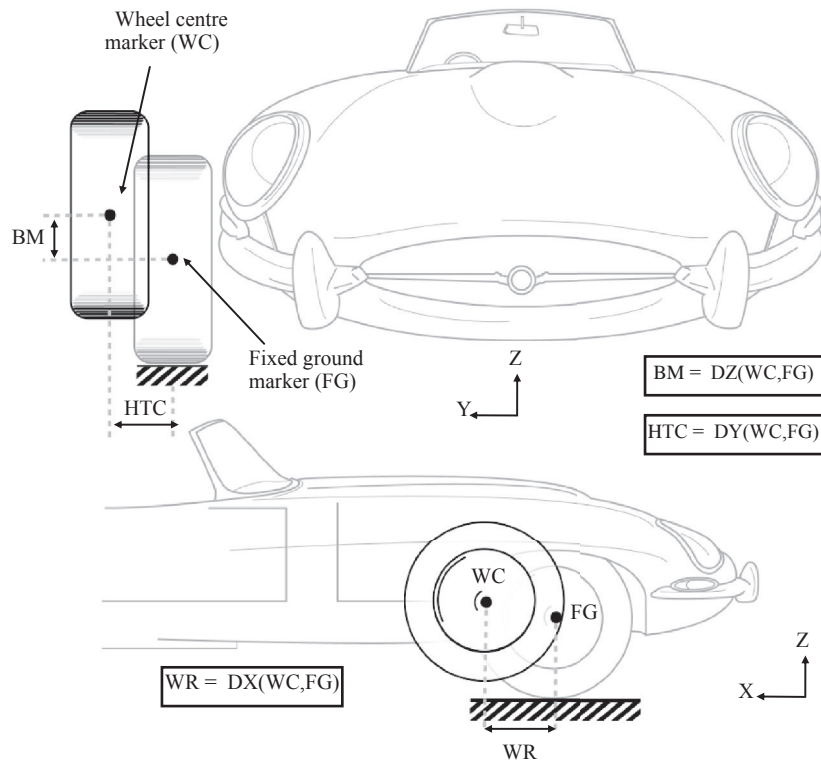
Camber angle,  $\gamma$ , is defined as the angle measured in the front elevation between the wheel plane and the vertical. Camber angle is measured in degrees and taken as positive if the top of the wheel leans outwards relative to the vehicle body as shown in [Figure 4.27](#).

The steer or toe angle,  $\delta$ , is defined as the angle measured in the top elevation between the longitudinal axis of the vehicle and the line of intersection of the wheel plane and road surface. Steer angle is taken here as positive if the front of the wheel toes in towards the vehicle.

Both camber and steer angle can be calculated using two frames located on the wheel spindle axis. In this case a frame is used at the wheel centre and another on the wheel spindle axis, taken in this example to be outboard of the wheel centre. The calculation of camber and steer angle is converted from radians to degrees by the factor  $(180/\pi)$ .

### 4.5.5 Castor angle and suspension trail

Castor angle,  $\phi$ , is defined as the angle measured in the side elevation between the steering (kingpin) axis and the vertical. Castor angle is measured in degrees and taken as positive if the top of the steering axis leans towards the rear as shown in [Figure 4.28](#). Castor is not to be confused with ‘caster’, which is a type of sugar for throwing (or ‘casting’) from a device specifically for that purpose (a ‘caster’).

**FIGURE 4.26**

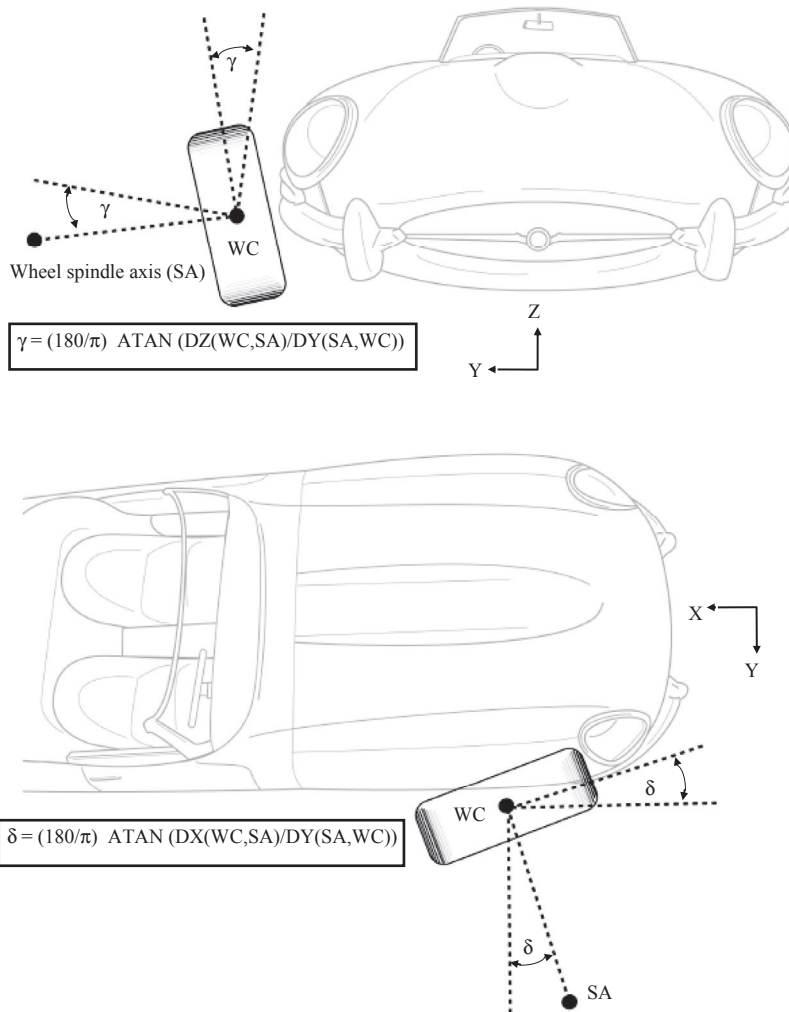
Bump movement (BM), wheel recession (WR) and half-track change (HTC).

Trail is the longitudinal distance in the x-direction between the wheelbase and the intersection between the steering axis and the ground. Trail generates a measure of stability providing a moment arm for lateral tyre forces that will cause the road wheels to 'centre'. Trail combines with tyre pneumatic trail, discussed in Chapter 5, and contributes to the steering 'feel'.

#### 4.5.6 Steering axis inclination and Ground offset

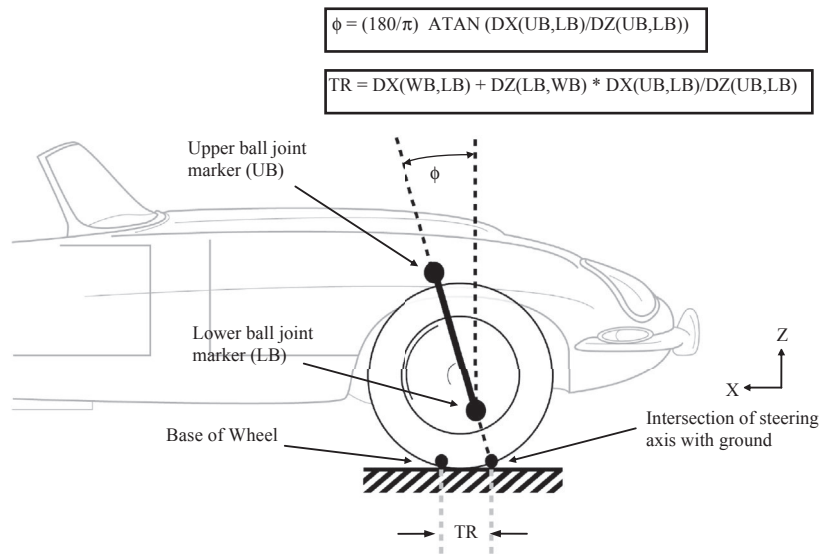
The steering axis inclination,  $\theta$ , is defined as the angle measured in the front elevation between the steering axis (sometimes referred to as the 'kingpin' axis from the days when a physical pin was used within a beam axle assembly) and the vertical. The angle is measured in degrees and taken as positive if the top of the steering axis leans inwards as shown in Figure 4.29.

Ground offset is the lateral distance in the y-direction between the wheelbase and the intersection between the steering axis and the ground. Ground offset is

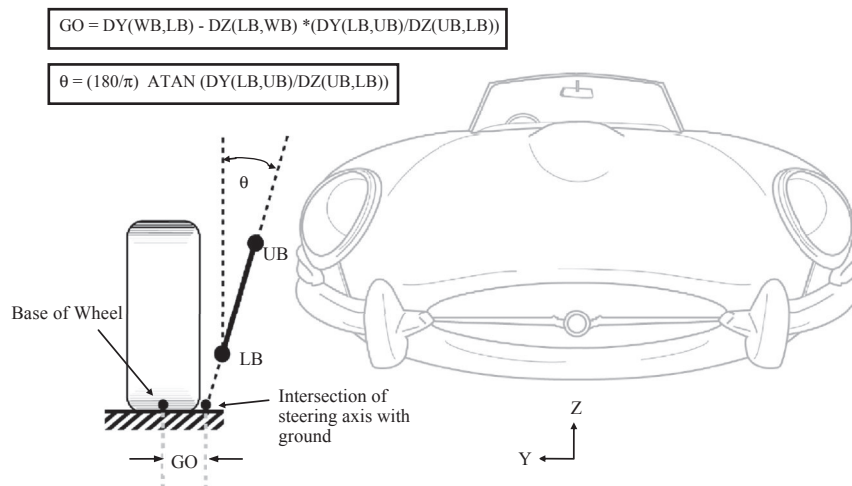
**FIGURE 4.27**

Calculation of Camber Angle  $\gamma$  and Steer Angle  $\delta$ .

sometimes also referred to as the scrub radius as the amount of ‘scrub’ in the tyre as it steers will depend on the Ground offset. Note that US practitioners sometimes use a different definition of scrub radius, which is the hypotenuse of a right triangle drawn using ground offset and trail from the steer axis intercept with the ground to the contact patch centre.


**FIGURE 4.28**

Calculation of Castor Angle  $\phi$  and Suspension Trail (TR).


**FIGURE 4.29**

Calculation of Steering Axis Inclination  $\theta$  and Ground Offset (GO).

### 4.5.7 Instant centre and roll centre positions

The determination of the instant centre and the roll centre position is more complicated than the previous calculations described here and the following is included to demonstrate an approach that can be used with MBS software to establish these positions.

The following text illustrates the traditional graphical (kinematic) type of construction as described by Gillespie (1992). The approach used to program the computations will require the definition of algebraic equations that calculate the gradients and intersection points of the lines used in the construction. There are two methods that may be used to achieve this in the absence of a dedicated program:

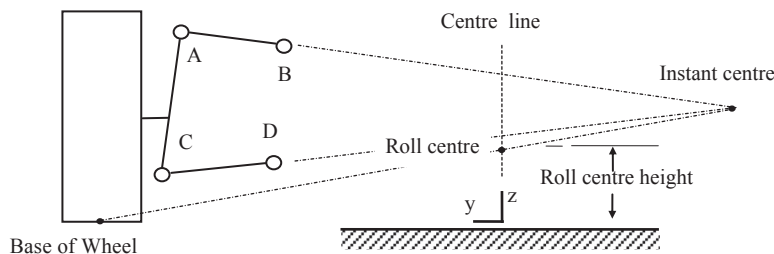
1. Programming in a run-time variable (solution state variable)
2. Preparing, compiling and linking a user-written subroutine

The methods used to formulate the construction will be dependent on the type of suspension system being considered. Contrast this with the generality of the half track change gradient method discussed in [Section 4.1.5](#). Examples are provided here for a double wishbone and a McPherson strut suspension system. For a double wishbone suspension the methods used to determine the instant centre and roll centre position for the front suspension are based on the construction shown in [Figure 4.30](#).

The instant centre is found by intersecting two lines projected along the upper and lower arms. The instant centre is the instantaneous centre of rotation for the complete suspension system. The suspension system can be thought of as an equivalent swing arm that pivots about the instant centre. As such the instant centre is also sometimes referred to as the effective swing arm pivot. The roll centre is found by projecting a line between the base of the wheel and the instant centre. The point at which this line intersects the centre line of the vehicle is taken to be the roll centre. Note earlier caveats on the physical significance of the roll centre.

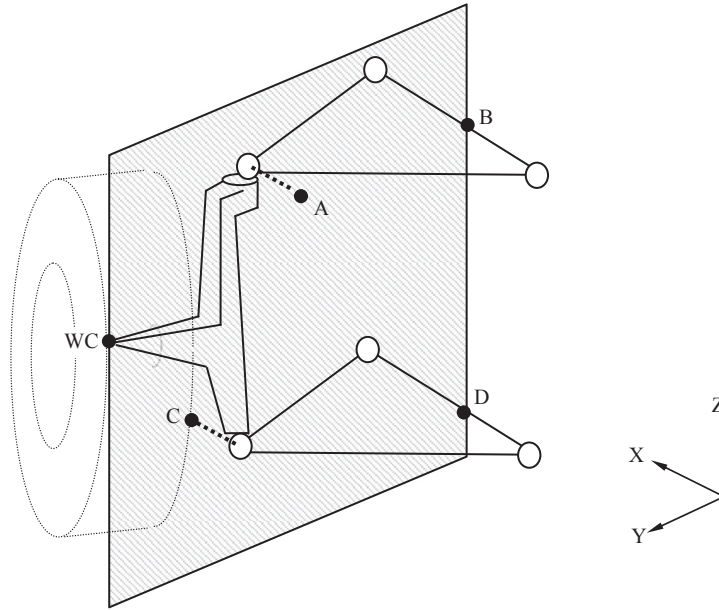
It should be noted that the two-dimensional representation shown in [Figure 4.30](#) is a simplification of the three-dimensional system and the graphical construction that takes place in a YZ plane passing through the wheel centre WC as shown in [Figure 4.31](#).

Since it cannot be assumed that the axes through the wishbone mount points are parallel to the x-axis the positions of points B and D will need to be obtained by



**FIGURE 4.30**

Instant centre and roll centre positions for a double wishbone suspension.

**FIGURE 4.31**

Position of instant centre construction points on wheel centre YZ plane.

interpolation to the wheel centre YZ plane. Positions A and C are found simply by projecting the upper and lower ball joints on to the same plane.

In order to program this construction algebraically the first step is to set up expressions for the gradients GR1 and GR2, of the upper and lower arms:

$$GR1 = (BZ - AZ) / (BY - AY) \quad (4.40)$$

$$GR2 = (DZ - CZ) / (DY - CY) \quad (4.41)$$

where

AY, AZ, BY, BZ, CY, CZ, DY, DZ are the y- and z-coordinates of points A, B, C and D.

The coordinates of the instant centre ICY and ICZ, can be established from two simultaneous equations based on the upper and lower arms:

$$ICZ = AZ + GR1 * (ICY - AY) \quad (4.42)$$

$$ICZ = CZ + GR2 * (ICY - CY) \quad (4.43)$$

Rearranging these two equations gives:

$$AZ + GR1 * ICY - GR1 * AY = CZ + GR2 * ICY - GR2 * CY \quad (4.44)$$

$$ICY * (GR1 - GR2) = GR1 * AY - GR2 * CY + CZ - AZ \quad (4.45)$$

which allows the instant centre to be located using:

$$ICY = (GR1 * AY - GR2 * CY + CZ - AZ) / (GR1 - GR2) \quad (4.46)$$

$$ICZ = AZ + GR1 * (ICY - AY) \quad (4.47)$$

The gradient of the line joining the wheelbase to the instant centre GR3, can be expressed as

$$GR3 = (ICZ - WBZ) / (ICY - WBY) \quad (4.48)$$

where WBY and WBZ are the y- and z-coordinates of the wheelbase.

This allows the roll centre to be located using:

$$RCY = 0.0 \quad (4.49)$$

$$RCZ = WBZ + GR3 * (RCY - WBY) \quad (4.50)$$

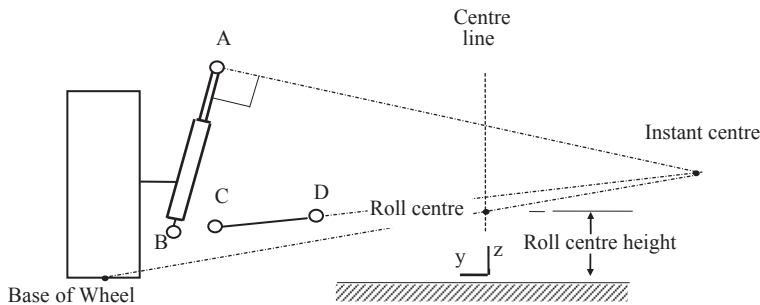
The roll centre height (RCH), can now be defined by

$$RCH = RCZ - RZ \quad (4.51)$$

where RZ is the z-coordinate of the road.

The methods used to determine the instant centre and roll centre position for a McPherson strut suspension are based on the construction shown in [Figure 4.32](#).

The instant centre is found by intersecting lines projected along the transverse arm and perpendicular to the axis of the strut. The roll centre is found by projecting a line between the base of the wheel and the instant centre. The point at which this line intersects the centre line of the vehicle is taken to be the roll centre. All calculations are assumed to take place in the a YZ plane containing the wheel centre.



**FIGURE 4.32**

Instant centre and roll centre positions for a McPherson Strut suspension.



The first step is to again set up expressions for the gradients GR1 for the line perpendicular to the strut and GR2 for the line projected along the transverse arm:

$$GR1 = (BY - AY) / (AZ - BZ) \quad (4.52)$$

$$GR2 = (DZ - CZ) / (DY - CY) \quad (4.53)$$

where

AY, AZ, BY, BZ, CY, CZ, DY, DZ are the y- and z-coordinates of points A, B, C and D.

The coordinates of the instant centre ICY and ICZ can be established from two simultaneous equations based on the upper and lower arms:

$$ICZ = AZ + GR1 * (ICY - AY) \quad (4.54)$$

$$ICZ = CZ + GR2 * (ICY - CY) \quad (4.55)$$

Rearranging these two equations gives:

$$AZ + GR1 * ICY - GR1 * AY = CZ + GR2 * ICY - GR2 * CY \quad (4.56)$$

$$ICY * (GR1 - GR2) = GR1 * AY - GR2 * CY + CZ - AZ \quad (4.57)$$

This allows the instant centre to be located using:

$$ICY = (GR1 * AY - GR2 * CY + CZ - AZ) / (GR1 - GR2) \quad (4.58)$$

$$ICZ = AZ + GR1 * (ICY - AY) \quad (4.59)$$

The gradient of the line joining the base of the wheel to the instant centre GR3 can be expressed as

$$GR3 = (ICZ - WBZ) / (ICY - WBY) \quad (4.60)$$

where WBY and WBZ are the y- and z-coordinates of the base of the wheel.

This allows the roll centre to be located using:

$$RCY = 0.0 \quad (4.61)$$

$$RCZ = WBZ + GR3 * (RCY - WBY) \quad (4.62)$$

The RCH can be defined by:

$$RCH = RCZ - RZ \quad (4.63)$$

where RZ is the z-coordinate of the road.

As stated earlier, the calculation of the instant centre and roll centre position can be implemented either by programming a run-time variable directly or by preparing

**Table 4.5** Calculation of Roll Centre Height Using a Run-Time Variable in a Solver-Level Set of Commands

```

VAR/14,IC=1,FU=DZ(1414,1411)/(DY(1414,1411)+1E-6)      ! GR1
VAR/15,IC=1,FU=DZ(1216,1213)/(DY(1216,1213)+1E-6)      ! GR2
VAR/16,IC=1,FU=((VARVAL(14)*DY(1411))
,-(VARVAL(15)*DY(1213))+DZ(1213)
,-DZ(1411))/(VARVAL(14)-VARVAL(15)+1E-6)
VAR/17,FU=DZ(1411)+VARVAL(14)*(VARVAL(16)-DY(1411))    ! ICZ
VAR/18,FU=(VARVAL(17)-DZ(1029))/(VARVAL(16)-
DY(1029)+1E-6)
VAR/19,FU=DZ(1029)+VARVAL(18)*(0.0-DY(1029))           ! RCZ
VAR/20,FU=VARVAL(19)+152.6                               ! RCH
REQ/1,F2=VARVAL(16)\F3=VARVAL(17)\F4=VARVAL(20)\
,TITLE=NULL:ICY:ICZ:RCH:NULL:NULL:NULL:NULL

```

a user-written subroutine. By way of example these methods are demonstrated for a front suspension system only. Using the run-time variable statement it is possible to program the equations laid out for the double wishbone system as shown in [Table 4.5](#).

Variables such as BZ-AZ are defined using system variables which measure components of displacements between markers, such as DZ(1414,1411). The REQUEST statement REQ/1 demonstrates how to access the information calculated by the VAR statements.

The alternative method of writing a subroutine is demonstrated in [Table 4.6](#) by the listing of a user written REQSUB developed specifically for a double wishbone suspension. The subroutine would be called from the main data set as follows:

```

REQUEST/id,FUNCTION=USER(1,par1,par2,par3,par4,par5,par6,par7,par8,
par9)

```

Where the parameters par1, par2,...par9 are the various items of data outlined in the subroutine.

#### 4.5.8 Calculation of wheel rate

The wheel rate for a suspension system can be thought of as the stiffness of an 'equivalent' spring acting between the WC and the vehicle body as shown in [Figure 4.33](#). This is the definition most useful for developing basic full vehicle MBS models where the wheel will be modelled as rigid with a separate tyre model.

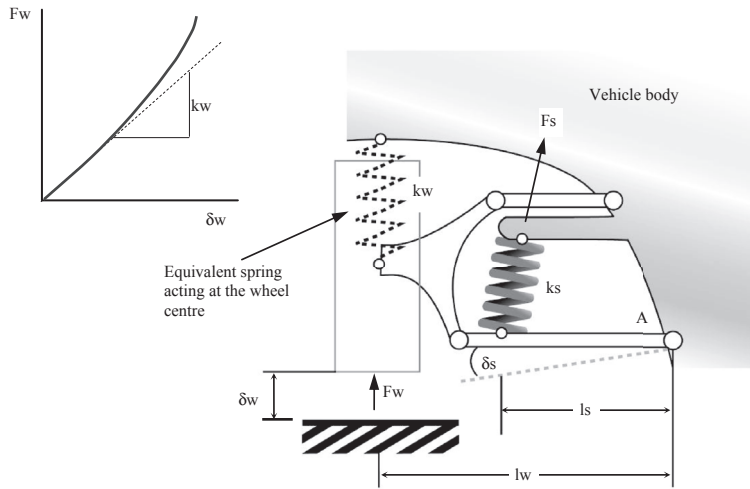
**Table 4.6** FORTRAN Subroutine to Calculate Roll Centre Height

```

SUBROUTINE REQSUB(ID,TIME,PAR,NPAR,
, IFLAG,RESULT)
C   M Blundell   Coventry University   Nov 1994
C   Calculation of Roll Centre Height and Instant
C   Centre Position -ROVER front suspension.
C
C   Definition of Parameters:
C   PAR(1) Subroutine id. Must be 1
C   PAR(2) WC marker
C   PAR(3) WB marker
C   PAR(4) Marker at point A
C   PAR(5) Marker at point B
C   PAR(6) Marker at point C
C   PAR(7) Marker at point D
C   PAR(8) Radius of wheel
C   PAR(9) RZ Height of Road in global Z
C
C   Results passed back to MSC.ADAMS as follows:
C   Note that the AView does not use
C   RESULT(1) or RESULT(5)
C
C   RESULT(2) Roll Centre Height above ground
C   RESULT(3) Roll Centre Z coordinate
C   RESULT(6) ICY coordinate
C   RESULT(7) ICZ coordinate
C
IMPLICIT DOUBLE PRECISION (A-H,O-Z)
DIMENSION PAR(*), RESULT(8)
LOGICAL IFLAG
DIMENSION DATA(6)
LOGICAL ERRFLG
C
IDWC=PAR(2)
IDWB=PAR(3)
IDA=PAR(4)
IDB=PAR(5)
IDC=PAR(6)
IDD=PAR(7)
RADIUS=PAR(7)
RZ =PAR(8)
CALL INFO ('DISP',IDWC,0,0,DATA,ERRFLG)
CALL ERRMES(ERRFLG,'WC ID',ID,'STOP')

WCX=DATA(1)
WCY=DATA(2)
WCZ=DATA(3)
CALL INFO('DISP',IDWB,0,0,DATA,ERRFLG)
CALL ERRMES(ERRFLG,'WB ID',ID,'STOP')
WBY=DATA(2)
WBZ=DATA(3)
CALL INFO ('DISP',IDA,0,0,DATA,ERRFLG)
CALL ERRMES(ERRFLG,'IDA',ID,'STOP')
AY=DATA(2)
AZ=DATA(3)
CALL INFO ('DISP',IDB,0,0,DATA,ERRFLG)
CALL ERRMES(ERRFLG,'IDB',ID,'STOP')
BY=DATA(2)
BZ=DATA(3)
CALL INFO ('DISP',IDC,0,0,DATA,ERRFLG)
CALL ERRMES(ERRFLG,'IDC',ID,'STOP')
CY=DATA(2)
CZ=DATA(3)
CALL INFO ('DISP',IDD,0,0,DATA,ERRFLG)
CALL ERRMES(ERRFLG,'IDD',ID,'STOP')
DY=DATA(2)
DZ=DATA(3)
GR1=(BZ-AZ)/(BY-AY)
GR2=(DZ-CZ)/(DY-CY)
RICY=((GR1*AY)-(GR2*CY)+CZ-AZ))
,/(GR1-GR2)
RICZ=AZ+GR1*(RICY-AY)
RCY=0.0
GR3=(RICZ-WBZ)/(RICY-WBY)
RCZ=WBZ+GR3*(RCY-WBY)
RCH=RCZ-RZ
RESULT(2)=RCH
RESULT(3)=RCZ
RESULT(6)=RICY
RESULT(7)=RICZ
RETURN
END

```

**FIGURE 4.33**

Equivalent spring acting at the wheel centre.

This differs slightly from other definitions sometimes used for wheel (or suspension rate) where the force displacement curve is measured at the centre of the tyre contact patch. In a quarter vehicle MBS model this would simply involve moving the point of jack contact with the wheel from the WC to the tyre contact patch.

The wheel rate should also not be confused with the term ride rate, which is associated with the force displacement relationship between the vehicle body, or sprung mass, and the ground. To derive ride rate with a quarter vehicle model it would be necessary to model an additional spring, representing the stiffness of the tyre, acting between the wheel centre and the jack with contact at the centre of the tyre contact patch. Note that the previously described quarter model implementation facilitates this readily.

The suspension outputs discussed until this point have been based on the suspension geometry and as such have not required the inclusion of the road spring in the model. By including the road spring and plotting the force against the displacement in the jack translational joint, the wheel rate may be obtained from the slope of the curve at the origin.

An estimate of the wheel rate may also be made as follows. Treating the road spring as linear gives the basic force displacement relationship

$$F_s = k_s \delta_s \quad (4.64)$$

For the equivalent spring we also have

$$F_w = k_w \delta_w \quad (4.65)$$

Taking moments about point A gives

$$F_w = (L_s/L_w) F_s \quad (4.66)$$

From the suspension geometry we can approximate the displacement in the road spring from

$$\delta s = (L_s/L_w) \delta w \quad (4.67)$$

This allows an estimate of the wheel rate,  $k_w$ , based on the road spring stiffness and suspension geometry from

$$k_w = F_w/\delta w = (L_s/L_w) F_s / (L_w/L_s) \delta s = (L_s/L_w)^2 k_s \quad (4.68)$$

The introduction of a square function in the ratio can be considered a combination of two effects:

1. The extra mechanical advantage in moving the road spring to the WC.
2. The extra spring compression at the WC.

---

## 4.6 The compliance matrix approach

The use of a compliance matrix, in programs such as the Milliken Research Vehicle Dynamic Modelling System (VDMS), is a method not commonly described in standard texts on vehicle dynamics but is well suited to an automated computer MBS analysis particularly when the influence of compliance requires consideration. The suspension compliance matrix relates incremental movements of the suspension to incremental forces applied at the wheel centres. The suspension compliance matrix is computed at each solution position as the suspension moves through its range of travel. Characteristics such as suspension ride rate and aligning torque camber compliance are computed based on the compliance matrix.

The compliance matrix for a suspension system,  $[C]$ , is defined as the partial derivatives of displacements with respect to applied forces

$$[C] = [\partial \Delta / \partial F] \quad (4.69)$$

If a system is assumed to be linear, the compliance matrix can be used to predict the system movement due to force inputs

$$\{\Delta\} = [C] \{F\} \quad (4.70)$$

Expanding Eqn (4.62) leads to a  $12 \times 12$  matrix relating the motion of the left and right wheel centres to unit forces and torques applied to the wheel centres. From this perspective, matrix element  $c_{ij}$  is the displacement of system degree of freedom  $i$  due to a unit force at degree of freedom  $j$  where the degrees of freedom

are the three displacements  $\Delta x$ ,  $\Delta y$  and  $\Delta z$  and the three rotations  $Ax$ ,  $Ay$  and  $Az$  at each of the left and right WCs.

$$\begin{bmatrix} \Delta x_{LW} \\ \Delta y_{LW} \\ \Delta z_{LW} \\ Ax_{LW} \\ Ay_{LW} \\ Az_{LW} \\ \Delta x_{RW} \\ \Delta y_{RW} \\ \Delta z_{RW} \\ Ax_{RW} \\ Ay_{RW} \\ Az_{RW} \end{bmatrix} = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} & C_{1,5} & C_{1,6} & C_{1,7} & C_{1,8} & C_{1,9} & C_{1,10} & C_{1,11} & C_{1,12} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} & C_{2,5} & C_{2,6} & C_{2,7} & C_{2,8} & C_{2,9} & C_{2,10} & C_{2,11} & C_{2,12} \\ C_{3,1} & C_{3,2} & C_{3,3} & C_{3,4} & C_{3,5} & C_{3,6} & C_{3,7} & C_{3,8} & C_{3,9} & C_{3,10} & C_{3,11} & C_{3,12} \\ C_{4,1} & C_{4,2} & C_{4,3} & C_{4,4} & C_{4,5} & C_{4,6} & C_{4,7} & C_{4,8} & C_{4,9} & C_{4,10} & C_{4,11} & C_{4,12} \\ C_{5,1} & C_{5,2} & C_{5,3} & C_{5,4} & C_{5,5} & C_{5,6} & C_{5,7} & C_{5,8} & C_{5,9} & C_{5,10} & C_{5,11} & C_{5,12} \\ C_{6,1} & C_{6,2} & C_{6,3} & C_{6,4} & C_{6,5} & C_{6,6} & C_{6,7} & C_{6,8} & C_{6,9} & C_{6,10} & C_{6,11} & C_{6,12} \\ C_{7,1} & C_{7,2} & C_{7,3} & C_{7,4} & C_{7,5} & C_{7,6} & C_{7,7} & C_{7,8} & C_{7,9} & C_{7,10} & C_{7,11} & C_{7,12} \\ C_{8,1} & C_{8,2} & C_{8,3} & C_{8,4} & C_{8,5} & C_{8,6} & C_{8,7} & C_{8,8} & C_{8,9} & C_{8,10} & C_{8,11} & C_{8,12} \\ C_{9,1} & C_{9,2} & C_{9,3} & C_{9,4} & C_{9,5} & C_{9,6} & C_{9,7} & C_{9,8} & C_{9,9} & C_{9,10} & C_{9,11} & C_{9,12} \\ C_{10,1} & C_{10,2} & C_{10,3} & C_{10,4} & C_{10,5} & C_{10,6} & C_{10,7} & C_{10,8} & C_{10,9} & C_{10,10} & C_{10,11} & C_{10,12} \\ C_{11,1} & C_{11,2} & C_{11,3} & C_{11,4} & C_{11,5} & C_{11,6} & C_{11,7} & C_{11,8} & C_{11,9} & C_{11,10} & C_{11,11} & C_{11,12} \\ C_{12,1} & C_{12,2} & C_{12,3} & C_{12,4} & C_{12,5} & C_{12,6} & C_{12,7} & C_{12,8} & C_{12,9} & C_{12,10} & C_{12,11} & C_{12,12} \end{bmatrix} \begin{bmatrix} Fx_{LW} \\ Fy_{LW} \\ Fz_{LW} \\ Tx_{LW} \\ Ty_{LW} \\ Tz_{LW} \\ Fx_{RW} \\ Fy_{RW} \\ Fz_{RW} \\ Tx_{RW} \\ Ty_{RW} \\ Tz_{RW} \end{bmatrix} \quad (4.71)$$

From Eqn (4.71) it can be seen that the coefficients on the leading diagonal of matrix  $[C]$  directly relate the displacement or rotation to the associated force or torque applied at that degree of freedom. For example, in the absence of any other forces or torques, the vertical motion of the left wheel centre due to a unit vertical force applied at the left wheel centre is given by  $\Delta z_{LW} = C_{3,3} Fz_{LW}$ . Figure 4.34 illustrates this for another example where the vertical motion of the left wheel centre due to the application only of a unit vertical force applied at the right wheel centre

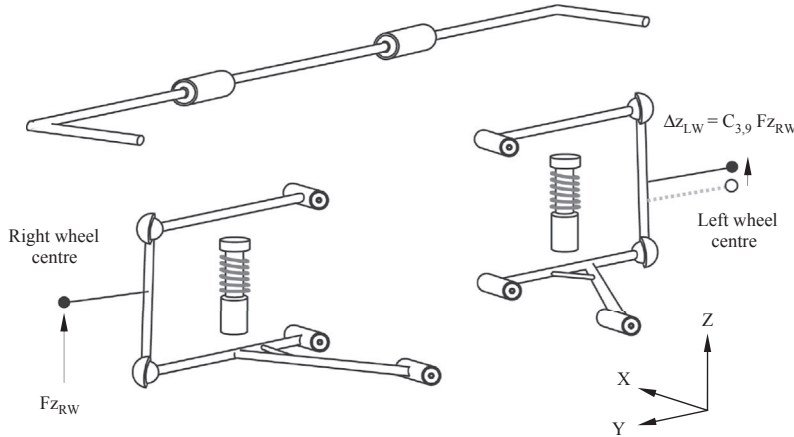


FIGURE 4.34

Application of compliance matrix to suspension system vehicle half model.

given by  $\Delta z_{LW} = C_{3,9} F_{zRW}$ . From Figure 4.34 it can be seen that for an independent suspension without a rollbar  $C_{3,9}$  would be zero in the absence of any mechanical coupling between the left and right suspension systems. The other elements of the compliance matrix are defined similarly.

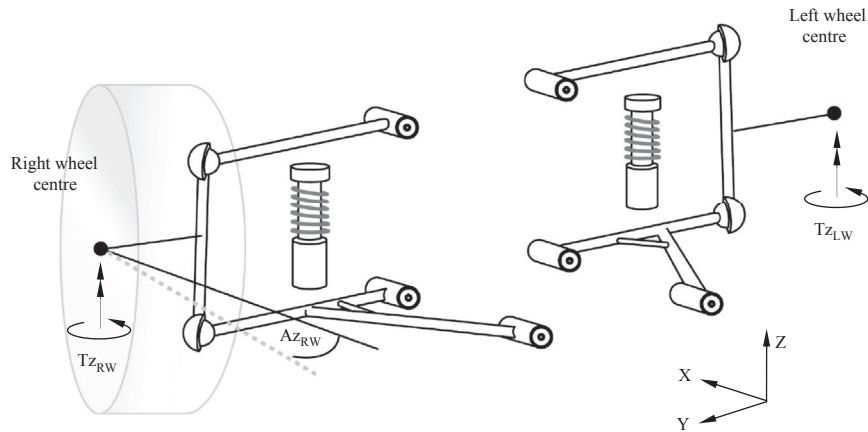
As stated, the compliance matrix approach is well suited to investigate the effects of suspension movement due to compliance. By way of further example consider the definition used in ADAMS/Car for the calculation of aligning torque — steer and camber compliance.

The aligning torque steer compliance is the change in steer angle due to unit aligning torques applied through the wheel centres. Similarly the aligning torque camber compliance is the change in camber angle due to unit aligning torques acting through the wheel centres.

Figure 4.35 illustrates the determination of steer angle resulting at the right wheel due to unit aligning torques acting through both the left and right wheel centres. Note that the usual symbol for steer angle is  $\delta$ . For the matrix approach used here however this is given by  $Az_{RW}$ . In this system a positive steer angle results when the wheel turns to the left, which in Figure 4.35 is consistent with a positive rotation  $Az_{RW}$  about the z-axis for the right wheel. In this case for the right wheel the steer angle would be given by  $Az_{RW} = C_{12,6} Tz_{LW} + C_{12,12} Tz_{RW}$ .

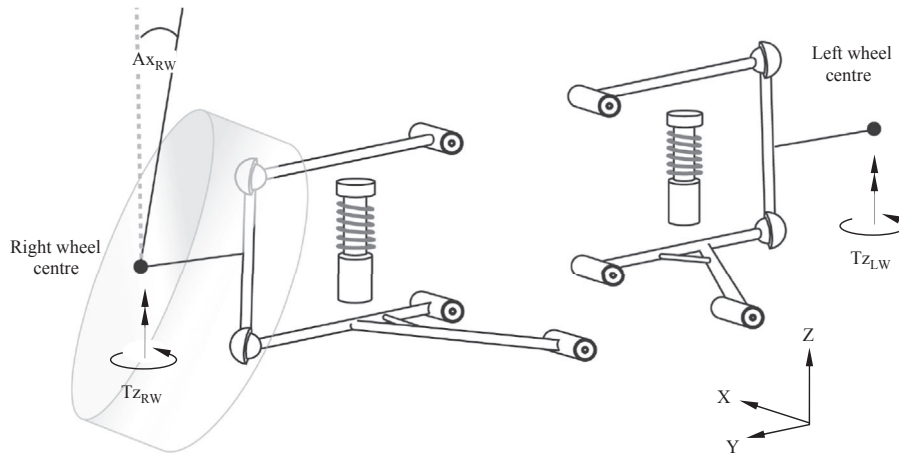
Similarly Figure 4.36 illustrates the determination of camber angle,  $Ax_{RW}$ , resulting at the right wheel due to unit aligning torques acting through both the left and right wheel centres.

In this system a positive camber angle results when the top of the wheel tilts away from the body, which in Figure 4.36 would actually be a negative rotation  $Ax_{RW}$



**FIGURE 4.35**

Steer angle at right wheel due to aligning torques at left and right wheels.

**FIGURE 4.36**

Camber angle at right wheel due to aligning torques at left and right wheels.

about the x-axis for the right wheel. In this case for the right wheel the camber angle would be given by  $A_{x_{RW}} = C_{10,12} T_{z_{RW}} - C_{10,6} T_{z_{LW}}$ .

The sign convention used to define positive steer and camber angles always requires careful consideration particularly when considering the definitions given here using a compliance matrix approach to measure movement of the road wheels relative to the vehicle body.

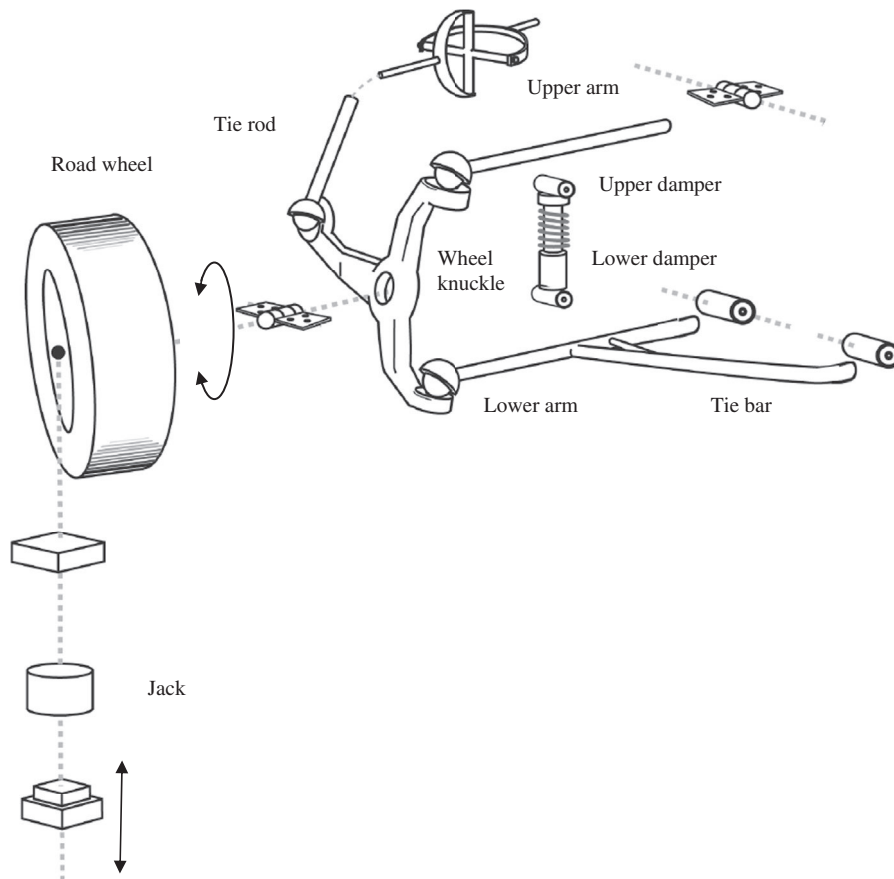
## 4.7 Case study 1 – suspension kinematics

The following case study is provided to illustrate the application of the method described in the previous sections to calculate the suspension characteristics as the suspension moves between the bump and rebound positions. Examples of the plotted outputs described here are shown in Figures 4.40–4.45. These plots were from a study based on the front suspension of a passenger car, considering the suspension connections to be joints, linear or nonlinear bushes. The assembly of parts used to make up the front suspension system is shown schematically in Figure 4.37. Example data sets for this model are provided in Appendix A together with more detailed system schematics.

The modelling of the suspension system using bushes is shown in Figure 4.38. The upper link is attached to the body using a connection that is rigid enough to be modelled as a revolute joint. Bushes are used to model the connection of the lower arm and the tie bar to the vehicle body.

Bushes are also used to model the connections at the top and bottom of the damper unit. Where the tie bar is bolted to the lower arm a fix joint has

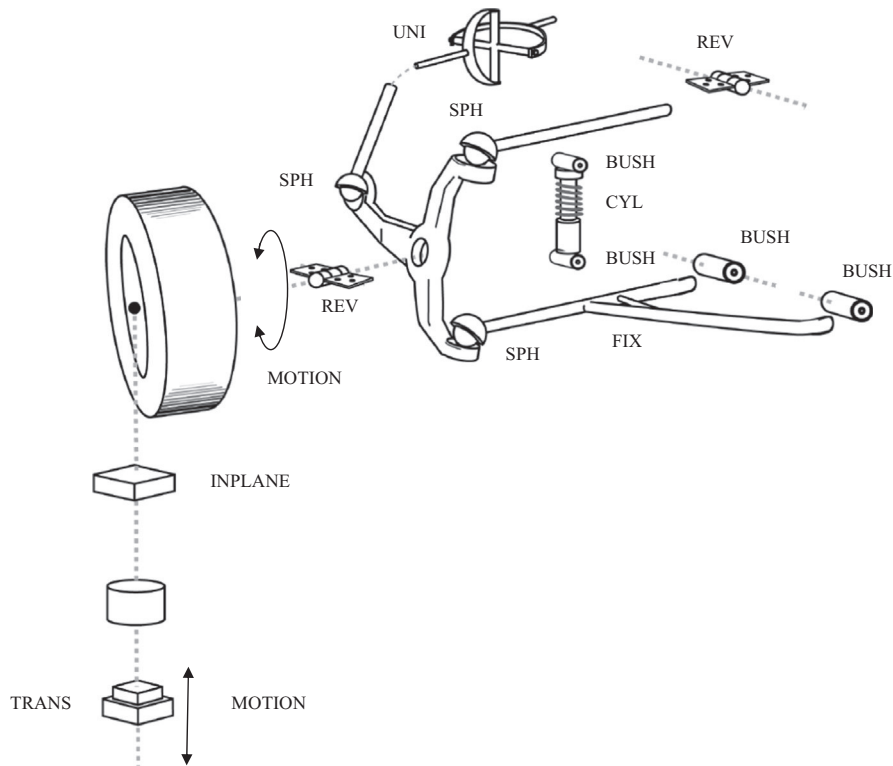


**FIGURE 4.37**

Assembly of parts in the front suspension system example.

been used to rigidly connect the two parts together. This joint removes all six relative degrees of freedom between the two parts, creating in effect a single lower control arm.

The modelling issue raised here is that rotation will take place about an axis through these two bushes where the bushes are not aligned with this axis. As rotation takes place the bushes must distort in order to accommodate this. The modelling of these connections as nonlinear, linear or as a rigid joint was therefore investigated to establish the effects on suspension geometry changes during vertical movement. For the suspension modelled in this manner it is possible to calculate the degrees of freedom for the system as follows:



**FIGURE 4.38**

Modelling the front suspension example using bushes. SPH, Spherical Joint; REV, Revolute Joint; UNI, Universal Joint; CYL, Cylindrical Joint; FIX, Fix Joint; TRANS, Translational Joint.

$$\text{Parts} \quad 9 \times 6 = 54$$

$$\text{Fix} \quad 1 \times -6 = -6$$

$$\text{Trans} \quad 1 \times -5 = -5$$

$$\text{Rev} \quad 2 \times -5 = -10$$

$$\text{Uni} \quad 1 \times -4 = -4$$

$$\text{Cyl} \quad 1 \times -4 = -4$$

$$\text{Sphs} \quad 3 \times -3 = -9$$

$$\text{Inplane} \quad 1 \times -1 = -1$$

$$\text{Motion} \quad 2 \times -1 = -2$$

$$\Sigma_{\text{DOF}} = 13$$

In order to produce a zero degree of freedom model for this suspension the bushes at the top and bottom of the strut have been replaced by a universal and a spherical joint.

The bushes that were used to connect the lower arm and the tie rod assembly to the vehicle body were replaced in this study by a revolute joint. The axis of this joint was aligned between the two bushes as shown in [Figure 4.39](#). For the suspension modelled in this manner using rigid joints it is possible to calculate the degrees of freedom for the system as follows:

$$\text{Parts} \quad 9 \times 6 = 54$$

$$\text{Fix} \quad 1 \times -6 = -6$$

$$\text{Trans} \quad 2 \times -5 = -10$$

$$\text{Rev} \quad 3 \times -5 = -15$$

$$\text{Uni} \quad 2 \times -4 = -8$$

$$\text{Sphs} \quad 4 \times -3 = -12$$

$$\text{Inplane} \quad 1 \times -1 = -1$$

$$\text{Motion} \quad 2 \times -1 = -2$$

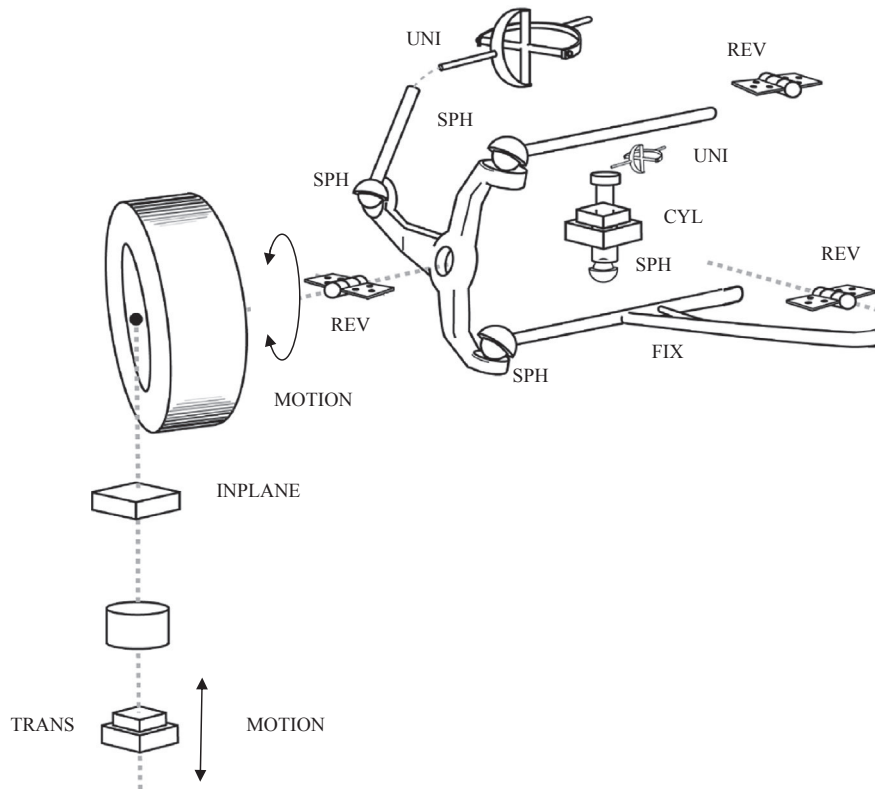
$$\Sigma_{\text{DOF}} = 0$$

For this suspension it was possible to compare the simulation results with measured suspension rig test data provided by the vehicle manufacturer for the variation of:

- Camber angle (deg) with Bump motion (mm) ([Figure 4.40](#))
- Caster angle (deg) with Bump motion (mm) ([Figure 4.41](#))
- Steer angle (deg) with Bump motion (mm) ([Figure 4.42](#))
- Roll centre height (mm) with Bump motion (mm) ([Figure 4.43](#))
- Half track change (mm) with Bump motion (mm) ([Figure 4.44](#))
- Vertical force (N) with Bump motion (mm) ([Figure 4.45](#))

Examination of the results shown here indicates that despite the alignment of the bushes on the lower arm assembly the calculated suspension characteristics agree well for models using rigid joints, linear bushes or nonlinear bushes. It is noticeable with the front suspension that the plots begin to deviate when approaching the full bump or full rebound positions. This is due to contact with the bump stop or rebound stop generating forces that are then reacted back through the suspension to the bushes. The reaction forces at the bushes lead to distortions that produce the changes in suspension geometry as shown in the plots. This effect is not present in the models using rigid joints that have zero degrees of freedom. Geometry changes are entirely dependent on the position and orientation of the joints.

Considering the merits of each modelling approach it appears from the curves plotted that for the range of vertical movement expected of a handling model there

**FIGURE 4.39**

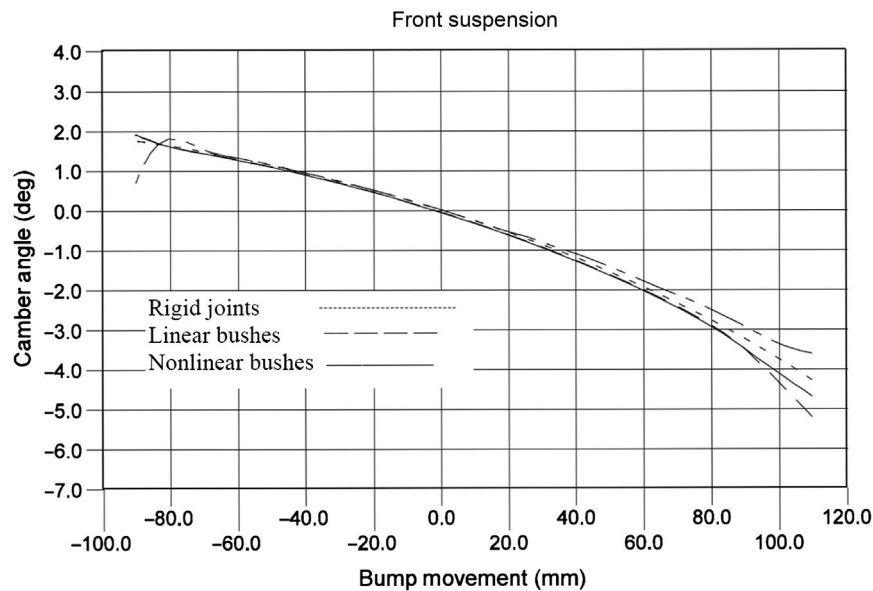
Modelling the front suspension example using rigid joints. SPH, Spherical Joint; REV, Revolute Joint; UNI, Universal Joint; CYL, Cylinder; FIX, Fix Joint; TRANS, Translational Joint.

is little difference between models using rigid joints, linear bushes or nonlinear bushes. The use of the nonlinear model will significantly increase the effort required to model the vehicle. This is evident from [Table 4.4](#) which compares the data inputs required to model the connection of the front suspension lower arm to the vehicle body.

## 4.8 Durability studies (component loading)

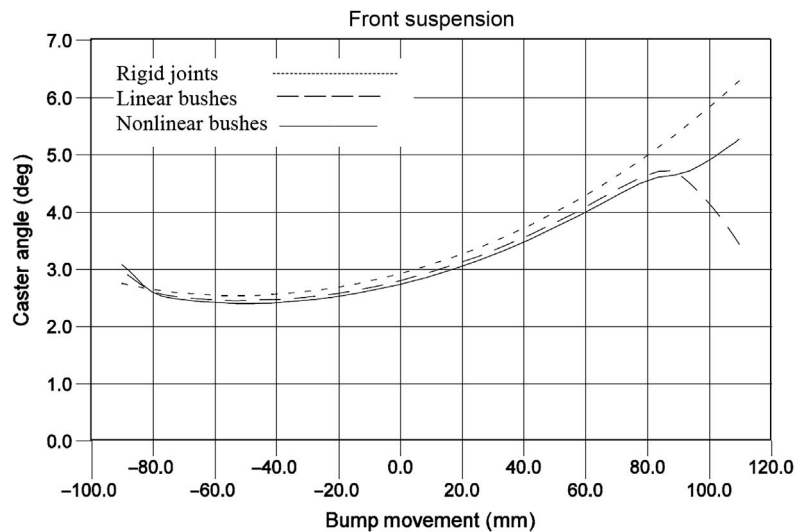
### 4.8.1 Overview

Early in the life of computer-aided engineering (CAE) tools, MBS programs were often used to determine the loads acting on suspension components and the body pickup points as inputs to subsequent FE analysis of the components or vehicle structure. Current technology allows a rather more synchronous process to be adopted, notionally allowing embedded FE models to recover stress states and even fatigue damage estimates during a multibody solution run. However, in order for



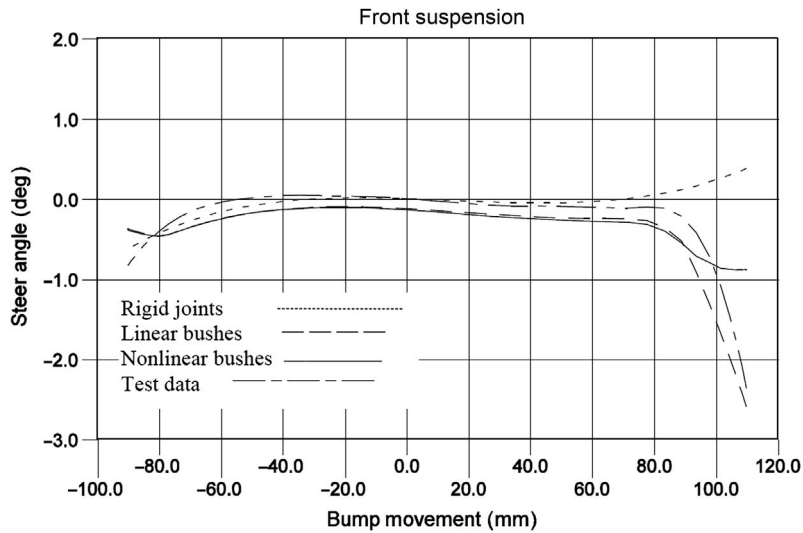
**FIGURE 4.40**

Front suspension — camber angle with bump movement.



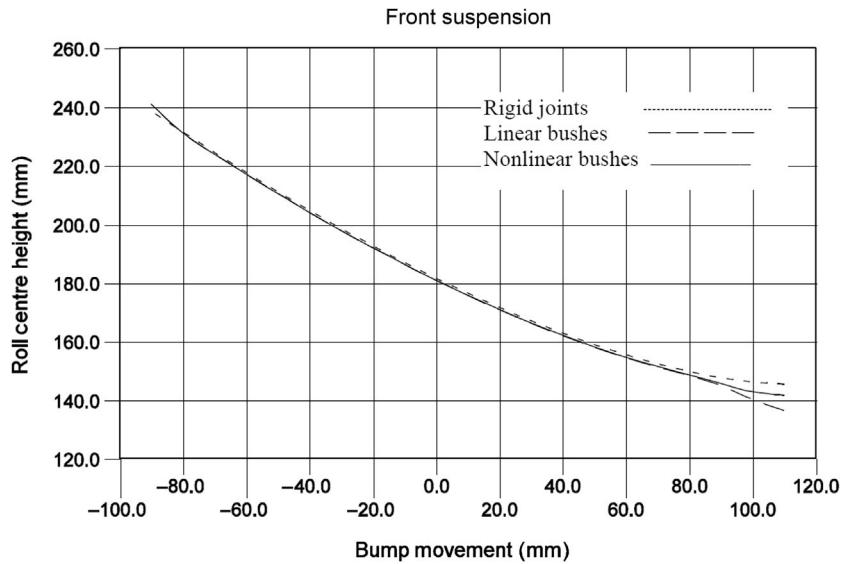
**FIGURE 4.41**

Front suspension — caster angle with bump movement.



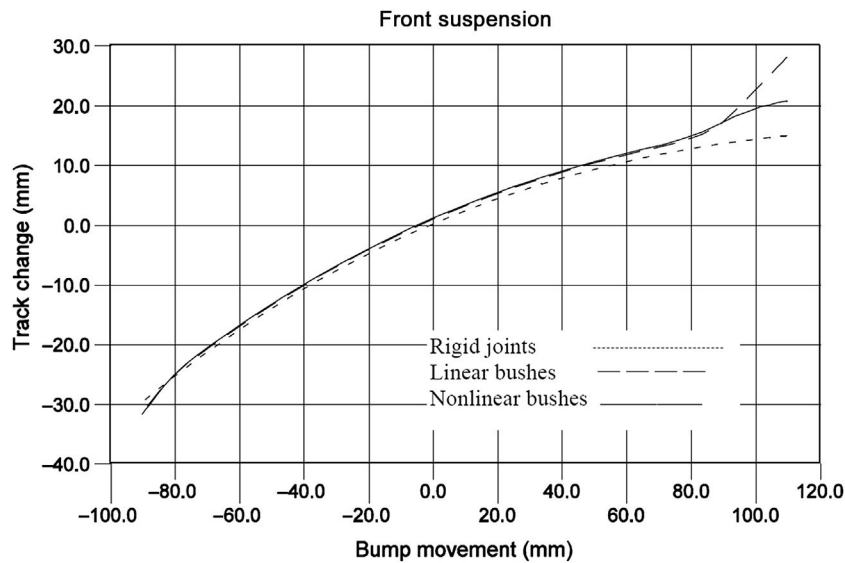
**FIGURE 4.42**

Front suspension — steer angle with bump movement.

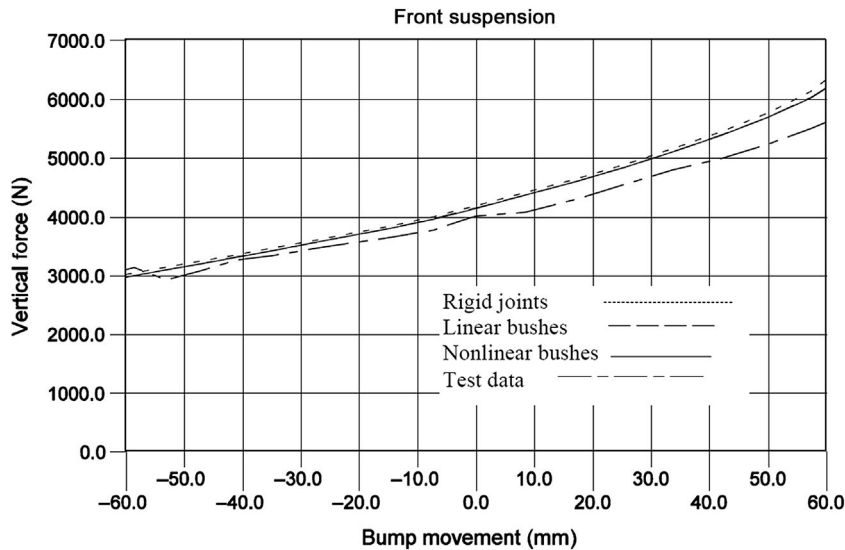


**FIGURE 4.43**

Front suspension — roll centre height with bump movement.



**FIGURE 4.44**  
Front suspension — half-track change with bump movement.



**FIGURE 4.45**  
Front suspension — vertical force with bump movement.

this to be possible, viable models — and therefore designs — of all the components of interest need to exist. There remains a place for what is sometimes referred to as ‘first-sight’ analysis to usefully estimate applied loads in order to size components of mechanisms, particularly when engineering a class of vehicle that is new to the organisation.

Such simulations intend to reflect the outlying (most extreme) events during the series of tests that a vehicle manufacturer would perform on the proving ground to test the durability of the vehicle. During the test the vehicle will be exposed to a variety of loads in various directions, the entire sequence of which is expected to leave the vehicle undamaged. The durability test is formulated to reflect the loads the vehicle will see in service and so these durability test loads might be called ‘service loads’. A substantial topic in its own right, the process of ensuring a durability test matches the actual loads imposed on the vehicle in service is a specialist discipline and not within the scope of this chapter.

Different manufacturers will implement their own procedures but typically these will involve establishing wheel centre loads resulting for example from accelerating, braking, cornering and driving on rough surfaces such as that shown in [Figure 4.46](#). The limited set of loads that notionally represent the outlying values from the service load history might be called ‘design loads’.

These loads are distinct from loads to which the vehicle might foreseeably be exposed to but from which it is not expected to emerge fully serviceable. Product liability legislation (and ethical vehicle engineering) requires that readily foreseeable events are managed in a safe way, even if the vehicle is damaged in the process. These events might be termed ‘abuse’ loads. In a mainstream automotive context, these include large, hard-edged potholes struck under braking at gross vehicle weight (GVW), jump events and so on. Note that in other contexts — for example rallying



**FIGURE 4.46**

Vehicle durability testing on a rough surface — the Mini WRC undergoes early development testing in February 2011. Picture: Harty.



or Baja racing — jumping may be regarded as a normal service load and not an abuse load.

It may be readily imagined that there is a somewhat blurred line between a large abuse event and a small crash. The large redistribution of loads that occurs when significant plastic deformation occurs is not typically suited to solution with MBS software, even when linear finite models are embedded and so there comes a point — usually determined by the amount of plasticity involved — when the analysis of such events is handed over to an explicitly formulated FE code. Such matters are beyond the scope of this chapter and so the subsequent discussion will confine itself to events where events are substantially elastic (i.e. deformations are recovered once loads are removed).

In rallying in particular, the neat definition of service and abuse loads is not really possible and so the somewhat fatalistic approach must be taken that ‘there will always be a bigger rock’; the selection of design loads becomes somewhat arbitrary but over time, guidelines emerge that allow designs which are robust enough for the intended application without becoming overly heavy.

In [Figure 4.47](#), note the total absence of the front left suspension and the complete but bent rear left suspension unit. The front suspension suffered a catastrophic overload when colliding with a large bank after the test driver, Markko Martin, lost control of the car; the subsequent rotation of the car gave a slightly less large impact at the rear, which was big enough to activate the mechanical ‘fuse’ — the rear toe link — without doing other damage to the suspension sub-frame. The author (Harty) observed all these events from the co-driver’s seat and is grateful (thanks, Jonathan Culwick!) for the roll cage optimisation work that went into the design. Nine hours after this picture was taken, the vehicle ran again to continue testing.

The loads that are applied to the suspension may be considered to act at the tyre contact patch or at the wheel centre depending on the type of loadcase. The loads shown in [Table 4.7](#) are typical of those that might be used for a static analysis on a vehicle of the type for which data are provided in this textbook. In this example, the loads are defined in the x-, y- and z-directions for a coordinate frame located at the centre of the tyre contact patch as shown in [Figure 4.48](#).

For the loads shown in [Table 4.7](#) it is possible to calculate values for cases such as cornering and braking using traditional vehicle dynamics and the principles of weight transfer. For cases involving impacts with kerbs and bumps it may be necessary to obtain instrumented road load measurements on the proving ground, or else arbitrary figures may be used. A particularly severe case involves braking while driving through a pothole. To simulate this sort of case the input loads at the contact patch may be set to produce forces, say acting along a tie rod, that are consistent with measured strains on the actual component during the proving ground tests.

The purpose of the MBS model of the suspension in this case is to obtain the distribution of the load through the suspension. This is illustrated in [Figure 4.48](#) where it is indicated that for a given set of loads at the tyre contact patch it is possible to predict the forces and moments that would for example act through the bushes mounting the suspension arm to the body of the vehicle.

**FIGURE 4.47**

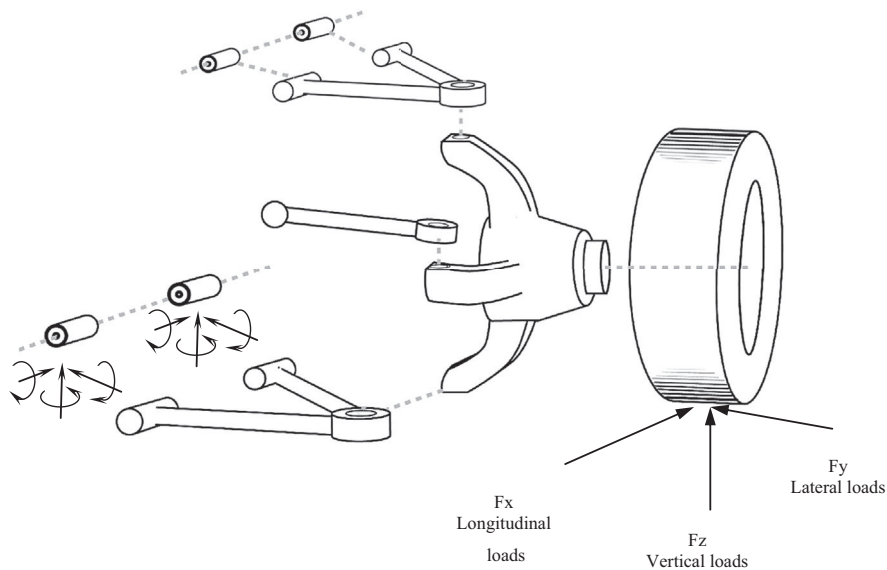
This is not a service load event, nor even a design load event — this is most definitely abuse.

These forces and moments can then be used as boundary conditions for FE models of vehicle structure or of suspension components. A typical FE model of a suspension arm is shown in [Figure 4.49](#).

Once generated the suspension model may be used in two ways. One of these is to apply the load and carry out a static analysis. This may result in the suspension system moving through relatively large displacement to obtain a static equilibrium position for the given load. The static reactions in joints, bushes and spring seats can then be extracted.

The use of equivalent static loads to represent real dynamic effects has been used by the automotive industry for some time, as previously noted in [Section 4.1.6](#). The disadvantage of the static analysis is that any velocity-dependent forces will not be transmitted through the damper without the inclusion of some static equivalent. The author's experience has been to include some static nominal force at the damper in order to exercise damper mounts in some representative manner; this is of less importance when the spring and damper act in a collinear way.

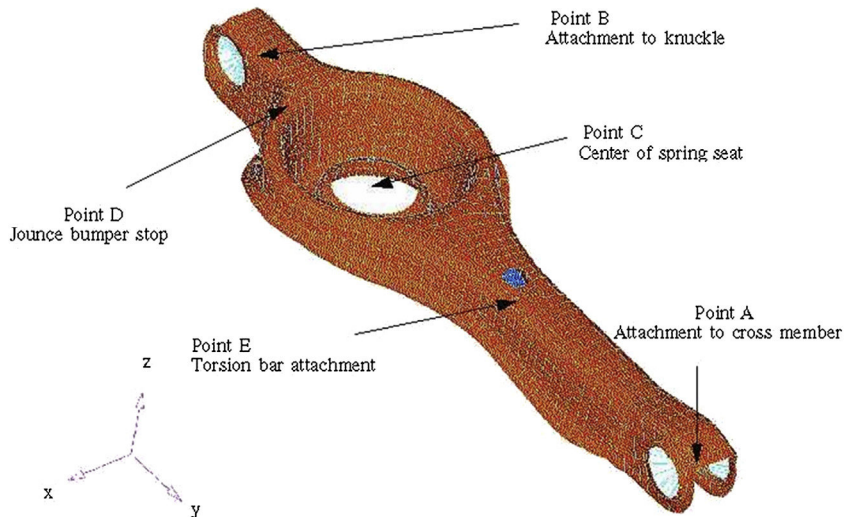
Table 4.7 Example Suspension Loadcases			
Loadcase	Fx (N)	Fy (N)	Fz (N)
3G bump			11180
2G rebound			−7460
0.75G cornering (outer wheel)		4290	5880
0.75G cornering (inner wheel)		−1180	1620
1G braking	5530		5530
0.35G reverse braking	−2150		3330
Kerb impact		9270	4120
Pothole braking	15900		12360



**FIGURE 4.48**

Application of road loads at the tyre contact patch.

Table 4.7 can easily be generated using a spreadsheet from basic vehicle parameters such as mass, mass centre height, wheelbase and track. With a little planning, the spreadsheet can be used to write out a solution command file to drive the software to perform the required commands automatically. Some ‘single-purpose general-purpose’ software packages (see Chapter 1) allow such factors to be entered automatically. This means the loading tables can be generated as soon as the basic proportions of the vehicle are known and allows immediate comparison with a known vehicle. This is important if, for example, some judgement is required on

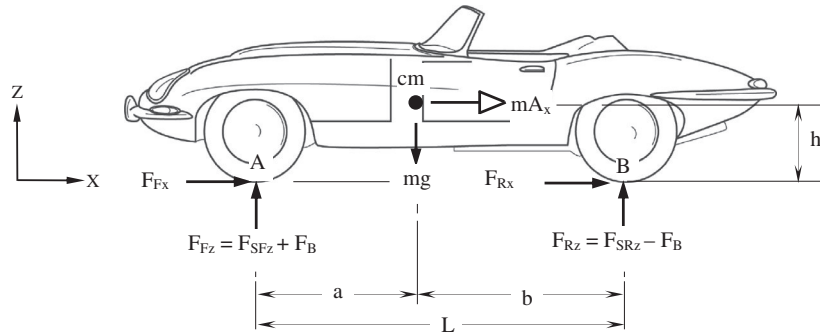
**FIGURE 4.49**

Finite element model of suspension arm.

*Provided courtesy of Jaguar Cars Ltd.*

the reuse of a suspension system from one platform to another; the judgement can be made instantly without waiting for weeks to know the results of the FE analysis, or previously known linear results can be scaled by load factors to give an immediate judgement. This kind of rapid feedback is deeply important in the early phases of a vehicle design when strategic choices are being made.

A second method, the extension of the static model to carry out a dynamic simulation, is possible. The model used may be a quarter suspension model or even a simulation of the complete vehicle. An example of this would be the simulation of a sports utility type vehicle in off-road conditions. An early example of this (Rai and Soloman, 1982) was the use of MSC ADAMS to carry out dynamic simulation of suspension abuse tests. A reaction force time history output generated at bush and joint positions may be used in downstream fatigue analysis for individual components. In order to be really useful, the enveloping behaviour of the tyre over small obstacles needs to be well captured — see Chapter 5 for discussion on this matter. As a pragmatic halfway house, load cells built into special purpose wheels can be used to capture the load history as seen by a wheel and this history ‘played in’ to a dynamic model in order to distribute the loads dynamically through the system and obtain load histories for each component. This is typically much more successful than the fully predictive approach but needs a vehicle of broadly representative proportions and a usefully close rim and tyre configuration to work. Some cautions remain, in particular the damping behaviour of the elastomers, about which more is written in Chapter 3.

**FIGURE 4.50**

Forward braking free body diagram.

The dynamic simulation of loads requires usefully accurate estimates of mass and inertia properties of components, which may not exist early in the design phase. Thus the static simulation – the so-called ‘first sight’ approach – remains useful in order to get started from a clean sheet. The concept of useful accuracy is discussed at some length in Chapter 1.

The input loads used to represent braking and cornering may be obtained using basic vehicle data and weight transfer analysis. As an example consider the free body diagram shown in Figure 4.50 where the wheel loads are obtained for a vehicle braking case. Note that in this example we are ignoring the effects of rolling resistance in the tyre, discussed later in Chapter 5, and the vehicle is on a flat road with no incline.

The vertical forces acting on the front and rear tyres when the vehicle is at rest can be found by the simple application of static equilibrium. These forces,  $F_{SFz}$  for the front wheel and  $F_{SRz}$  for the rear wheel, are given by

$$F_{SFz} = \frac{m g b}{2 L} \qquad F_{SRz} = \frac{m g a}{2 L} \qquad (4.72)$$

It can be noted that the division of the loads by 2 in Eqn (4.72) is simply to reflect that we are dealing with a symmetric case so that half of the mass is supported by the wheels on each side of the vehicle. In this analysis the vehicle will brake with a deceleration  $A_x$  as shown in Figure 4.50. During braking weight transfer will result in an increase in load on the front tyres by an additional load  $F_B$  and a corresponding reduction in load  $F_B$  on the rear tyres. This can be obtained by taking moments about either wheel, for  $m A_x$  only and not  $mg$ , to give

$$F_B = \frac{m A_x h}{2 L} \qquad (4.73)$$

Note that in determining the longitudinal braking forces  $F_{Fx}$  and  $F_{Rx}$  we have a case of indeterminacy with four unknown forces and only three equations of static equilibrium. The solution is found using another relationship to represent the

relationship between the braking and vertical loads. At this stage we will assume that the braking system has been designed to proportion the braking effort so that the coefficient of friction  $\mu$  is the same at the front and rear tyres. The generation of longitudinal braking force in the tyre is generally not this straightforward and will be covered in the next chapter. We can now combine the static and dynamic forces acting vertically on the wheels to give the full set of forces:

$$F_{Fz} = F_{SFz} + F_B = \frac{m g b}{2 L} + \frac{m A_x h}{2 L} \quad (4.74)$$

$$F_{Rz} = F_{RFz} - F_B = \frac{m g a}{2 L} - \frac{m A_x h}{2 L} \quad (4.75)$$

$$F_{Fx} + F_{Rx} = m A_x \quad (4.76)$$

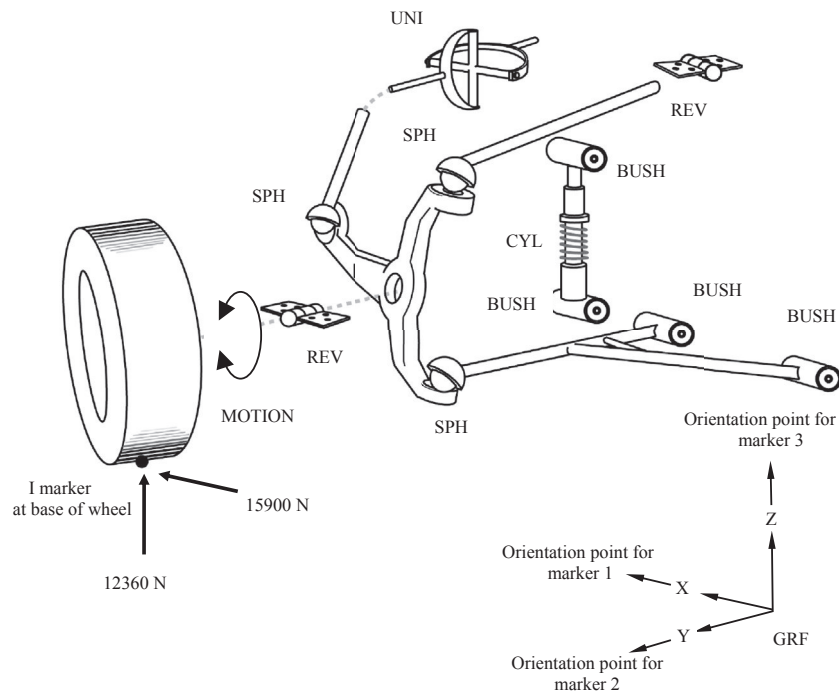
$$\frac{F_{Fx}}{F_{Fz}} = \frac{F_{Rx}}{F_{Rz}} = \mu \quad (4.77)$$

Note that the inboard reaction of drive torques means that loads are applied at the wheel centre rather than at ground height for tractive cases involving traditional inboard torque generation and shaft outputs to the wheels. For in-wheel motors the situation is the same as the brake case discussed here.

#### 4.8.2 Case study 2 — static durability loadcase

In order to demonstrate the application of road input loads to the suspension model, a case study is presented here based on the same front suspension system described in [Section 4.7](#) for case study 1. The loading to be applied is for the pothole braking case outlined in [Table 4.7](#). Due to the severity of the loading, the suspension model used here is one that includes the full nonlinear definition of all the bushes, the bump stop (spring aid) and a rebound stop. The model also includes a definition of the dampers and the damping terms in the bushes. These will be required later for an analysis that demonstrates the dynamic input of a road load but are not used for the initial phase where the load is applied quasi-statically.

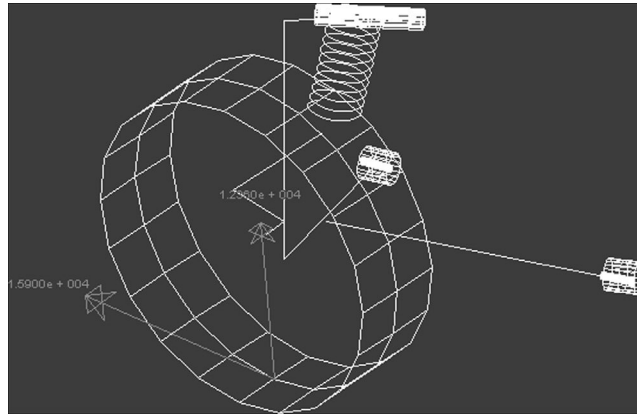
A schematic for the model is shown in [Figure 4.51](#) where it can be seen that the jack that was used earlier to move the suspension between the rebound and bump positions has been replaced by applied forces acting on a marker located at the bottom of the road wheel. An alternative approach can be taken where longitudinal force and a pitch moment equivalent to the longitudinal force multiplied by the wheel radius is applied at the wheel hub. This simplifies the loading model by removing the wheel part and the motion, saving the solution of six equations of motion to their constrained values; the choice of working practice remains with the practitioner, who must as always take responsibility for the formulation of his/her own model.

**FIGURE 4.51**

Application of Pothole Braking loads to suspension model. GRF, Ground Reference Frame; SPH, Spherical Joint; REV, Revolute Joint; UNI, Universal Joint; CYL, Cylindrical Joint.

The loads are applied as action-only single forces acting on the I marker at the contact patch. In this example the wheel is treated as a single rigid body and any compliance in the tyre is ignored. The I frame is in this case located at an undeformed radius directly below the wheel centre. Some practitioners can become obsessed with calculating the actual deformed radius of the tyre under these conditions but to complicate the calculations with second-order effects is to miss the approximate nature of the loading conditions, which are typically expressed to one figure ('3G bump') or the nearest quarter g ('0.75G cornering'); the pursuit of accuracy under such conditions is inappropriate. The motion statement associated with the road wheel revolute joint has a function set to zero to effectively lock the rotation of the wheel. The loads are applied parallel to the axes of the ground reference frame (GRF) at the start and remain parallel to the GRF during the simulation. They do not rotate with the wheel as the suspension deforms under the loading. Different software codes have different ways of achieving this but the nonrotation of the applied forces is important. Again, some practitioners can become obsessed with attempting to compensate for the change in vehicle platform attitude during such a manoeuvre but this again misses the approximate nature of the design conditions.



**FIGURE 4.52**

MSC ADAMS graphics of suspension at maximum Pothole Braking caseload. Graphics are deliberately shown in a 'minimalist' style to emphasise the ability to calculate component loads before the existence of CAD geometry.

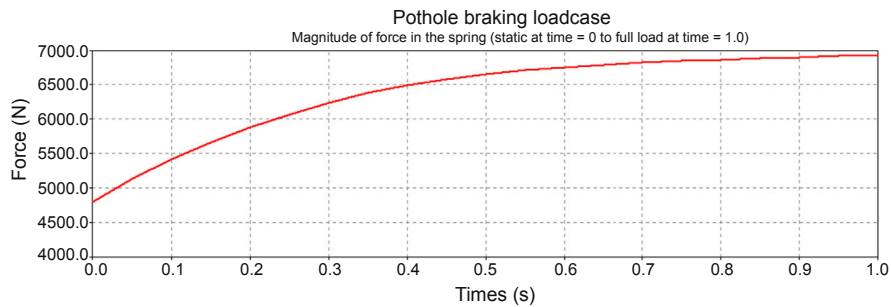
Three separate forces are used to define the pothole braking case in this example. It is typically true that solutions starting a long way from equilibrium can have difficulty converging and so it can be desirable to start with the self-weight force, which typically resembles the condition under which the suspension model is defined, and to ramp on the desired additional load in several static equilibrium steps. Each step is in fact a true static equilibrium; the 'time' component of the solution simply serves as a lookup variable to select the level of load application.

```
X Force: FUNCTION = 15900 * TIME
Y Force: FUNCTION = 0
Z Force: FUNCTION = 3727 + 8633 * TIME
```

From this it can be seen that for the pothole braking case an additional longitudinal load in the x-direction and an additional vertical load in the z-direction are applied in comparison to the design definition condition. Note also that different organisations have different definitions for the design condition and so to presume that the design condition is the same as the kerb condition may well be incorrect. The lateral load in the y-direction is set to zero. The functions are set for this example so that for the initial static analysis a vertical load of 3727 N is applied with the additional components due to pothole braking being added over 1 s. It can be seen from this that the functions used to define the forces can be quickly changed to correspond with each of the loadcases given in [Table 4.7](#).

The graphics showing the suspension deformed under full load are shown in [Figure 4.52](#) with additional graphics showing the force components at the contact patch. An XY plot showing the development of force magnitude in the spring is shown in [Figure 4.53](#). Examination of the numerical values associated with the components of this force at full load, after 1 s, would provide the inputs for any subsequent FE models.



**FIGURE 4.53**

MSC ADAMS plot of spring load for Pothole braking case.

It should be noted that with the quasi-static example used here there is no velocity-dependent load transmission through the damper. To increase the validity of the results it would be necessary to estimate an equivalent load and apply this as an additional static force. An alternative would be to develop the analysis of the suspension to apply the force as a function of time and carry out a dynamic simulation as described next.

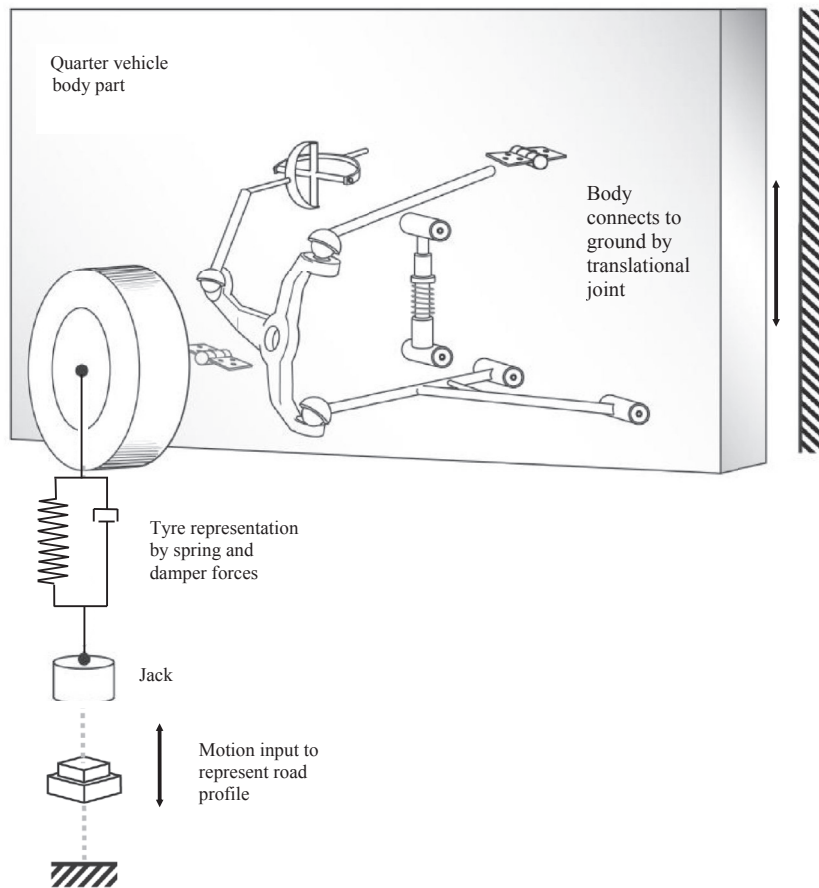
### 4.8.3 Case study 3 — dynamic durability loadcase

In this case study we extend the model of the single suspension system to include an additional part representing the corner of the vehicle body, quarter model, to which the suspension linkages attach as shown in [Figure 4.54](#).

The vehicle body is attached to the ground part by a translational joint that allows the body to move vertically in response to the loads transmitted through the suspension system. A jack part is reintroduced to reproduce vertical motion inputs representative of road conditions. An additional complexity in the model is the introduction of stiffness and damping terms in a force element that represents the behaviour of the tyre. The force element acts between the centre of the wheel and a point on the jack coincident with the centre of the tyre contact patch. In this case this is assumed to be directly below the wheel centre. It is important that the force element for the tyre acts only in compression and allows the tyre to lift off the top of the jack if the input is severe enough.

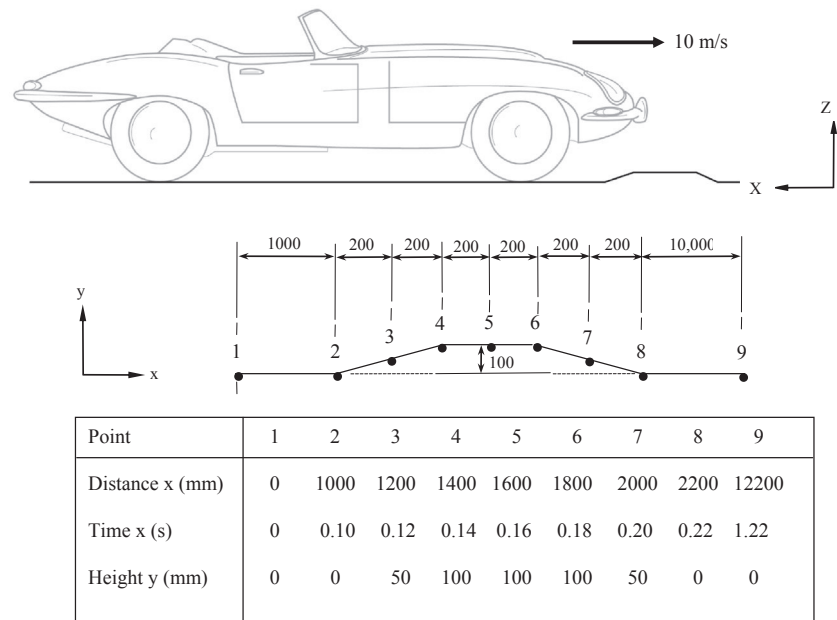
The tyre is taken to have linear radial stiffness of 160 N/mm and a damping coefficient of 0.5 Ns/mm. The undeformed radius of the tyre, equivalent to the free length of the tyre spring, is taken here to be 318 mm. A step function can also be used to zero the tyre force if contact with the jack is lost. The full definition can be accomplished using the following function statement that is taken to act between the frame at the wheel centre and the frame on the jack part.

```
FUNCTION=STEP(318-DZ(wheel_frame,jack_frame), 0, 0, 0.1, VARVAL(1) )
VARIABLE/1, FU=160*(318-DZ(wheel_frame,jack_frame))- 0.5*VZ(wheel_
frame,jack_frame)
```

**FIGURE 4.54**

Quarter vehicle body and suspension model.

The STEP function is common to several multibody programs and consists of a continuously differentiable blend between two states, based on the value of a variable or expression. In this case the expression is  $318-DZ(\text{wheel\_frame}, \text{jack\_frame})$ , which may be understood as the distance of the wheel centre above the presumed floor, subtracted from the undeformed radius. When the result of this expression is zero or more, the force resulting has a constant value of zero. When the result of the expression is less than zero it means the wheel is off the floor and the force returned is zero. When the expression is greater than 0.1 mm, a linear stiffness and viscous damping component is enacted. Between 0 and 0.1 mm, the STEP function blends the two functions together in a manner that is sympathetic to the numerical integrators in use.



**FIGURE 4.55**  
Road profile for ‘sleeping policeman’ speed bump.

It can be seen there is a trade-off between the numerical friendliness, which is assured with a large transition region, and absolute rigour of implementation of the transition at any point above zero deformation of the modelled tyre forces. Given that, as George Box famously said, “All models are wrong but some models are useful” (Box and Draper, 1987), an obsessive desire to capture the immediacy of transition is perhaps overlooking the inaccuracy of the linear model when inspected closely in that very transition region. Since it does not dominate the overall behaviour of the system (the reader is invited to verify this assertion by varying the transition region and checking whether it changes the outcome of the analysis in any meaningful way) it is apparently a candidate for another home-grown aphorism: “I’d rather be simple-and-wrong than complicated-and-wrong.” This is another way of saying that simple-and-useful is more valuable than complicated-and-useful.

The next step is to define the motion imparted to the jack part to represent the input from the road surface. This is illustrated in Figure 4.55 where the profile for a ‘sleeping policeman’ road obstacle is given. The profile is defined as a set of xy pairs. Note here that the xy values are local to the definition of the obstacle profile and not associated with the X and Y axes of the GRF. The vehicle is assumed to be moving with a forward speed of 10 m/s so that the x values associated with distance can be converted to time.

One method that could be used to input a motion associated with the profile of the road surface would be to use a form of spline. Most multibody packages offer

cubic splines and some offer other spline schemes. All of them have in common that they repeatedly fit an analytical function through a subset of all the points provided in order to provide a continuously differentiable and infinitely resolvable version of the discrete table of points.

Caution is needed here, however, for although the data points may be sufficient to capture the profile of the bump there may not be enough to ensure a good spline fit, as noted in Chapter 3.

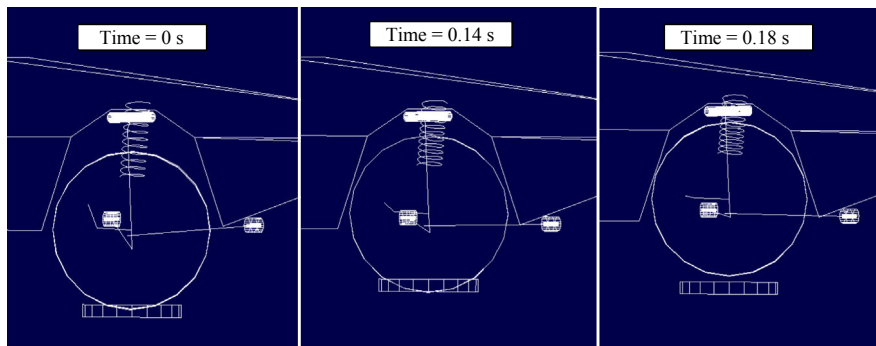
A more elegant but elaborate method might be to forgo the use of interpolation and use a combination of arithmetic IF and step functions. This method requires care in formatting but may be applied as follows.

```
FUNCTION=IF(TIME - 0.14: STEP(TIME, 0.1, 0, 0.14, 100), 100,STEP
(TIME, 0.16, 100, 0.22, 0))
```

MSC ADAMS graphics showing the suspension deflecting on the jack and the subsequent departure of the tyre from the road surface are shown in Figure 4.56. A plot showing the time histories for the vehicle body and road wheel vertical displacement is shown in Figure 4.57. The force in the bump stop is provided by way of example Figure 4.58. It should be noted that at this stage the analysis only represents vertical force input and not longitudinal force input from the road surface.

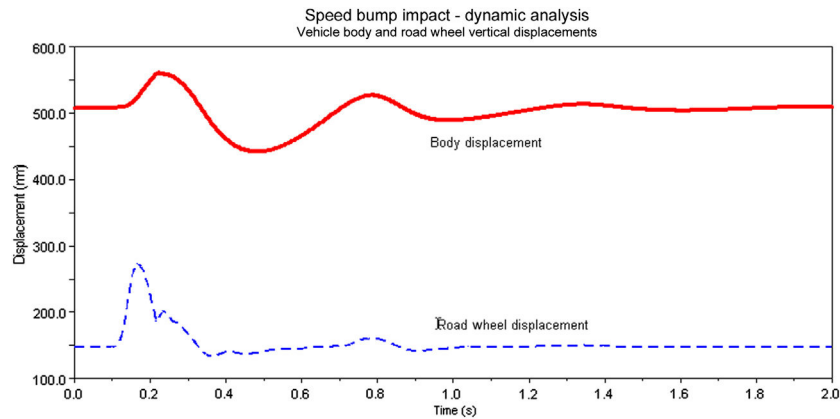
This is no different to the case in the real world, where vertical-only inputs do not capture the full set of inputs to which the real vehicle is subject. Just as in the real world, ever-more elaborate rigs can be used with half and full vehicle models, culminating in running the full vehicle over a complete description of the test surface.

It is tempting but somewhat naive to argue for the most detailed possible model at all times. To do so is fall prey to a common error in predictive engineering: the belief that the model is the product. In fact, the purpose of the model is to support design decisions and to allow a monotonic progression of the design from a concept to a finished product, ready to manufacture, with the minimum number of iterative processes and feedback loops along the way. The model is *not* the product.

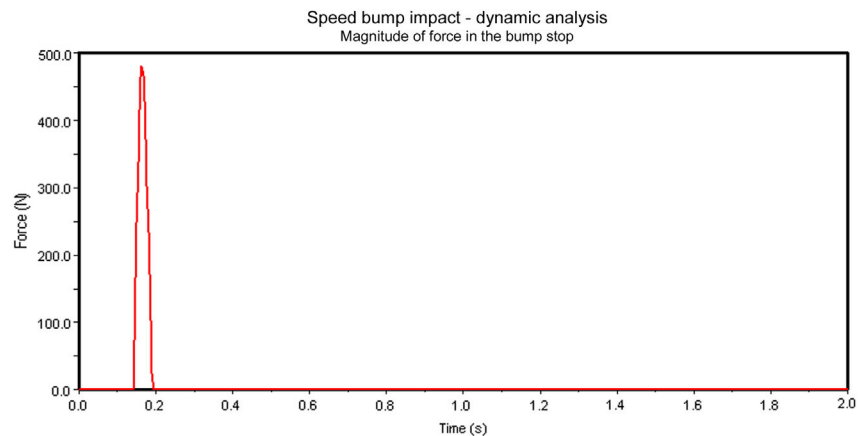


**FIGURE 4.56**

MSC ADAMS graphics of suspension deflecting on a speed bump.

**FIGURE 4.57**

MSC ADAMS plot vehicle body and road wheel displacements for speed bump strike.

**FIGURE 4.58**

MSC ADAMS plot of bump stop force time history for speed bump strike.

Some practitioners (Tang et al, 2000) advocate the use of explicit FE solutions instead of the combined multibody/implicit FE approach used here. In durability terms, the use of fully featured explicit FE mesh suffers from most of the same drawbacks that running a real prototype vehicle suffers from: by the time we have all the information to define it, it is too late to make substantial changes. So the question is ‘how do we get to a design in which we have confidence that it will not need substantial changes by the time we sign it off?’ The answer, it appears to these

authors, is a rational and staged approach consisting of a sequence looking something like this:

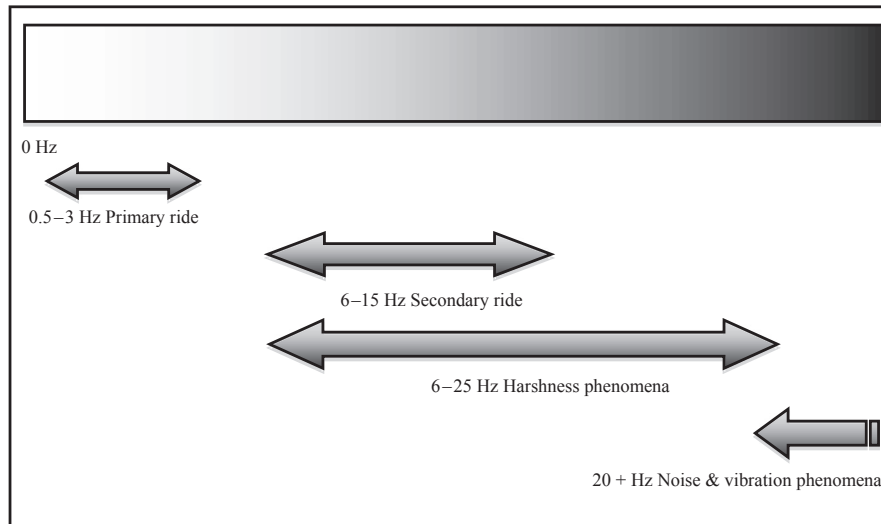
1. Establish conceptual design based on cost constraints, package constraints and so on.
2. Sketch the conceptual design in a multibody environment with all of the elements represented by sparse line graphics.
3. Perform a functional analysis that distributes static design loads through the various components, as in case study 2.
4. Use the loads to broadly size the components in a manner consistent with their anticipated manufacturing method, sketching proportions in a parametric CAD environment but avoiding final detailing (e.g. paint drain holes) and then solving for linear stress response; the use of linear stress response being consistent with the practices of the organisation (e.g. ‘no stresses over 75% yield for design loads’ or similar).
5. Detailed progress based on other attributes — ride, refinement and so on — to define details not strongly related to durability.
6. Full dynamic simulation of a model over the expected durability sign-off using all the design data available up to that point to discern dynamic load histories; these being used with detailed material models to search out areas of the design that are not suitably robust using the linear stress sizing criteria. Case study 3 represents the simplest possible approach to this task.
7. Prototyping and real-world sign-off with high confidence.
8. Review sizing in step 4 for future activities; refine if necessary.

Note that if no areas emerge in step 6 then the sizing criteria in step 4 are unnecessarily conservative and the product is probably overweight; if too many emerge then the sizing criteria are not strenuous enough, systematically resulting in an unnecessary redesign for every product.

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## 4.9 Ride studies (body isolation)

The determination of vehicle ‘ride’ quality is associated with the extent to which the occupants of the vehicle are affected by vehicle motion. Ride is often described as a mainly vertical phenomenon but this can be unhelpful; a better description might be that it is the study of uncommanded vehicle motions, primarily through the interaction of the vehicle with the surface. Automotive companies will often have departments concerned with ride and handling where MBS analysis will be deployed to support design and analysis work. Another area of activity is referred to as noise, vibration and harshness, frequently referred to as ‘NVH’. Many authors distinguish between these different phenomena by using arbitrary frequency separations. While superficially helpful, it should be kept in mind that the numbers used represent the centre of some ‘transition band’ in a continuous spectrum and that the phenomena merge into one another rather like the colours in a rainbow. [Figure 4.59](#) represents this idea graphically.

**FIGURE 4.59**

Ideas of the continuous frequency spectrum and different events on it.

For the purposes of this text, the different dynamic phenomena might be described as follows:

*Primary ride*: the motion of the whole vehicle on its suspension ‘bodily’; excited by transient, unsteady forcing from the road surface.

*Secondary ride*: the motion of substantial masses within the vehicle such as powertrain-on-mounts or individual suspension assemblies; excited by underlying energy in the road surface in a pseudo-steady state manner and easily recognisable as ‘large masses in motion’ when *operating shape* information is recorded. Operating shape is a specific type of dynamic measurement in which accelerometers are used with sophisticated postprocessing software to capture the dynamically deformed shape of a system and show how it varies at different frequencies. When a system is in resonance, the operating shape is dominated by the mode shape for the resonance in question; it follows that if the shape does not resemble a mode shape, it is not therefore in resonance but rather being forced statically. See earlier in this chapter for a fuller description.

*Harshness*: complex motion, sometimes within an individual component, that is touched intermittently or continuously by the operator to give a sense of annoyance (no touch = no harshness); often excited in a broadly steady state manner by powertrain or road surface energy.

*Noise & vibration*: complex motion, with coupling to the acoustic cavity in the case of noise, detectable by a combination of hearing and touch but occurring at low energy levels; often excited in a broadly steady state manner by powertrain or road surface energy.

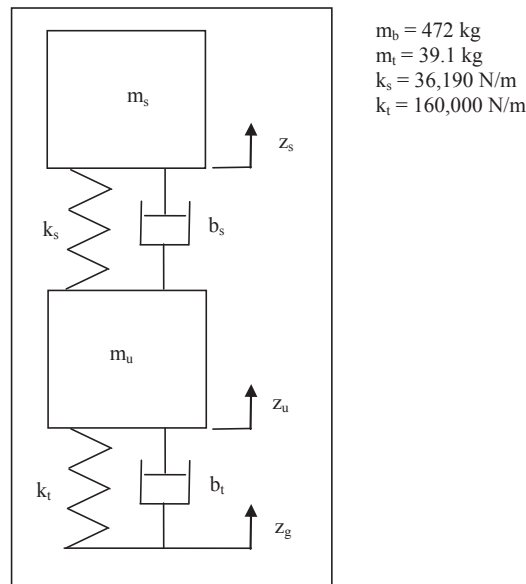
Primary and secondary ride vibrations are usually amenable to analysis with the multibody techniques described in this textbook. Some aspects of harshness can be

assessed with multibody models when they include intracomponent resonances in the form of flexible representations of components using modal components, a technique described later. Above 20 Hz or so and with complex motions involving distortions of individual components, the analysis of these modes and other acoustic type problems is more in the domain of advanced FE analysis and as such is not covered here.

Vehicles should be thought of as dynamic systems, a mixture of masses, springs and dampers, where vibration is exhibited in response to excitation. The source of excitation may be due to out of balance loads from rotating bodies such as the road wheel or from other sources in the vehicle including the engine and driveline. The other main source of vibration will be associated with the profile of the road surface. At this stage it is easy to envisage that the excitation of vehicle pitch may be in response to a road with an undulating type profile of relatively long wavelength whereas the excitation of a smaller mass such as the road wheel will occur at higher frequencies. This might for example occur whilst driving on a cobbled type of road surface.

#### 4.9.1 Case study 4 – quarter vehicle dynamic performance analysis

A great deal of insight can be gained from the classical study of the quarter vehicle model shown in Figure 4.60. This study can easily be replicated in a commercial multibody program to reproduce the results obtained here, as described later in this section.



**FIGURE 4.60**

Quarter vehicle model in mathematical form.



A sprung DoF,  $m_s$ , is connected to an unsprung DoF,  $m_u$  in series and both are driven by a road excitation profile,  $u$ . To perform the calculations, the equations of motion can be written largely by inspection. For the sprung mass:

$$\ddot{z}_s = \frac{b_s(\dot{z}_u - \dot{z}_s) + k_s(z_u - z_s)}{m_s} \quad (4.78)$$

For the unsprung mass there are some more terms since it is acted upon by both the tyre and suspension but the principle is identical:

$$\ddot{z}_u = \frac{b_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) + b_t(\dot{z}_g - \dot{z}_u) + k_t(z_g - z_u)}{m_u} \quad (4.79)$$

These terms are convenient for direct numerical solution as is, to obtain time-domain results. Note that the formulation of the tyre forces can be subject to the same sort of conditional switching described in case study 3 to allow the wheel to leave the ground. Before solving the model in the time domain, it is good practice to estimate the natural frequencies for the body on the suspension and for the unsprung mass between the road spring and the tyre spring. Figure 4.61 shows a 2 DoF quarter vehicle model with the data used to support the calculations.

The undamped natural frequencies for the body,  $f_b$ , and unsprung mass,  $f_t$ , can be estimated using the following equations. Note that for the body we determine an equivalent stiffness,  $k_{eqv}$ , to represent the combined contribution of the road and tyre springs.

$$k_{eqv} = \frac{k_s k_t}{k_s + k_t} \quad (4.80)$$

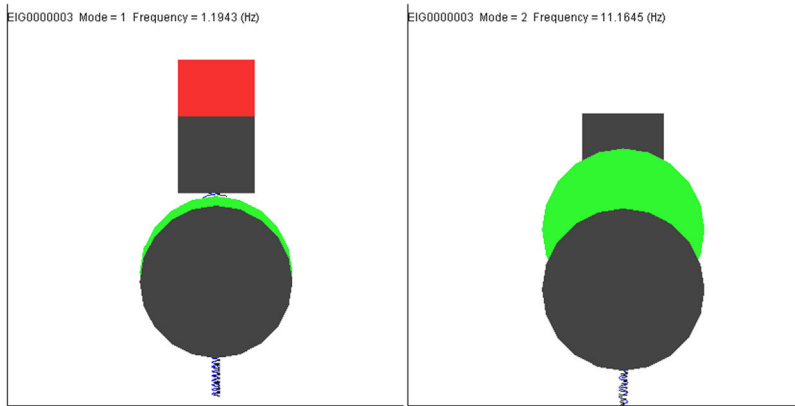


FIGURE 4.61

MSC ADAMS prediction of the modes of vibration of the simplified quarter vehicle model.

$$f_s = \frac{1}{2\pi} \sqrt{\frac{k_{eqv}}{m_s}} \quad (4.81)$$

$$f_u = \frac{1}{2\pi} \sqrt{\frac{k_s + k_t}{m_t}} \quad (4.82)$$

Performing the calculations using the data given results in values as follows:

$$k_{eqv} = 26639 \text{ N/mm}$$

$$f_b = 1.196 \text{ Hz}$$

$$f_t = 11.15 \text{ Hz}$$

A multibody model consisting exactly of the model as sketched above can be used to calculate undamped linear modes in a more exact fashion, giving two modes of vibration as might be expected but slightly differing numerical results to the classical calculations.

This particular modal solution uses the ADAMS/Linear™ product, which in turn uses numerical perturbation methods to estimate mass and stiffness matrices and assemble the Jacobian matrix about an operating point before solving for eigenvalues by finding its determinant in the normal fashion.

For purely mechanical systems such as the one modelled, such methods can be relied upon to give good quality results. However for systems where forces are time-dependent and are modelled in specific ways (e.g. as differential equations for modelling turbocharger behaviour as described in Chapter 6, or tyre relaxation length modelling) the results are not necessarily reliable at the time of writing and must be examined on an individual basis before confidence is placed in them. A test solution is recommended to establish confidence in the eigensolution methods for a given software code before critical design decisions are based on it. Note that these types of calculation behaviours are the worst possible kind – the results are plausible but misleading; obvious nonsense can easily be spotted but something which looks reasonable is more difficult to pick up.

All software packages are subject to ongoing modification and development and so functionality of this nature should be evaluated periodically; such evaluations should be part of the software commissioning process within individual organisations, particularly if critical decisions are to be based on software output.

The estimates given above are a simplification of the actual analytical solution to the system. Such a solution is obtained by first dropping the damping terms and external inputs from Eqns (4.69) and (4.70):

$$\ddot{z}_s = \frac{k_s(z_u - z_s)}{m_s} \quad (4.83)$$

$$\ddot{z}_u = \frac{k_s(z_s - z_u) + k_t(-z_u)}{m_u} \quad (4.84)$$

These can be arranged more conveniently as:

$$m_s \ddot{z}_s = z_u k_s + z_s (-k_s) \quad (4.85)$$

$$m_u \ddot{z}_u = z_u (-k_t - k_s) + z_s (k_s) \quad (4.86)$$

If a solution is assumed of the form

$$x = X e^{\lambda t} \quad (4.87)$$

then

$$\ddot{x} = \lambda^2 X e^{\lambda t} = \lambda^2 x \quad (4.88)$$

thus

$$m_u \lambda^2 z_u = z_u (-k_t - k_s) + z_s (k_s) \quad (4.89)$$

$$m_s \lambda^2 z_s = z_s (k_s) + z_s (-k_s) \quad (4.90)$$

which may be rearranged into the familiar eigenvalue problem:

$$\begin{bmatrix} -k_t - k_s - m_u \lambda^2 & k_s \\ k_s & -k_s - m_s \lambda^2 \end{bmatrix} \begin{bmatrix} z_u \\ z_s \end{bmatrix} = 0 \quad (4.91)$$

in which the determinant of the matrix can be used to find the eigensolution when set to zero:

$$(-k_t - k_s - m_u \lambda^2)(-k_s - m_s \lambda^2) - k_s^2 = 0 \quad (4.92)$$

$$(m_s m_u) \lambda^4 + (k_s m_u + (k_t + k_s) m_s) \lambda^2 + k_t k_s + k_s^2 - k_s^2 = 0 \quad (4.93)$$

which may be recognised as a quadratic in  $\lambda^2$  and solved in the normal manner:

$$\lambda^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (4.94)$$

$$a = m_s m_u \quad (4.95)$$

$$b = k_s m_u + (k_t + k_s) m_s \quad (4.96)$$

$$c = k_t k_s + k_s^2 - k_s^2 = k_t k_s \quad (4.97)$$

The calculated roots using this method are

$\lambda^2$ :

-56.3075

-4920.87

$\lambda$ :

7.50383i

70.1489i

frequencies:

1.19427 Hz (“Primary Ride”)

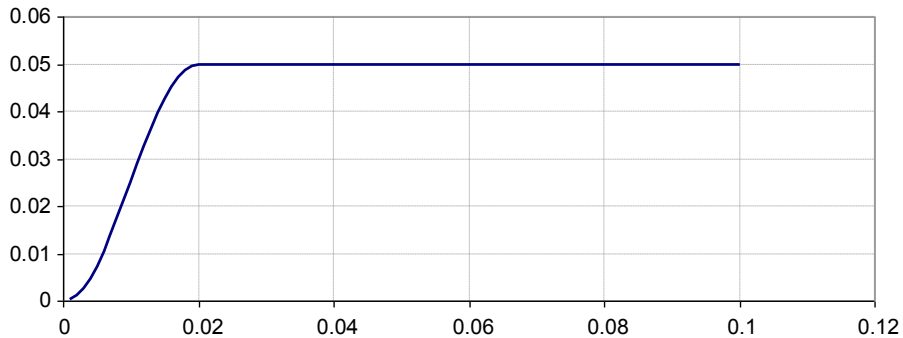
11.1645 Hz (“Wheel Hop”)

which can be seen to agree exactly with the MSC ADAMS model. However, the differences between the exact method and the approximate method are small — less than 0.15%. Thus the approximate method is a ‘good enough’ check for this system. This is generically true for quarter vehicle models, where the second mode of vibration is typically an order of magnitude higher than the first. However, for particularly stiff suspensions or compliant tyres as may be used on circuit cars, the suitability of the approximate method breaks down and therefore it should be used with some care.

Before leaving the simplified model, it is worth examining its behaviour in the time domain in the context of real vehicle behaviour. A step-up ramp can be implemented by (Figure 4.62):

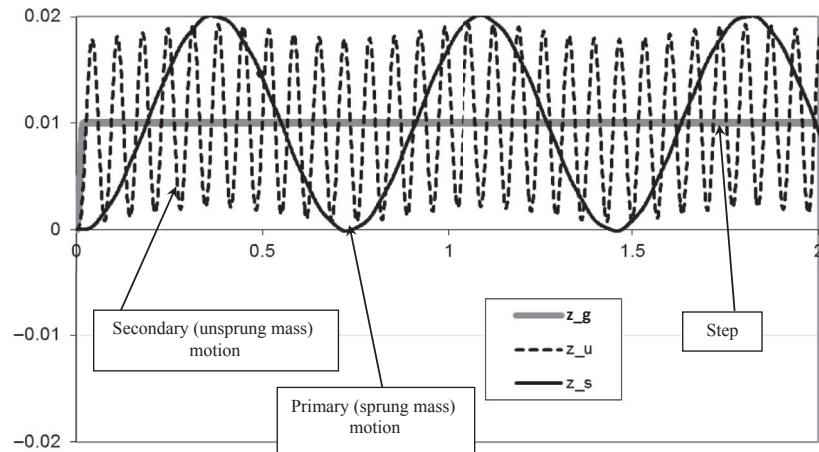
$$\dot{z}_g(t) = \frac{A}{2}(1 - \cos(\pi t/t_e)\pi), t < t_e \quad (4.98)$$

$$\dot{z}_g(t) = A, t \geq t_e \quad (4.99)$$



**FIGURE 4.62**

Cosine step up, a continuously differentiable input function.

**FIGURE 4.63**

Undamped solution to a step input for illustration.

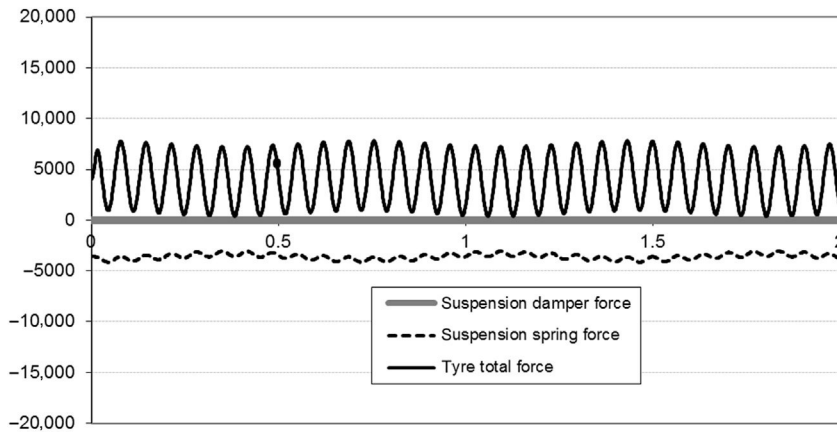
where  $A$  is the height of the step and  $t_e$  is the time at which the vehicle reaches the end of the step, given by step length/velocity. This formulation can be used for both step up and step down; a large, e.g. 1 m, step down is a jump landing simulation.

The step up is strictly a cosine ramp-up as described by Eqn (4.98), since the enveloping behaviour of the tyre means a genuine mathematical step that is never really seen by a real suspension. The length of the ramp is set at 0.5 m. The speed of the vehicle is set at 25 m/s (90 kph) so that the obstacle is encountered in 0.02 s, significantly quicker than the wheel hop dynamics and therefore it may be expected to excite both wheel hop and primary ride. Initially the obstacle height is set at a mere 10 mm.

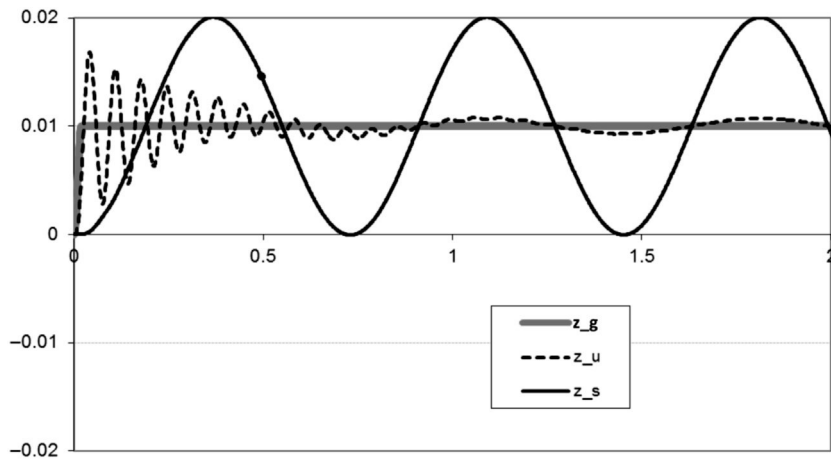
For illustration, the system was run with zero damping levels to show the base behaviour of the different parts. Figure 4.63 shows how the step input excites both resonances very clearly.

Features of interest in the undamped solution are the primary (sprung) mass displacement which is double the initial step height and the combined response of the secondary (unsprung) mass, which has a 'long wave' component connected to the primary mass motion as well as its own, faster oscillation — the so-called 'wheel hop' mode.

When considering the tyre load variation in this system, it can be seen that it is substantially dominated by the faster, wheel hop vibration. This is unsurprising given the high spring rate of the tyre. It can be seen that the spring force variation (and hence the body accelerations) are very low and thus this notional vehicle might not be as uncomfortable as the somewhat dramatic graphical results in Figure 4.64 suggest.

**FIGURE 4.64**

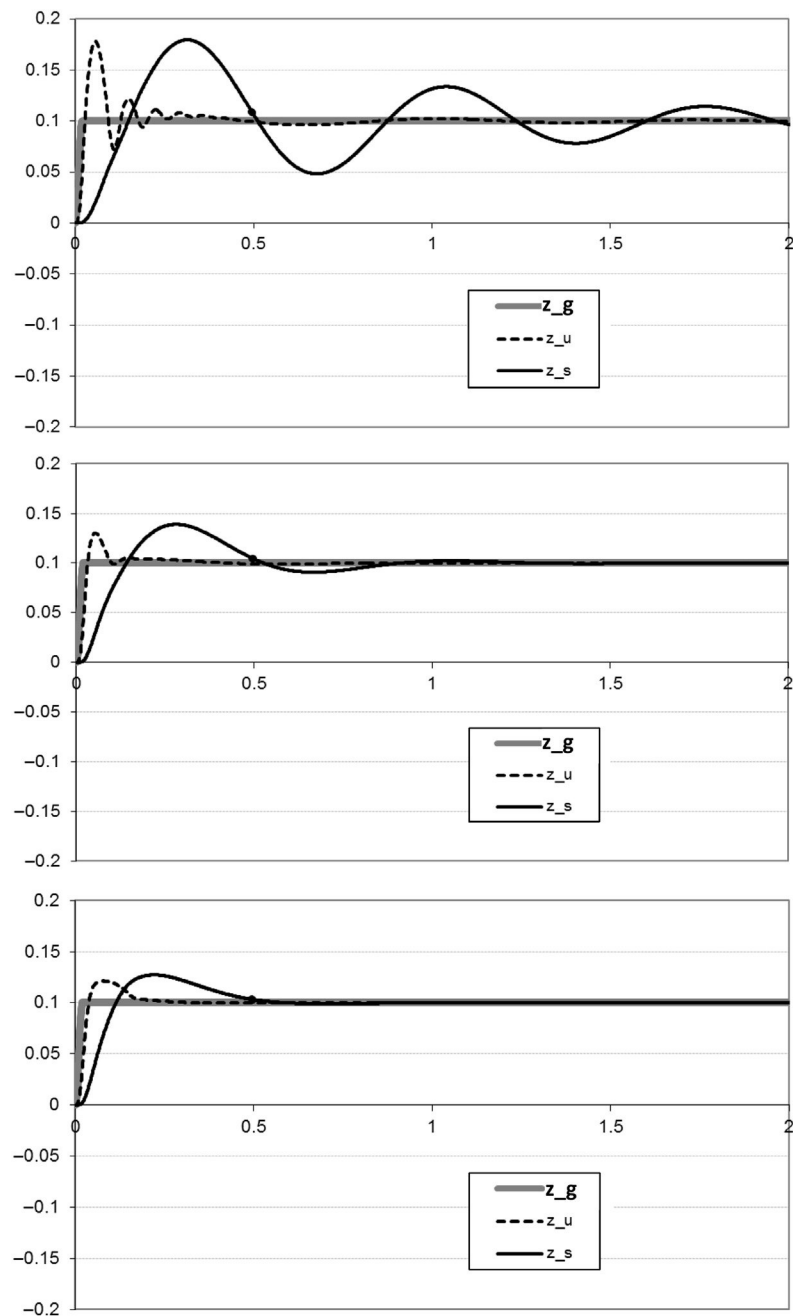
Forces generated in the undamped solution.

**FIGURE 4.65**

The addition of tyre damping.

Adding a typical level of tyre damping (estimated at 0.4 Ns/mm) controls the secondary motion, albeit only lightly, but not the primary motion as shown in [Figure 4.65](#).

The road step input is adjusted to 0.1 m height, a typical kerb height. This is an aggressive event at 25 m/s and promotes separation of the tyre from the road. Three results are presented for initial discussion of the suspension damping calculations in [Figure 4.66](#).

**FIGURE 4.66**

Suspension damping influence on system performance.

The first damped results, [Figure 4.66](#) (top), are calculated with a suspension damping level of 1 Ns/mm, giving an overall damping ratio of 0.2. The continued oscillation of the primary mass can be clearly seen, although the secondary vibration is clearly reduced compared to the tyre damping alone. Despite this, there are repeated land-and-bounce events experienced by the secondary mass.

The results shown in [Figure 4.66](#) (middle) are calculated with a suspension damping level of 3 Ns/mm, a damping ratio around 0.7 and quite a sensible value for a performance-biased road vehicle. While there is some overshoot, it can be seen that the secondary mass quickly returns to the surface.

The results shown in [Figure 4.66](#) (bottom) are with a suspension damping level of 5 Ns/mm, damping ratio of around 1.1 and ‘too hard’ by most measures. While the control of the primary mass is better, it seems clear that the ‘time in flight’ for the secondary mass is longer than the middle solution.

It can thus be seen that the damper calibration has a substantial influence on the behaviour of the system. However, it is less immediately obvious what constitutes ‘good’ behaviour.

An approach can be taken that looks at several aspects of system performance and attempts to calculate scores for each of them. The desire is to obtain good performance in all areas, with an overall rating being possible if weightings are assigned to the different scores. With enough discipline, it is possible to formulate a genuine multidimensional optimisation problem using these measures.

Quarter vehicle model dynamic performance can be broadly split into:

- Active safety: the ability to stop and steer in emergency situations
- Ride: the ability of the vehicle to absorb disturbances

‘Active’ safety in this context means the ability to avoid a collision through manoeuvring the vehicle. It is distinct from ‘passive’ safety, which is the ability of the vehicle to protect the occupants in the event of a collision and includes so-called crumple zones and airbag technology. It does not specifically imply the presence of electronic or other control systems. Other performance measures can be added at will — the response to out-of-travel events, jump landings and so on. Not all items are uniformly important in every application. Active Safety and Ride are considered in detail below and a method for determining a key performance indicator (KPI) is illustrated. It should be noted that these indicators are entirely arbitrary and are a way of using numerical analysis in some meaningful way to influence the product. Indicators should be invented and discarded at will, as long as care is taken to document them carefully and to only compare like with like.

For the KPIs there are some preferable qualities:

- Each should be a ‘more is better’ measure to allow easy assimilation
- Each should be on a reasonable scale, e.g. 1–10
- Each should have some sort of meaning associated with significant points on the scale, e.g. 0, 10.



To achieve these qualities, KPIs are not always in common engineering units, although they are always traceable to more normal engineering quantities.

Formulating performance metrics in this way means they can be judged easily by nonexperts (since most people have a good idea of the relative difference between 6/10 and 8/10 but many people struggle to assimilate a reduction of  $0.5 \text{ m/s}^2$  ride acceleration). By expending a little effort on the indicators in their own right, the somewhat arbitrary application of cost functions in an optimisation can be avoided since they are all set to unity if the third requirement is met in some sensible way.

### 4.9.2 An active safety KPI

The ability to stop and steer the vehicle is completely dominated by the ability of the tyres to generate forces in the ground plane, sometimes referred to as ‘grip’. As discussed in Chapter 5, variation in vertical load has the effect of degrading available forces in the ground plane.

Root mean square (RMS) load variation is therefore a useful response variable and can be nondimensionalised by dividing it by the static load. If the static load is first divided by  $\sqrt{2}$  then a KPI of less than zero can be seen to induce momentary ‘flight’ of the wheel under sinusoidal excitation. For smooth road use this is not typical but it is much more common than might be intuitively expected for even moderately degraded surfaces without necessarily running out of suspension travel; some reflection shows that if steps down are more than the corner load divided by the tyre stiffness then the load will indeed get to zero; for a typical modern run-flat tyre this is of the order of 10 mm. It should be noted that the degradation of lateral force does not require a period of zero load — only a variation in vertical load. Finally, the value is subtracted from 1 to give a more-is-better measure and then multiplied by 10

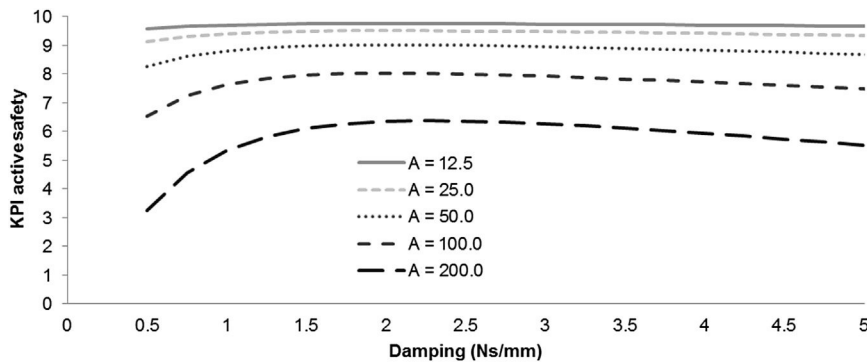


FIGURE 4.67

Variation of Active Safety Key Performance Indicator (KPI) ('Grip') with damping levels over varying steps up — calculated with a 2 degree-of-freedom model.

to give a maximum value of 10 (no load variation). For a rough road, negative values of the KPI are unsurprising.

$$\text{KPI} = \left( 1 - \frac{\sqrt{\frac{\int_0^t (F_z - F_{z0})^2 dt}{t}}}{\frac{F_{z0}}{\sqrt{2}}} \right) \cdot 10 \quad (4.100)$$

If this definition is used with a linear quarter vehicle model then a clear optimum emerges in terms of damping level for the modelled step up, as can be seen in Figure 4.67.

Note that Figure 4.67 represents the results of multiple solutions to the model – 100 different solutions, each consisting of 10,000 integration steps<sup>3</sup>. In this case, the picture is worth a million calculations. Several lessons emerge from this exercise:

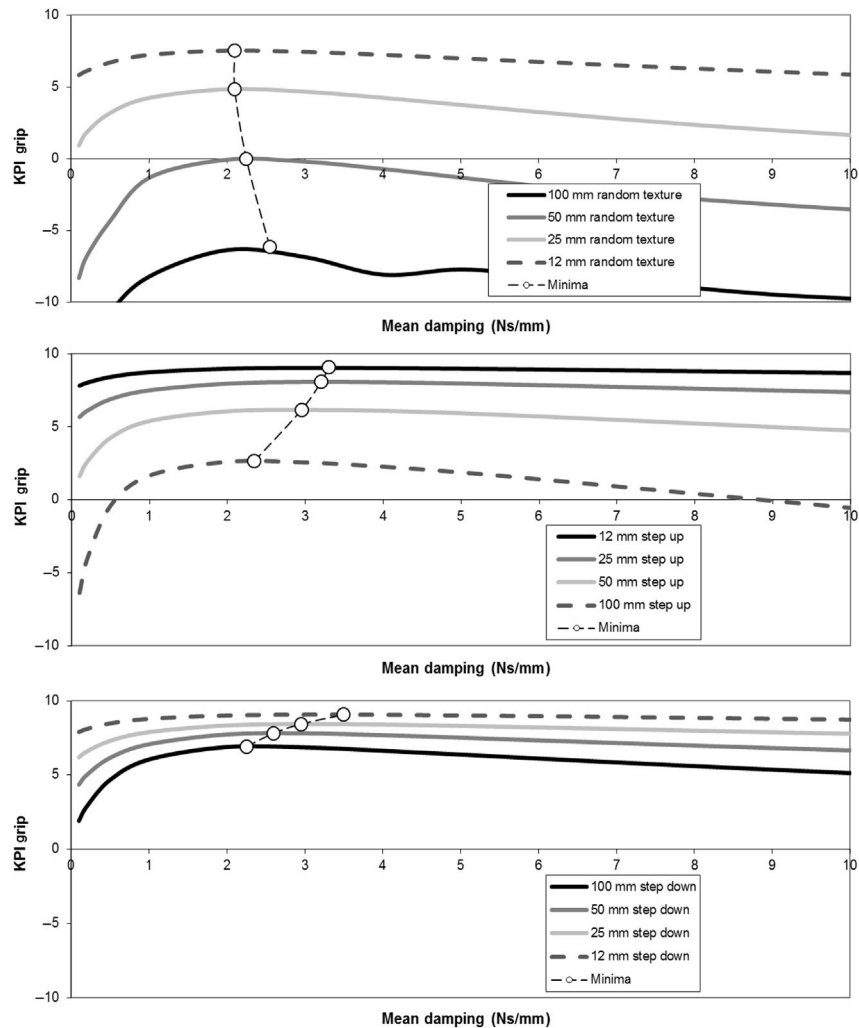
- Increasing the amplitude of the surface increases the load variation in a manner that cannot be compensated by damper level
- A clear optimum exists that is remarkably similar in level whatever the amplitude of the surface input to the model
- Too much damping degrades the KPI significantly less than too little
- The smaller the input, the less influence damping has on the KPI

These are interesting conclusions for this surface but it is unclear from these results how general these findings might be; it might be argued that the nature of the grip index is very dependent on the surface chosen.

However, this does not appear true. Results were computed as part of a separate exercise using the quarter vehicle corner model for a step up, step down and ‘random’ surface input. They are shown in Figure 4.68. While the levels of KPI grip are clearly different between the surfaces, they all point to a similar optimum (2–3 Ns/mm in this case), a strong penalty for insufficient damping and only a small penalty for excessive damping. Therefore the KPI is seen as a fairly robust measure that is not unduly influenced by the inputs chosen. The parameters for this exercise were not the same as the reference model provided, suggesting the conclusions drawn are not overly sensitive to the surface profile chosen, nor are they particular artefacts of a specific data set.

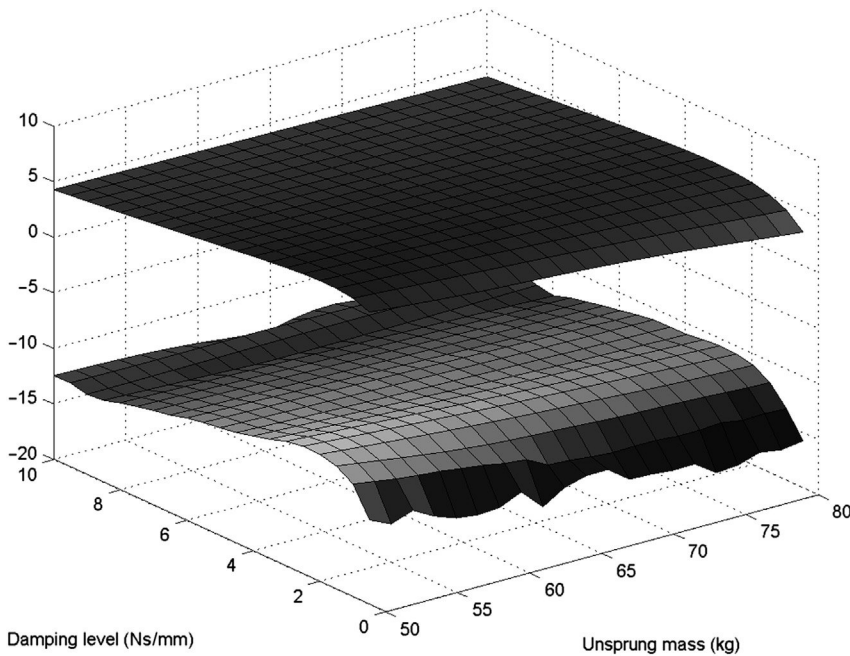
Further investigations were made by Harty (2010) to understand the influence of road profile scaling and unsprung mass.

<sup>3</sup>The number of steps is high because rather dumb rectangular integration is in use in the Microsoft Excel spreadsheet used to generate this example. This is not a reflection on Microsoft but was convenient to code for the author.

**FIGURE 4.68**

Key Performance Indicator (KPI) Grip variation for three different surface profiles.

It can be seen in [Figure 4.69](#) that the same type of optimum level appears and that the optimum is very insensitive to unsprung mass. It is also apparent that surface roughness significantly degrades the active safety KPI regardless of damper setting or unsprung mass — as was previously observed. These analyses were carried out using scaled versions of a synthesised textured road profile, one representing a fairly smooth road with low-level inputs, typical of many made roads in the Western world, the second representing a surface with large inputs comparable to the



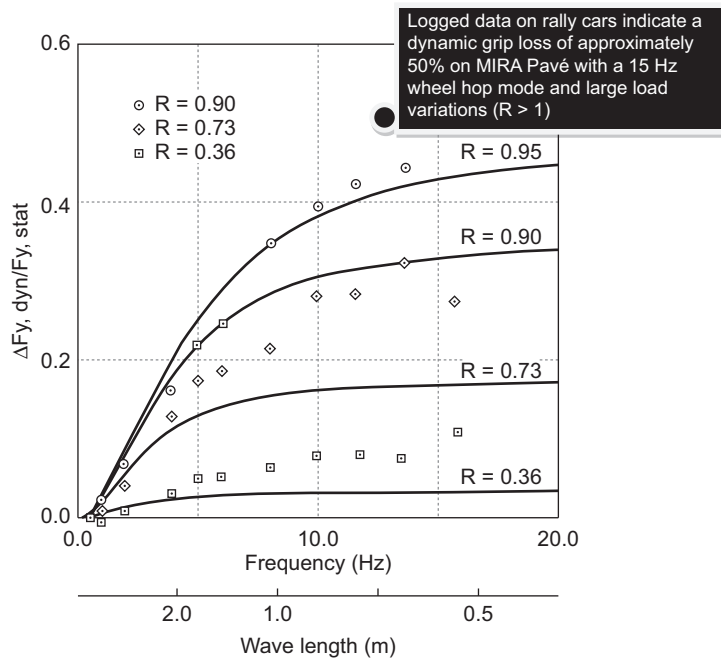
**FIGURE 4.69**

The Variation of Key Performance Indicator Grip with unsprung mass and damping level for two different road surface roughness levels. The lower surface represents a rougher road.

durability surface known as Pavé or Belgian Block. The rough surface drives the KPI negative, as was previously declared possible, well past the point where the wheel just unloads in some notional sinusoidal cycle.

Without precise tyre characteristics — specifically relaxation length variation with load, which is not typically published or present in empirical data sets — it is impossible to be certain about the effect of load variation on lateral performance, although a useful estimate may be made if lateral acceleration levels and vehicle and wheel motions are logged on a smooth and a rough surface. Care must be taken to avoid damaging the vehicle on particularly rough surfaces at maximum lateral acceleration. The authors' experience testing the handling behaviour of gravel rally cars on the Pavé surface at MIRA suggests that for this level of degradation, the maximum lateral acceleration capability of the vehicle is roughly halved, which helps calibrate the scales in engineering units. This level of degradation is broadly consistent with published data such as that shown in [Figure 4.70](#).

Note the previous observation that the conclusions drawn from such metrics do not seem unduly sensitive to the exact surface profile used. This leads us to the

**FIGURE 4.70**

Measured data showing the loss of lateral force over non-smooth surfaces with an indication of rally car behaviour on the MIRA Pavé surface (Pacejka and Takahashi, 1992).

fortunate position where we can optimise what seems like a useful surrogate for grip *without* detailed knowledge of tyre characteristics apart from knowledge of the vertical dynamics.

### 4.9.3 Ride KPIs

The vehicle ride environment is dominated by vertical acceleration. Expert assessors often break down ride behaviour into ‘primary’ and ‘secondary’ ride. Primary ride is the description given for the vehicle body motions on the suspension and is usually in the window below 3 Hz. Secondary ride is the description given to a number of higher frequency phenomena in the vehicle, such as powertrain shake and wheel vibration on the tyre stiffness, around 6 and 12 Hz, respectively. These ideas were illustrated in Figure 4.59.

The primary ride KPI for this study is RMS vertical acceleration in the 0–3 Hz band for the 2 DoF corner model running over a known road profile. However, since more acceleration is bad, the RMS acceleration in metres per square second is subtracted from  $5 \text{ m/s}^2$ ; this reflects the ISO 2631 severe discomfort threshold in the 1–2 Hz region. Finally the value is multiplied by 2 to scale the KPI.

$$\text{KPI} = \left( 5 - \sqrt{\frac{\int_0^t (\text{filt}(\ddot{z}_s))^2 dt}{t}} \right) \cdot 2 \quad (4.101)$$

For a completely immobile ride the RMS acceleration would be zero and therefore the KPI is 10. For an RMS value of  $5 \text{ m/s}^2$  (approximately 0.5 g), the ride would be at the severe discomfort level for 1-h exposure and the KPI would be zero. When considering the ISO 2631 curves it may be seen that the progression through the different comfort levels from ‘perception’ to ‘comfort’ and onto ‘severe discomfort’ is substantially linear and therefore the KPI is a useful indicator.

The secondary ride KPI is very similar but just in a different frequency band – 3 to 20 Hz – and using different constants. A value of  $3 \text{ m/s}^2$  reflects the ISO 2631 severe discomfort threshold in the 12–15 Hz region.

$$\text{KPI} = \left( 3 - \sqrt{\frac{\int_0^t (\text{filt}(\ddot{z}_s))^2 dt}{t}} \right) \cdot \frac{10}{3} \quad (4.102)$$

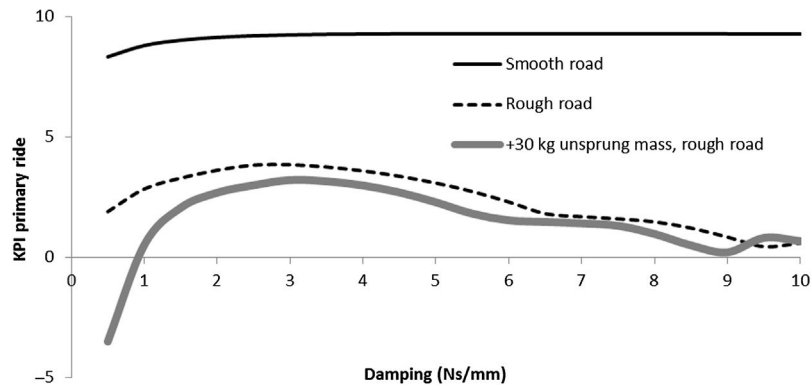
Again using the simplified model, these KPIs can be investigated to illustrate certain aspects of ride behaviour that may not be obvious. Ride KPIs were investigated as part of the same exercise as that used to generate [Figure 4.69](#) (Harty, 2010).

Smooth road primary ride gets very high KPI scores, close to 10, suggesting that the primary ride environment is largely indistinguishable from a completely quiescent vehicle, which matches well with experience of motorway travel where primary events are confined to specifically notable features in the road such as bridge joints or repair patches ([Figure 4.71](#)).

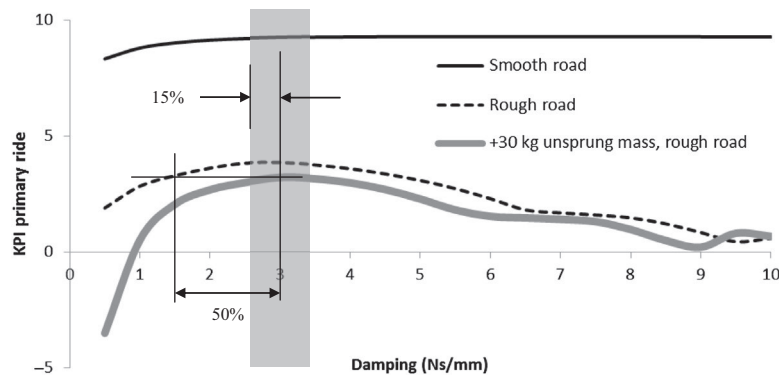
On a rough road the KPI scores fall to around 3, reflecting the fact that surfaces such as the MIRA Pavé are somewhat uncomfortable on prolonged exposure.

Particularly light levels of damping, such as might be present in a vehicle with faulty dampers, produce a larger change of response. The degradation in primary ride with unsprung mass is most marked on a rough road and with light levels of damping.

In general the degradation in rough road primary ride due to 30 kg additional unsprung mass is similar to a slightly miscalibrated damper. (The process capability for a typical automotive damper delivers dampers to around  $\pm 15\%$  of the nominal characteristic; the miscalibration required to reproduce the unsprung mass effect is about 50%, see [Figure 4.72](#)).

**FIGURE 4.71**

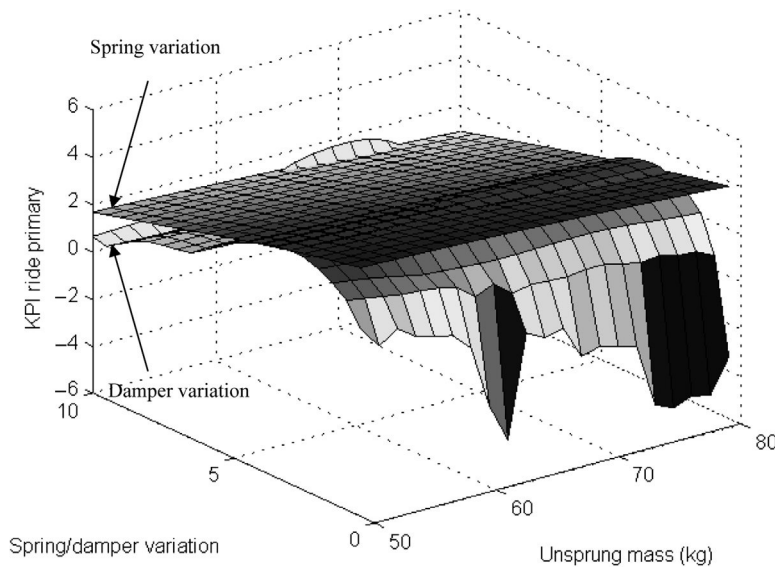
Variation in Primary Ride Key Performance Indicator (KPI) between rough and smooth roads.

**FIGURE 4.72**

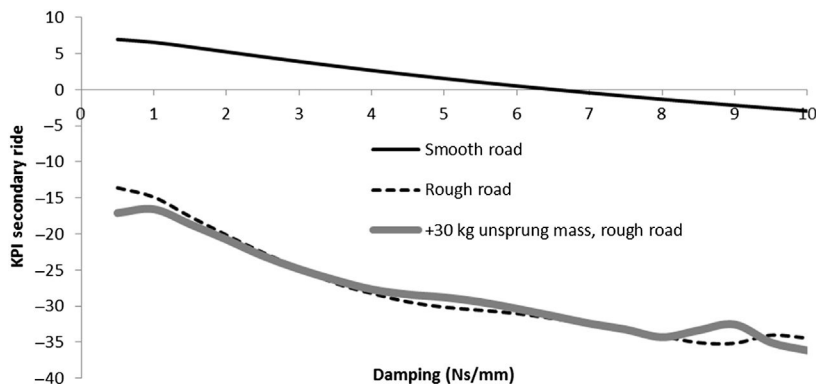
Typical damper calibration process capability in mass production compared to Primary Ride Key Performance Indicators (KPIs).

The influence of spring calibration on primary ride is also significant (Figure 4.73). In general softer springs produce a better ride with no optimum. Were RMS acceleration the only consideration, then the springs on every vehicle would be driven as soft as possible; motion sickness considerations generally prevent primary ride frequencies being set below about 0.6 Hz as described in Section 4.1.2.

Secondary ride KPI values are illustrated in Figure 4.74 and show a very different trend; immediately noticeable is that for typical levels of damping (around 3 Ns/mm) even the smooth road KPI is less than 4, suggesting that secondary ride is very noticeable and verging on uncomfortable for prolonged exposure even on a smooth road. This may seem a strange assertion but the sceptical reader is invited to spend some time sitting uncushioned in the load area of a panel van to test its

**FIGURE 4.73**

Relative influence of springs, dampers and unsprung mass on primary ride Key Performance Indicator (KPI).

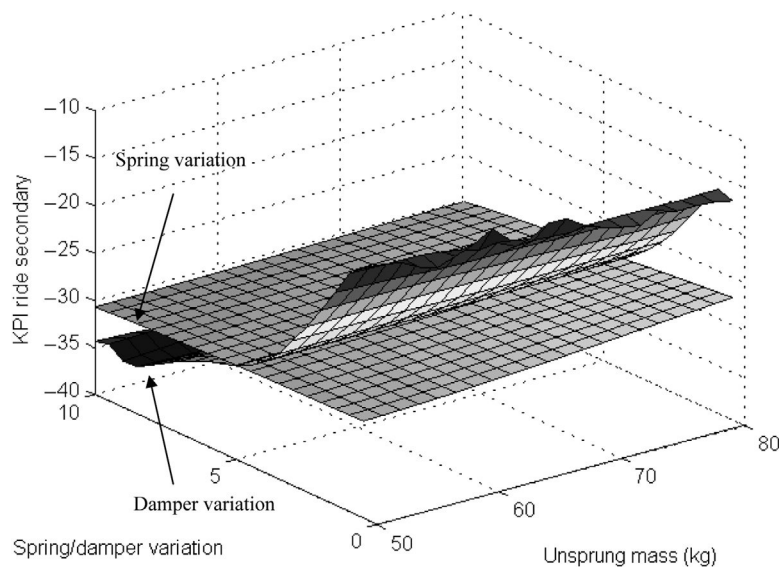
**FIGURE 4.74**

Key performance indicator (KPI) results show that secondary ride starts from a lower base and is more degraded by suspension damping on smooth roads than rough.

veracity. Modern automotive seats are primarily aimed at isolating the secondary ride issue but even so, secondary ride — a restless jiggle — is noticeable on all but the very best of modern cars even on the motorway.

It is also apparent that any level of suspension damping degrades secondary ride in a very pronounced manner. For this reason, road vehicle damper curves tend to have



**FIGURE 4.75**

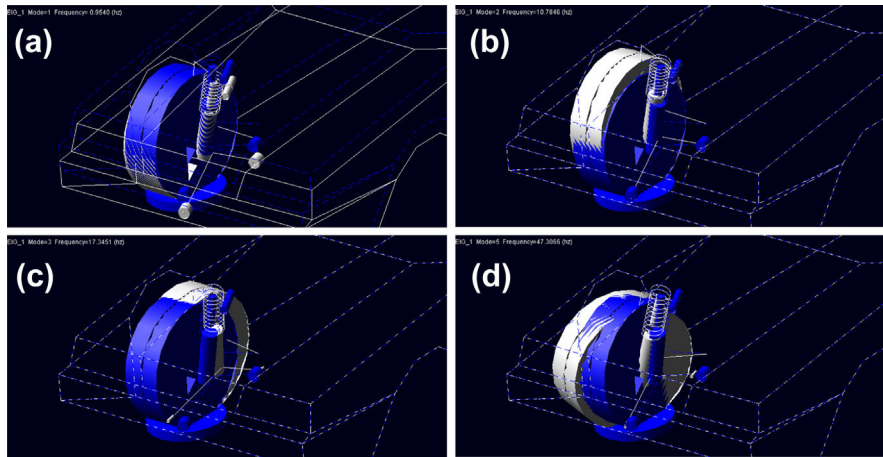
Relative influence of spring and damper calibration on secondary Ride Key Performance Indicator (KPI).

an initial slope around the level identified for primary ride damping but then show a characteristic ‘knee point’ when the damping force slope rises much less steeply. This characteristic is sometimes referred to as ‘digressive’ or ‘blow-off’. Rough road secondary ride is shown to be well below the severe discomfort threshold and this is no surprise to anyone who has driven on the MIRA Pavé for any length of time; staying on it for an hour continuously is a deeply unpleasant prospect.

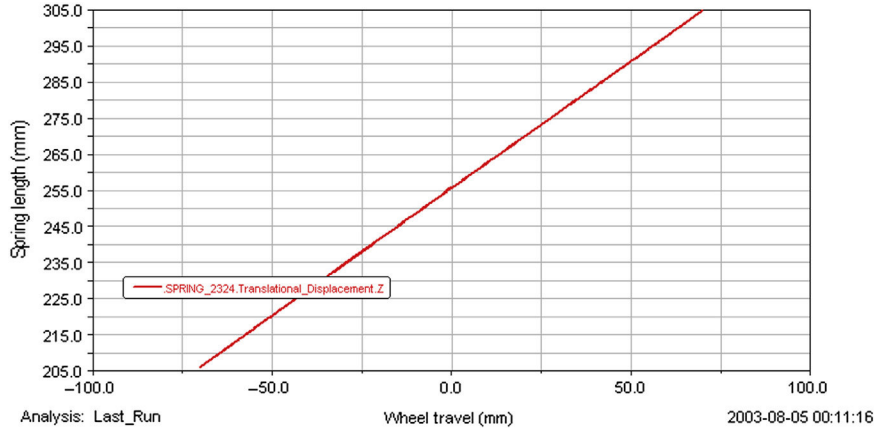
Suspension springs are shown in [Figure 4.75](#) to have almost no influence on secondary ride; this is unsurprising because the stiffness in the wheel hop mode is entirely dominated by the tyre stiffness.

When a real suspension system is analysed in MSC ADAMS, the first observation is that the eigenvalues predicted using the same data but including real wishbone and elastomer geometry can be seen to be somewhat different ([Figure 4.76](#)).

The first and most obvious source of error is the use of the spring rate directly from the detailed model in the simplified 2 DoF models. In the literal model, as in the real vehicle, the motion of the wheel does not directly correspond to the motion of the spring. Using the model, the so-called ‘motion ratio’ can be examined between spring and wheel. It can be seen ([Figure 4.77](#)) that the spring changes length by 1.43 mm for every 1 mm of wheel vertical motion. Some practitioners quote this as a motion ratio of 1.43:1 and some as 0.699:1. It is very important that the definition of motion ratio as quoted is understood before the number is used since it is not always true that the spring moves less than the wheel.

**FIGURE 4.76**

MSC ADAMS Prediction of the modes of vibration for the full linkage quarter vehicle model with elastomers and individual component mass & inertia data. Primary ride mode (a) 0.95 Hz. Wheel hop mode (b) 10.78 Hz. Fore-Aft compliance mode (c) 17.35 Hz. Unsprung mass lateral mode (d) 47.31 Hz.

**FIGURE 4.77**

Spring motion with respect to wheel motion from the model.

Using the motion ratio, the wheel rate due to the spring can be seen to be  $31.96/1.43^2 = 15.63 \text{ N/mm}$ . Reassessing the classical calculations, the estimated ride frequency is now 0.874 Hz, with a ride rate of 14.24 N/mm. The estimated frequency is now less than that calculated using the full model (which was 0.954 Hz), suggesting there is some additional stiffness or reduced mass in the mode of

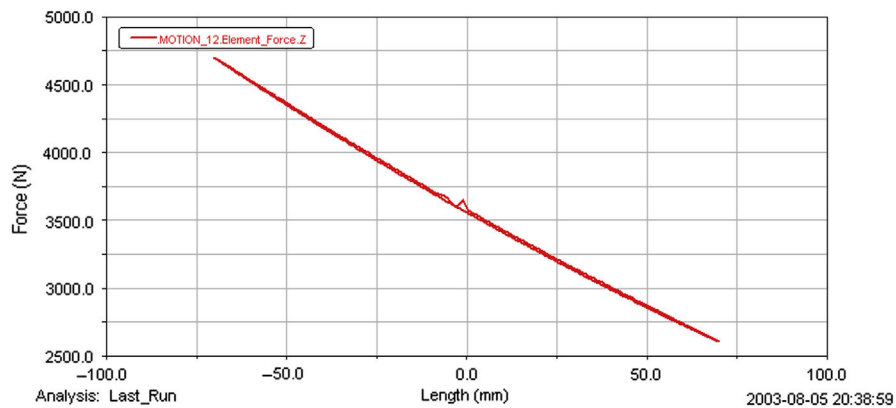


FIGURE 4.78

Wheel rate measured from full linkage model.

vibration. Examining the model again, the so-called ‘ride rate’ ( $k_{eqv}$ ) can be taken directly (Figure 4.78). In the case of this particular suspension geometry, the ride rate is 14.97 N/mm. The difference between the two ride rates may be attributed to additional rates arising from the suspension bushes; these are commonly referred to as ‘parasitic’ rates. Although that term implies something undesirable about them, they are generally small and do not degrade the suspension behaviour unduly.

It may be noted, however, that the difference between the two results is not fully accounted for by the difference in wheel rate. For this particular example, using the analytical solution for the two-mass solution we may calculate ‘effective’ masses and stiffnesses for each mode of vibration. In the results from the full-linkage model, these effective masses and stiffnesses are reported in the results file. They are known as ‘modal’ mass and stiffness values and the interested reader is referred to Brüel and Kjær (2003) for further information on the theory of modal decomposition. For the model in question, these values are given as:

Primary Ride Mode :	$17.2232 \text{ Nmm}^{-1}$	479.355 kg
Wheel Hop Mode :	$177.019 \text{ Nmm}^{-1}$	38.552 kg

It is clear then that there is some other stiffness influence on the ride rate within the model that is not readily apparent to the user. Similarly, the mass is larger than the mass associated with the body alone. This acknowledges the fact that several components are in motion with different velocities when the modes of vibration are excited and thus are storing kinetic energy.

For an appreciation of these and other differences between the simplified and detailed model, the kinetic and strain energy tables calculated by the software can be examined in some detail. Table 4.8 shows the output from the calculations directly. It can be seen that the kinetic energy for the primary ride mode is contained almost

**Table 4.8** ADAMS/Linear Output Tables

```
*****
Mode number           =      1
Damping ratio         = 2.20914E-11
Undamped natural freq.= 9.54000E+01 Cycles per second
Generalized stiffness = 1.72232E+01 User units
Generalized mass      = 4.79355E+02 User units
Kinetic energy        = 2.18135E-01
```

	Percentage distribution of Kinetic energy									
	X	Y	Z	RXX	RYY	RZZ	RXY	RXZ	RYZ	
PART/20										
PART/1			98.50							
PART/10		0.06	0.04	0.02						
PART/11		0.03	0.02							
PART/12			0.22	0.05						
PART/13			0.30							
PART/14			0.34	0.03	0.01		0.01			
PART/15			0.04	0.01						
PART/16			0.08							
PART/17		0.01	0.20	0.01						

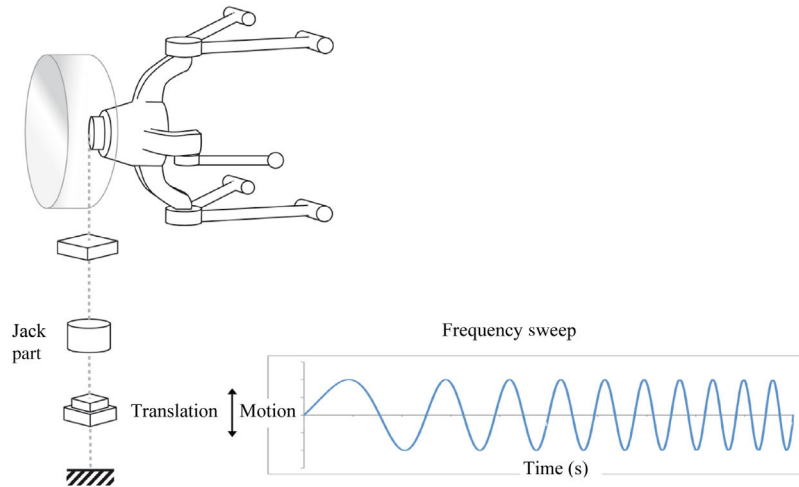
  

	Percentage distribution of Strain energy						
	Total	X	Y	Z	RXX	RYY	RZZ
BUSH/16	0.01				0.01		
BUSH/17	0.01				0.01		
BUSH/19	0.14				0.14		
BUSH/21	16.32	0.06	2.47	13.78			
SFOR/1029	5.35	0.01	0.55	4.79			
SFOR/2728							
SFOR/2526							
SFOR/3233							
SPRI/2324	37.71	0.19	0.25	37.27			
GFOR/16	20.16	2.38	17.27	0.02	0.49		
GFOR/17	2.56	0.22	1.79	0.13	0.42		

entirely in the body mass. However, it will be noted that  $472/0.985 = 479$  kg; thus the kinetic energy table shows the contributions of the other components in the model to the modal mass. The largest of the other contributors to the modal mass are the suspension arms and damper. For the strain energy results, it can be seen that the compliance in the lower arm bushes and the damper lower bush contribute significantly to strain energy storage in the primary ride mode of vibration.

The preceding results were all generated using an undamped eigensolution. However, for vehicle ride work, the influence of the dampers is critical. Moreover, the dampers are typically highly nonlinear devices. Therefore, a further treatment of the existing MSC ADAMS model is required once the modal 'positioning' (i.e. the undamped frequencies for primary ride, wheel hop and fore-aft compliance) is established.

In terms of the modelling approach the only modification here is a change in the motion applied to the jack where the function now represents a sinusoidal input with fixed amplitude but with a frequency that increases as a function of time as illustrated in

**FIGURE 4.79**

Input of frequency sweep via jack motion.

**Figure 4.79.** The motion input is referred to as a frequency sweep, sometimes described as a ‘chirp’ — for reasons that are obvious if the resulting signal is audible.

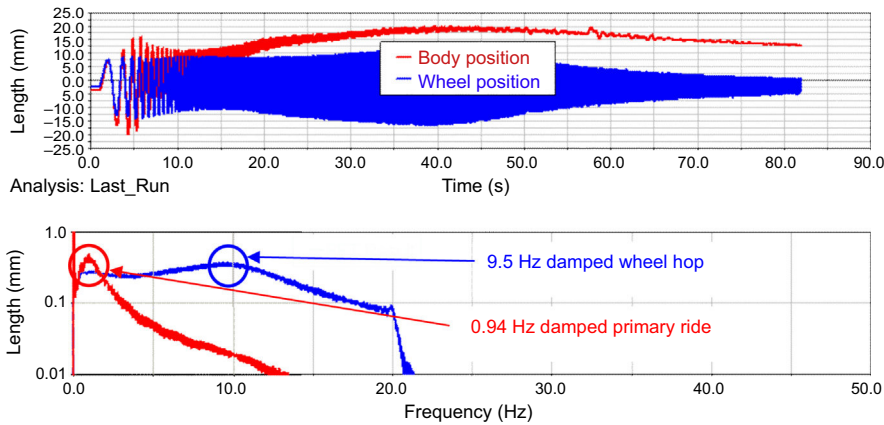
The following motion statement is an example of a suitable input function where the amplitude of the road input is fixed at 10 mm and the frequency is increased using the following function from 0 to 20 Hz after 80 s.

```
FUNCTION=10.0*SIN(TIME/8*TIME*360D)
```

Using this simulation we can produce a time history plot showing the change in vertical acceleration of the sprung and unsprung masses of the quarter vehicle model as the simulation progresses. Examination of the response shown in [Figure 4.80](#) reveals that excitation of a system resonance (a ‘mode of vibration’) occurs at two points during the simulation. The first of these corresponds with the natural frequency of the body and the second with the natural frequency of the unsprung mass.

Another interpretation of the results obtained here is to perform a fast Fourier transform so that the results can be plotted in the frequency rather than the time domain as shown in the lower half of [Figure 4.80](#). The simulation was allowed to run for 81.91 s with an output sample rate of 100 Hz, giving 8192 points including the zero time point. A single buffer transform was performed on the entire record for each of the signals. Although flawed, this method is adequate for identifying frequency peaks. However, for quantifying amplitude content the underlying presumption that the signal repeats itself after the end of the observation buffer is clearly in error; therefore the magnitude results of this exercise should not be used further.

From [Figure 4.80](#) it can be seen that the damped natural frequency of the body occurs at around 0.94 Hz and that the natural frequency of the unsprung mass is about 9.5 Hz. Comparing these values with the previous values, it can be seen



**FIGURE 4.80**

Quarter vehicle body vertical acceleration time history. (For colour version of this figure, the reader is referred to the online version of this book.)

they are systematically low. This is to be expected since the addition of a damping ratio  $\zeta$  reduces the damped natural frequency,  $\omega_d$ , when compared with the undamped natural frequency,  $\omega_n$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \quad (4.103)$$

The greatest reduction in frequency comes with the heavily damped wheel hop mode. Observing the change between undamped and damped frequency allows an estimate of the damping ratio to be made. For the two modes the damping ratios can be estimated as 0.07 and 0.48 for primary ride and wheel hop, respectively. The damping of the primary ride is low. However, if the exercise were to be repeated at different amplitudes of excitation, the level of damping in the primary ride is certain to vary since the damper characteristics are highly nonlinear. For this reason, the time-domain method and subsequent processing are the preferred methods for evaluating ride behaviour once simplified undamped positioning calculations have been carried out.

## 4.10 Case study 5 – suspension vector analysis comparison with MBS

### 4.10.1 Problem definition

The following study is intended to demonstrate the application of the vector theory outlined in Chapter 2 to a range of suspension analyses. Before the advent of computer programs to analyse the motion of suspension linkages, vehicle designers resorted to graphical methods or simplified calculations often using two-dimensional representations to study the suspension in a fore-aft or transverse plane.

These methods are still taught and included in many texts addressing vehicle dynamics. They can develop an understanding of suspension design, and the effect on total vehicle performance, that can be lost using the automated methods in modern computer aided analysis.

The aim of this textbook is to bridge the gap between traditional vehicle dynamics theories and the MBS approach. The following calculations are typical of the processes carried out using MBS software. Whilst the methods used here do not represent exactly the internal machinations of, for example, the MSC ADAMS software, they do give an indication of the computational process involved. The example chosen here is based on a typical double wishbone suspension system. The answers obtained are compared with those running an MSC ADAMS model using the same data. The calculations will include a series of analyses including:

- Geometry analysis
- Velocity analysis
- Acceleration analysis
- Static force analysis
- Dynamic force analysis

The geometric data required to define this problem is defined in Figure 4.81. Note that in this example the x-axis is orientated towards the front of the vehicle rather than pointing to the rear as used generally throughout this chapter. This is therefore the front left suspension system on the vehicle.

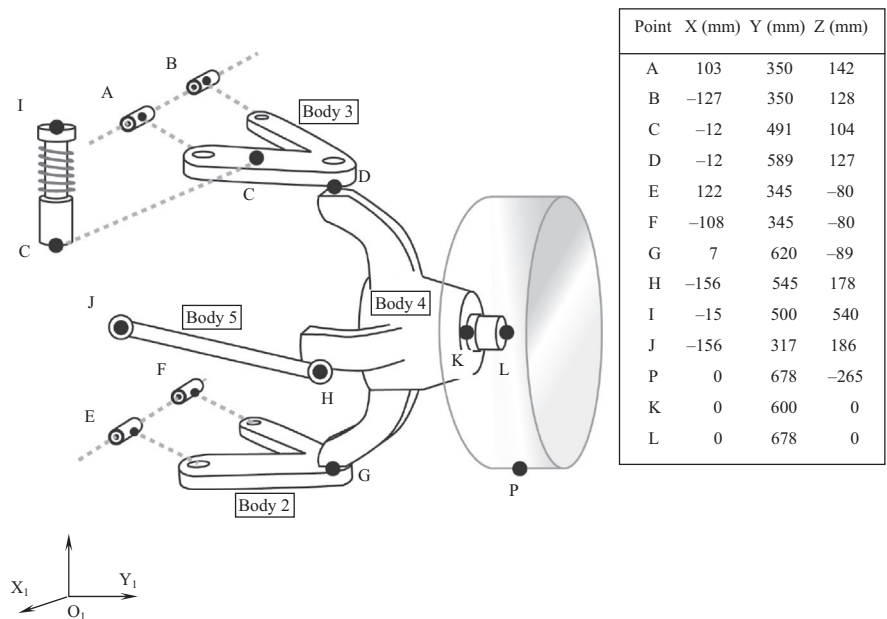
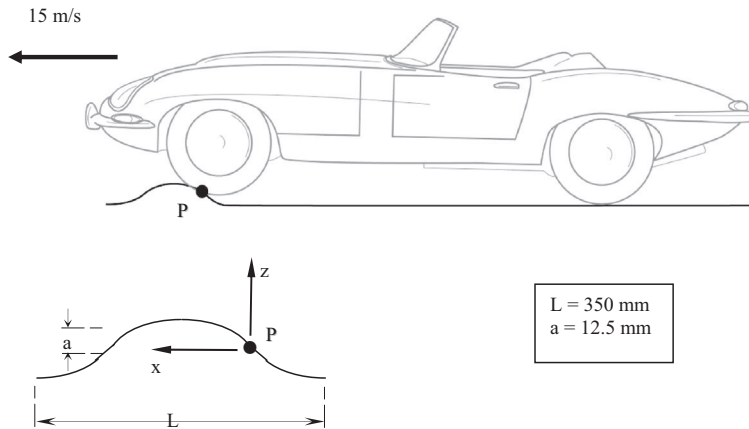


FIGURE 4.81

Double wishbone suspension example geometry data.



**FIGURE 4.82**

Road input definition for velocity analysis.

### 4.10.2 Velocity analysis

The starting point for this analysis is to establish a boundary condition, or road input, at the tyre contact patch, P, as the vehicle negotiates the road hump shown in Figure 4.82 with a forward speed of 15 m/s. The analysis is simplified by ignoring the compliance in the tyre and the profile of the road hump is taken as a sine function.

The local x-z axis taken to reference the geometry of the road hump is located at a point where the vertical velocity,  $V_{Pz}$ , of the contact point P reaches a maximum with a corresponding vertical acceleration,  $A_{Pz}$  equal to zero. The profile of the road hump can be defined using

$$z = a \sin\left(\frac{2\pi x}{L}\right) \quad (4.104)$$

Working in millimetres and taking the wavelength L as 350 mm and the amplitude a as 12.5 mm, for a total bump height of 25 mm, gives

$$z = 12.5 \sin\left(\frac{\pi x}{175}\right) \quad (4.105)$$

We are after the vertical velocity of point P, which can be expressed as:

$$\frac{dz}{dt} = \frac{dz}{dx} \frac{dx}{dt} \quad (4.106)$$

where

$$\frac{dx}{dt} = 15000 \text{ mm/s}$$



and

$$\frac{dz}{dx} = \frac{12.5\pi}{175} \cos\left(\frac{\pi x}{175}\right) \quad (4.107)$$

giving

$$\frac{dz}{dt} = \frac{12.5\pi}{175} \cos\left(\frac{\pi x}{175}\right) 15000 \quad (4.108)$$

The maximum value of  $\frac{dz}{dt}$  occurs when  $\cos\left(\frac{\pi x}{175}\right) = 1$  and occurs at values of  $x = 0, 350, 700 \dots$  giving

$$\left(\frac{dz}{dt}\right)_{\max} = \frac{12.5\pi}{175} 15000 = 3366 \text{ mm/s} \quad (4.109)$$

The acceleration  $\frac{d^2z}{dt^2}$  is given by

$$\frac{d^2z}{dt^2} = -\frac{12.5\pi^2}{175^2} \sin\left(\frac{\pi x}{175}\right) 15000 \quad (4.110)$$

and has a value of zero at  $x = 0, 350, 700 \dots$

This provides inputs for the following velocity and acceleration analyses of

$$V_{Pz} = 3366 \text{ mm/s}$$

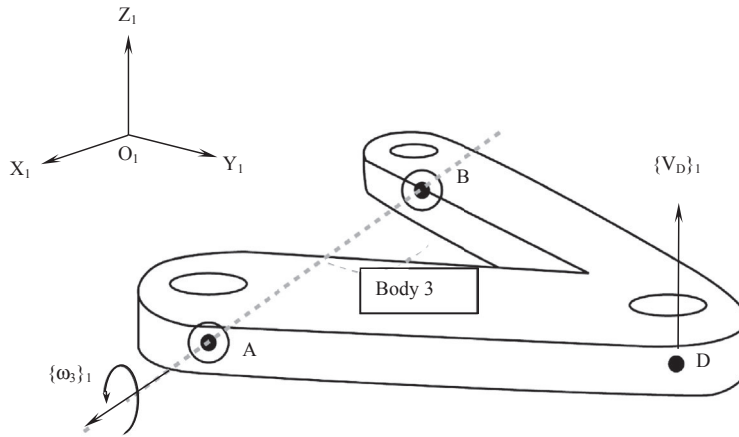
$$A_{Pz} = 0 \text{ mm/s}^2$$

The approach taken here is to initially ignore the spring damper assembly between points C and I shown in [Figure 4.82](#). Solving for the rest of the suspension system will deliver the velocity  $\{V_C\}_1$  of point C, thus providing a boundary condition allowing a separate analysis of the spring damper to follow.

Before proceeding with the velocity analysis it is necessary to identify the unknowns that define the problem and the same number of equations as unknowns leading to a solution. The angular velocities of the rigid bodies representing suspension components can be used to find the translational velocities at points within the system. An example of this is shown for the upper wishbone in [Figure 4.83](#).

Referring back to the earlier treatment in Chapter 2 we can remind ourselves that as the suspension arm is constrained to rotate about the axis AB, ignoring at this stage any possible deflection due to compliance in the suspension bushes, the vector  $\{\omega_3\}_1$  for the angular velocity of Body 3 will act along the axis of rotation through AB. The components of this vector would adopt signs consistent with producing a positive rotation about this axis as shown in [Figure 4.83](#).

When setting up the equations to solve a velocity analysis it will be desirable to reduce the number of unknowns based on the knowledge that a particular body is constrained to rotate about a known axis as shown here. The velocity vector,  $\{\omega_3\}_1$ , could for example be represented as follows.



**FIGURE 4.83**

Angular and translational velocity vectors for the upper wishbone.

$$\{\omega_3\}_1 = f\omega_3 \{\mathbf{R}_{AB}\}_1 \quad (4.111)$$

Since  $\{\omega_3\}_1$  is parallel to the relative position vector  $\{\mathbf{R}_{AB}\}_1$  a scale factor  $f\omega_3$  can be introduced. This reduces the problem from the three unknown components,  $\omega_{x3}$ ,  $\omega_{y3}$  and  $\omega_{z3}$  of the vector  $\{\omega_3\}_1$  to a single unknown  $f\omega_3$ .

Once the angular velocities of Body 3 have been found it follows that the translational velocity of, for example, point D can be found from

$$\{\mathbf{V}_{DA}\}_1 = \{\omega_3\}_1 \times \{\mathbf{R}_{DA}\}_1 \quad (4.112)$$

It also follows that since point A is considered fixed with a velocity  $\{\mathbf{V}_A\}_1$  equal to zero that the absolute velocity  $\{\mathbf{V}_D\}_1$  of point D can be found from a consideration of the triangle law of vector addition giving

$$\{\mathbf{V}_D\}_1 = \{\mathbf{V}_{DA}\}_1 \quad (4.113)$$

A consideration of the complete problem indicates that the translational velocities throughout the suspension system can be found if the angular velocities of all the rigid bodies 2, 3, 4 and 5 are known. Clearly the same approach can be taken with the lower wishbone, Body 2, as with the upper wishbone using a single scale factor  $f\omega_2$  to replace the three unknown components,  $\omega_{x2}$ ,  $\omega_{y2}$  and  $\omega_{z2}$  of the vector  $\{\omega_2\}_1$ . Finally a consideration of the boundaries of this problem reveals that while points A, B, E, F and J are fixed, the longitudinal velocity,  $V_{Px}$  and the lateral velocity,  $V_{Py}$  at the contact point P remain as unknowns.

Thus this analysis can proceed if we can develop 10 equations to solve the 10 unknowns:

$$f\omega_2, f\omega_3, \omega_{x4}, \omega_{y4}, \omega_{z4}, \omega_{x5}, \omega_{y5}, \omega_{z5}, V_{Px}, V_{Py}$$

Working through the problem it will be seen that a strategy can be developed using the triangle law of vector addition to generate sets of equations. As a starting point we will develop a set of three equations using

$$\{\mathbf{V}_{DG}\}_1 = \{\mathbf{V}_D\}_1 - \{\mathbf{V}_G\}_1 \quad (4.114)$$

In this form Eqn (4.114) does not introduce any of the 10 unknowns listed above. It will therefore be necessary to initially define  $\{\mathbf{V}_{DG}\}_1$ ,  $\{\mathbf{V}_D\}_1$  and  $\{\mathbf{V}_G\}_1$  in terms of the angular velocity vectors that contain unknowns requiring solution. The first step in the analysis can therefore proceed as follows.

Determine an expression for the velocity  $\{\mathbf{V}_G\}_1$  at point G

$$\{\mathbf{V}_G\}_1 = \{\mathbf{V}_{GE}\}_1 = \{\omega_2\}_1 \times \{\mathbf{R}_{GE}\}_1 \quad (4.115)$$

$$\{\omega_2\}_1 = f\omega_2 \{\mathbf{R}_{EF}\}_1 = f\omega_2 \begin{bmatrix} 230 \\ 0 \\ 0 \end{bmatrix} \text{ rad/s} \quad (4.116)$$

$$\begin{bmatrix} V_{Gx} \\ V_{Gy} \\ V_{Gz} \end{bmatrix} = f\omega_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -230 \\ 0 & 230 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 275 \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 2070f\omega_2 \\ 63250f\omega_2 \end{bmatrix} \text{ mm/s} \quad (4.117)$$

Determine an expression for the velocity  $\{\mathbf{V}_D\}_1$  at point D

$$\{\mathbf{V}_D\}_1 = \{\mathbf{V}_{DA}\}_1 = \{\omega_3\}_1 \times \{\mathbf{R}_{DA}\}_1 \quad (4.118)$$

$$\{\omega_3\}_1 = f\omega_3 \{\mathbf{R}_{AB}\}_1 = f\omega_3 \begin{bmatrix} 230 \\ 0 \\ 14 \end{bmatrix} \text{ rad/s} \quad (4.119)$$

$$\begin{bmatrix} V_{Dx} \\ V_{Dy} \\ V_{Dz} \end{bmatrix} = f\omega_3 \begin{bmatrix} 0 & -14 & 0 \\ 14 & 0 & -230 \\ 0 & 230 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 239 \\ -15 \end{bmatrix} = \begin{bmatrix} -3346f\omega_3 \\ 1840f\omega_3 \\ 54970f\omega_3 \end{bmatrix} \text{ mm/s} \quad (4.120)$$

Determine an expression for the relative velocity  $\{\mathbf{V}_{DG}\}_1$  of point D relative to point G

$$\{\mathbf{V}_{DG}\}_1 = \{\omega_4\}_1 \times \{\mathbf{R}_{DG}\}_1 \quad (4.121)$$

$$\begin{bmatrix} V_{DGx} \\ V_{Dy} \\ V_{DGz} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{4z} & \omega_{4y} \\ \omega_{4z} & 0 & -\omega_{4x} \\ -\omega_{4y} & \omega_{4x} & 0 \end{bmatrix} \begin{bmatrix} -19 \\ -31 \\ 216 \end{bmatrix} = \begin{bmatrix} 31\omega_{4z} + 216\omega_{4y} \\ -19\omega_{4z} - 216\omega_{4x} \\ 19\omega_{4y} - 31\omega_{4x} \end{bmatrix} \text{ mm/s} \quad (4.122)$$

We can now apply the triangle law of vector addition to equate the expression for  $\{V_{DG}\}_1$  in Eqn (4.122) with  $\{V_D\}_1$  in Eqn (4.120) and  $\{V_G\}_1$  in Eqn (4.117).

$$\{V_{DG}\}_1 = \{V_D\}_1 - \{V_G\}_1 \quad (4.123)$$

$$\begin{bmatrix} 31\omega_{4z} + 216\omega_{4y} \\ -19\omega_{4z} - 216\omega_{4x} \\ 19\omega_{4y} - 31\omega_{4x} \end{bmatrix} = \begin{bmatrix} -3346f_{\omega 3} \\ 1840f_{\omega 3} \\ 54970f_{\omega 3} \end{bmatrix} - \begin{bmatrix} 0 \\ 2070f_{\omega 2} \\ 63250f_{\omega 2} \end{bmatrix} \text{ mm/s} \quad (4.124)$$

Rearranging Eqn (4.124) yields the first three equations required to solve the analysis.

$$\text{Equation 1} \quad 3346 f_{\omega 3} + 216 \omega_{4y} + 31 \omega_{4z} = 0 \quad (4.125)$$

$$\text{Equation 2} \quad 2070 f_{\omega 2} - 1840 f_{\omega 3} - 216 \omega_{4x} - 19 \omega_{4z} = 0 \quad (4.126)$$

$$\text{Equation 3} \quad 63250 f_{\omega 2} - 54970 f_{\omega 3} - 31 \omega_{4x} + 19 \omega_{4y} = 0 \quad (4.127)$$

We can now proceed to set up the next set of three equations working from point H to point D and using the triangle law of vector addition.

$$\{V_{DH}\}_1 = \{V_D\}_1 - \{V_H\}_1 \quad (4.128)$$

Determine an expression for the velocity  $\{V_H\}_1$  at point H

$$\{V_H\}_1 = \{V_{HJ}\}_1 = \{\omega_5\}_1 \times \{R_{HJ}\}_1 \quad (4.129)$$

$$\begin{bmatrix} V_{Hx} \\ V_{Hy} \\ V_{Hz} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{5z} & \omega_{5y} \\ \omega_{5z} & 0 & -\omega_{5x} \\ -\omega_{5y} & \omega_{5x} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 228 \\ -8 \end{bmatrix} = \begin{bmatrix} -228\omega_{5z} - 8\omega_{5y} \\ 8\omega_{5x} \\ 228\omega_{5x} \end{bmatrix} \text{ mm/s} \quad (4.130)$$

We already have an expression for  $\{V_D\}_1$  in Eqn (4.120) and can determine an expression for the relative velocity  $\{V_{DH}\}_1$  of point D relative to point H using

$$\{V_{DH}\}_1 = \{\omega_4\}_1 \times \{R_{DH}\}_1 \quad (4.131)$$

$$\begin{bmatrix} V_{DHx} \\ V_{DHy} \\ V_{DHx} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{4z} & \omega_{4y} \\ \omega_{4z} & 0 & -\omega_{4x} \\ -\omega_{4y} & \omega_{4x} & 0 \end{bmatrix} \begin{bmatrix} 144 \\ 44 \\ -51 \end{bmatrix} = \begin{bmatrix} -44\omega_{4z} - 51\omega_{4y} \\ 144\omega_{4z} + 51\omega_{4x} \\ -144\omega_{4y} + 44\omega_{4x} \end{bmatrix} \text{ mm/s} \quad (4.132)$$

We can now apply the triangle law of vector addition to equate the expression for  $\{V_{DH}\}_1$  in Eqn (4.132) with  $\{V_D\}_1$  in Eqn (4.120) and  $\{V_H\}_1$  in Eqn (4.130).

$$\{V_{DH}\}_1 = \{V_D\}_1 - \{V_H\}_1 \quad (4.133)$$

$$\begin{bmatrix} -44\omega_{4z} - 51\omega_{4y} \\ 144\omega_{4z} + 51\omega_{4x} \\ -144\omega_{4y} + 44\omega_{4x} \end{bmatrix} = \begin{bmatrix} -3346f_{\omega 3} \\ 1840f_{\omega 3} \\ 54970f_{\omega 3} \end{bmatrix} - \begin{bmatrix} -228\omega_{5z} - 8\omega_{5y} \\ 8\omega_{5x} \\ 228\omega_{5x} \end{bmatrix} \text{ mm/s} \quad (4.134)$$

Rearranging Eqn (4.134) yields the next three equations required to solve the analysis.

$$\text{Equation 4} \quad 3346 f_{\omega 3} - 51 \omega_{4y} - 44 \omega_{4z} - 8 \omega_{5y} - 228 \omega_{5z} = 0 \quad (4.135)$$

$$\text{Equation 5} \quad -1840 f_{\omega 3} + 51 \omega_{4x} + 144 \omega_{4z} + 8 \omega_{5x} = 0 \quad (4.136)$$

$$\text{Equation 6} \quad -54970 f_{\omega 3} + 44 \omega_{4x} - 144 \omega_{4y} + 228 \omega_{5x} = 0 \quad (4.137)$$

We can now proceed to set up the next set of three equations working from point G to point P and using the triangle law of vector addition.

$$\{V_{PG}\}_1 = \{V_P\}_1 - \{V_G\}_1 \quad (4.138)$$

Determine an expression for the relative velocity  $\{V_{PG}\}_1$  of point P relative to point G

$$\{V_{PG}\}_1 = \{\omega_4\}_1 \times \{R_{PG}\}_1 \quad (4.139)$$

$$\begin{bmatrix} V_{PGx} \\ V_{PGy} \\ V_{PGz} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{4z} & \omega_{4y} \\ \omega_{4z} & 0 & -\omega_{4x} \\ -\omega_{4y} & \omega_{4x} & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 58 \\ -176 \end{bmatrix} = \begin{bmatrix} -58\omega_{4z} - 176\omega_{4y} \\ -7\omega_{4z} + 176\omega_{4x} \\ 7\omega_{4y} + 58\omega_{4x} \end{bmatrix} \text{ mm/s} \quad (4.140)$$

We already have an expression for  $\{V_G\}_1$  in Eqn (4.117) and we can define the vector  $\{V_P\}_1$  in terms of the known vertical velocity component,  $V_{Pz}$  and the unknown components,  $V_{Px}$  and  $V_{Py}$ .

$$\{V_P\}_1 = \begin{bmatrix} V_{Px} \\ V_{Py} \\ 3366 \end{bmatrix} \text{ mm/s} \quad (4.141)$$

We can now apply the triangle law of vector addition to equate the expression for  $\{V_{PG}\}_1$  in Eqn (4.140) with  $\{V_P\}_1$  in Eqn (4.141) and  $\{V_G\}_1$  in Eqn (4.117).

$$\{V_{PG}\}_1 = \{V_P\}_1 - \{V_G\}_1 \quad (4.142)$$

$$\begin{bmatrix} -58\omega_{4z} - 176\omega_{4y} \\ -7\omega_{4z} + 176\omega_{4x} \\ 7\omega_{4y} + 58\omega_{4x} \end{bmatrix} = \begin{bmatrix} V_{Px} \\ V_{Py} \\ 3366 \end{bmatrix} - \begin{bmatrix} 0 \\ 2070f_{\omega 2} \\ 63250f_{\omega 2} \end{bmatrix} \text{ mm/s} \quad (4.143)$$

Rearranging Eqn (4.143) yields the next set of three equations required to solve the analysis.

$$\text{Equation 7} \quad 176 \omega_{4y} + 58 \omega_{4z} + V_{Px} = 0 \quad (4.144)$$

$$\text{Equation 8} \quad -2070 f_{\omega 2} - 176 \omega_{4x} + 7 \omega_{4z} + V_{Py} = 0 \quad (4.145)$$

$$\text{Equation 9} \quad -63250 f_{\omega 2} - 58 \omega_{4x} - 7 \omega_{4y} = -3366 \quad (4.146)$$

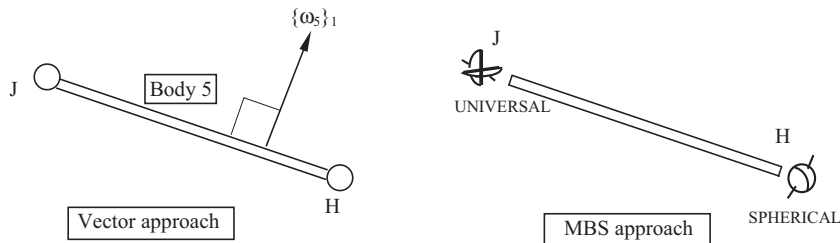
This leaves us with nine equations and ten unknowns. The last equation is obtained by constraining the rotation of the tie rod (Body 5) to prevent spin about its own axis. This degree of freedom has no bearing on the overall kinematics of the suspension linkage. If there is to be no spin component of angular velocity parallel to the axis of the tie rod, spin, then the angular velocity vector  $\{\omega_5\}_1$  must be perpendicular to the line HJ as shown in Figure 4.84. Note that this is equivalent to a common practice in MBS modelling where a universal or Hooke's joint is used at one end of the link with a spherical joint at the other end. The universal joint allows the tie rod to articulate in a way that does not constrain overall suspension movement but constrains the spin freedom that would exist if a spherical joint were used at each end.

Using the vector dot product to enforce perpendicularity, as described in Chapter 2, yields the tenth and final equation required to solve this part of the problem.

$$\{\omega_5\}_1 \bullet \{R_{HJ}\}_1 = [\omega_{5x} \ \omega_{5y} \ \omega_{5z}] \begin{bmatrix} 0 \\ 228 \\ -8 \end{bmatrix} = 0 \text{ mm/s} \quad (4.147)$$

$$\text{Equation 10} \quad 228 \omega_{5y} - 8 \omega_{5z} = 0 \quad (4.148)$$

The 10 equations can now be set up in matrix form ready for solution. The solution of a  $10 \times 10$  matrix will require access to a mathematical or spreadsheet program that offers the capability to invert the matrix.



**FIGURE 4.84**

Constraining the spin degree of freedom of the tie rod. MBS, multibody systems.

$$\begin{bmatrix}
 0 & 3346 & 0 & 216 & 31 & 0 & 0 & 0 & 0 & 0 \\
 2070 & -1840 & -216 & 0 & -19 & 0 & 0 & 0 & 0 & 0 \\
 63250 & -54970 & -31 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 3346 & 0 & -51 & -44 & 0 & -8 & -228 & 0 & 0 \\
 0 & -1840 & 51 & 0 & 144 & 8 & 0 & 0 & 0 & 0 \\
 0 & -54970 & 44 & -144 & 0 & 228 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 176 & 58 & 0 & 0 & 0 & 1 & 0 \\
 -2070 & 0 & -176 & 0 & 7 & 0 & 0 & 0 & 0 & 1 \\
 -63250 & 0 & -58 & -7 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 228 & -8 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 f_{\omega 2} \\
 f_{\omega 3} \\
 \omega_{4x} \\
 \omega_{4y} \\
 \omega_{4z} \\
 \omega_{5x} \\
 \omega_{5y} \\
 \omega_{5z} \\
 V_{Px} \\
 V_{Py}
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 -3366 \\
 0
 \end{bmatrix}
 \quad (4.149)$$

Solving Eqn (4.149) yields the following answers for the 10 unknowns:

$$f_{\omega 2} = 5.333 \times 10^{-2} \text{ rad/mm.s}$$

$$f_{\omega 3} = 6.104 \times 10^{-2} \text{ rad/mm.s}$$

$$\omega_{4x} = -8.774 \times 10^{-3} \text{ rad/s}$$

$$\omega_{4y} = -0.945 \text{ rad/s}$$

$$\omega_{4z} = -1.446 \times 10^{-3} \text{ rad/s}$$

$$\omega_{5x} = 14.121 \text{ rad/s}$$

$$\omega_{5y} = 3.881 \times 10^{-2} \text{ rad/s}$$

$$\omega_{5z} = 1.106 \text{ rad/s}$$

$$V_{Px} = 166.468 \text{ mm/s}$$

$$V_{Py} = 108.859 \text{ mm/s}$$

It is now possible to use the two scale factors found,  $f_{\omega 2}$  and  $f_{\omega 3}$ , to calculate the angular velocity vectors  $\{\omega_2\}_1$  and  $\{\omega_3\}_1$ .

$$\{\omega_2\}_1 = f_{\omega 2} \{\mathbf{R}_{EF}\}_1 = 5.333 \times 10^{-2} \begin{bmatrix} 230 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 12.266 \\ 0 \\ 0 \end{bmatrix} \text{ rad/s} \quad (4.150)$$

$$\{\omega_3\}_1 = f_{\omega 3} \{\mathbf{R}_{AB}\}_1 = 6.1041 \times 10^{-2} \begin{bmatrix} 230 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} 14.039 \\ 0 \\ 0.855 \end{bmatrix} \text{ rad/s} \quad (4.151)$$

In summary the angular velocity vectors for the rigid bodies are as follows:

$$\begin{aligned}\{\omega_2\}_1^T &= [12.266 \ 0 \ 0] \text{ rad/s} \\ \{\omega_3\}_1^T &= [14.039 \ 0 \ 0.855] \text{ rad/s} \\ \{\omega_4\}_1^T &= [-8.774 \times 10^{-3} \ -0.945 \ -1.446 \times 10^{-3}] \text{ rad/s} \\ \{\omega_5\}_1^T &= [14.121 \ 3.881 \times 10^{-2} \ 1.106] \text{ rad/s}\end{aligned}$$

We can now proceed to calculate the translational velocities at all the moving points, C, D, G, H and P, within this part of the model.

$$\{V_C\}_1 = \{V_{CA}\}_1 = \{\omega_3\}_1 \times \{R_{CA}\}_1 \quad (4.152)$$

$$\begin{bmatrix} V_{Cx} \\ V_{Cy} \\ V_{Cz} \end{bmatrix} = \begin{bmatrix} 0 & -0.855 & 0 \\ 0.855 & 0 & -14.039 \\ 0 & 14.039 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 141 \\ -38 \end{bmatrix} = \begin{bmatrix} -120.555 \\ 435.157 \\ 1979.499 \end{bmatrix} \text{ mm/s} \quad (4.153)$$

$$\{V_D\}_1 = \{V_{DA}\}_1 = \{\omega_3\}_1 \times \{R_{DA}\}_1 \quad (4.154)$$

$$\begin{bmatrix} V_{Dx} \\ V_{Dy} \\ V_{Dz} \end{bmatrix} = \begin{bmatrix} 0 & -0.855 & 0 \\ 0.855 & 0 & -14.039 \\ 0 & 14.039 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 239 \\ -15 \end{bmatrix} = \begin{bmatrix} -204.345 \\ 112.260 \\ 3355.321 \end{bmatrix} \text{ mm/s} \quad (4.155)$$

$$\{V_G\}_1 = \{V_{GE}\}_1 = \{\omega_2\}_1 \times \{R_{GE}\}_1 \quad (4.156)$$

$$\begin{bmatrix} V_{Gx} \\ V_{Gy} \\ V_{Gz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -12.266 \\ 0 & 12.266 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 275 \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ 110.394 \\ 3373.150 \end{bmatrix} \text{ mm/s} \quad (4.157)$$

$$\{V_H\}_1 = \{V_{HJ}\}_1 = \{\omega_5\}_1 \times \{R_{HJ}\}_1 \quad (4.158)$$

$$\begin{bmatrix} V_{Hx} \\ V_{Hy} \\ V_{Hz} \end{bmatrix} = \begin{bmatrix} 0 & -1.1062 & 3.881 \times 10^{-2} \\ 1.1062 & 0 & -14.121 \\ -3.881 \times 10^{-2} & 14.121 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 229 \\ -9 \end{bmatrix} = \begin{bmatrix} -252.524 \\ 112.968 \\ 3219.588 \end{bmatrix} \text{ mm/s} \quad (4.159)$$



The velocity vector  $\{V_P\}_1$  is already available. In summary the velocity vectors for the moving points are as follows:

$$\{V_C\}_1^T = [-120.555 \quad 435.157 \quad 1979.499] \text{ mm/s}$$

$$\{V_D\}_1^T = [-204.345 \quad 112.260 \quad 3355.321] \text{ mm/s}$$

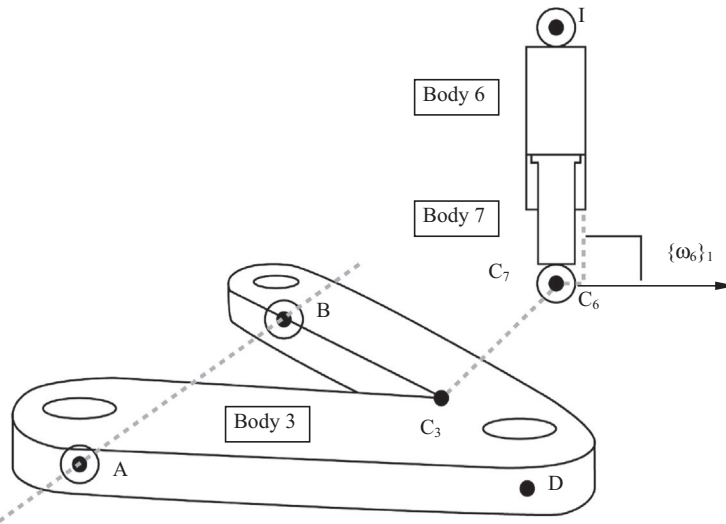
$$\{V_G\}_1^T = [0.0 \quad 110.394 \quad 3373.150] \text{ mm/s}$$

$$\{V_H\}_1^T = [-252.524 \quad 112.968 \quad 3219.588] \text{ mm/s}$$

$$\{V_P\}_1^T = [166.468 \quad 108.859 \quad 3366.0] \text{ mm/s}$$

Having found the velocity  $\{V_C\}_1$  at the bottom of the springdamper unit point C we can now proceed to carry out a separate analysis of the unit to find the sliding component of velocity acting along the axis CI. For this phase of the analysis we introduce two new bodies, Body 6 and Body 7, to represent the upper and lower parts of the damper as shown in Figure 4.85.

This phase of the analysis can be facilitated by modelling three coincident points,  $C_3$  on Body 3,  $C_6$  on Body 6 and  $C_7$  on Body 7, all located at point C. While points  $C_3$  and  $C_7$  can be considered physically located at point C, point  $C_6$  is a virtual extension of the upper damper as shown in Figure 4.85. The sliding velocity in the damper can then be determined from the relative velocity  $\{V_{C_6C_7}\}_1$  of points  $C_6$  and  $C_7$ . Since  $C_6$  and  $C_7$  are coincident, the relative velocity vector can only act along the direction of sliding, axis CI, allowing us to adopt a scale factor and reduce the number of unknowns.



**FIGURE 4.85**

Modelling the damper unit for a velocity analysis.

$$\{V_{C6C7}\}_1 = f_{vs} \{R_{CI}\}_1 \quad (4.160)$$

Since I is a fixed point and applying the triangle law of vector addition gives

$$\{V_{C6}\}_1 = \{V_{C6I}\}_1 = \{V_{C7}\}_1 + \{V_{C6C7}\}_1 \quad (4.161)$$

Note also that since points  $C_3$  and  $C_7$  move together and are physically located at point C we already have the velocity for this boundary condition from the preceding velocity analysis of the double wishbone linkage.

$$\{V_{C7}\}_1^T = \{V_{C3}\}_1^T = \{V_C\}_1^T = [-120.555 \quad 435.157 \quad 1979.499] \text{ mm/s} \quad (4.162)$$

Combining these last three equations to substitute Eqns (4.160) and (4.162) into Eqn (4.161) gives

$$\{V_{C6}\}_1 = \{V_{C6I}\}_1 = \begin{bmatrix} -120.555 \\ 435.157 \\ 1979.499 \end{bmatrix} + f_{vs} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} \text{ mm/s} \quad (4.163)$$

As the suspension moves and the strut component rotates it is also clear that as the only degree of freedom between Body 6 and Body 7 is relative sliding motion then  $\{\omega_6\}_1 = \{\omega_7\}_1$ .

The velocity vector  $\{V_{C6I}\}_1$  can also be defined using

$$\{V_{C6}\}_1 = \{V_{C6I}\}_1 = \{\omega_6\}_1 \times \{R_{CI}\}_1 \quad (4.164)$$

$$\begin{bmatrix} V_{C6Ix} \\ V_{C6Iy} \\ V_{C6Iz} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{6z} & \omega_{6y} \\ \omega_{6z} & 0 & -\omega_{6x} \\ -\omega_{6y} & \omega_{6x} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = \begin{bmatrix} 9\omega_{6z} - 436\omega_{6y} \\ 3\omega_{6z} + 436\omega_{6x} \\ -3\omega_{6y} - 9\omega_{6x} \end{bmatrix} \text{ mm/s} \quad (4.165)$$

Equating the expressions for  $\{V_{C6I}\}_1$  in Eqns (4.162) and (4.164) gives

$$\begin{bmatrix} 9\omega_{6z} - 436\omega_{6y} \\ 3\omega_{6z} + 436\omega_{6x} \\ -3\omega_{6y} - 9\omega_{6x} \end{bmatrix} = \begin{bmatrix} -120.555 \\ 435.157 \\ 1979.499 \end{bmatrix} + f_{vs} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} \text{ mm/s} \quad (4.166)$$

Rearranging Eqn (4.166) yields three equations that can be used to solve this part of the analysis.

$$\text{Equation 1} \quad 3 f_{vs} + 436 \omega_{6y} - 9 \omega_{6z} = 120.555 \quad (4.167)$$

$$\text{Equation 2} \quad -9 f_{vs} - 436 \omega_{6x} - 3 \omega_{6z} = -435.157 \quad (4.168)$$

$$\text{Equation 3} \quad -436 f_{vs} + 9 \omega_{6x} + 3 \omega_{6y} = -1979.499 \quad (4.169)$$

This leaves us with four unknowns,  $\omega_{6x}$ ,  $\omega_{6y}$ ,  $\omega_{6z}$  and  $f_{vs}$  but only three equations. We can use the same approach here as used with the tie rod in the preceding analysis. Since the spin degree of freedom of Body 6 about the axis CI has no bearing on the overall solution we can again use the vector dot product to enforce perpendicularity of  $\{\omega_6\}_1$  to  $\{R_{CI}\}_1$  as shown in Figure 4.85. This will yield the fourth equation as follows.

$$\{\omega_6\}_1 \bullet \{R_{CI}\}_1 = 0 \quad (4.170)$$

$$\begin{bmatrix} \omega_{6x} & \omega_{6y} & \omega_{6z} \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = 0 \text{ mm/s} \quad (4.171)$$

$$\text{Equation 4} \quad 3 \omega_{6x} - 9 \omega_{6y} - 436 \omega_{6z} = 0 \quad (4.172)$$

The four equations can now be set up in matrix form ready for solution. The solution of a four by four matrix will require a lengthy calculation or access as before to a program that offers the capability to invert the matrix.

$$\begin{bmatrix} 3 & 0 & 436 & -9 \\ -9 & -436 & 0 & -3 \\ -436 & 9 & 3 & 0 \\ 0 & 3 & -9 & -436 \end{bmatrix} \begin{bmatrix} f_{vs} \\ \omega_{6x} \\ \omega_{6y} \\ \omega_{6z} \end{bmatrix} = \begin{bmatrix} 120.555 \\ -435.157 \\ -1979.499 \\ 0 \end{bmatrix} \quad (4.173)$$

Solving Eqn (4.173) yields the following answers for the four unknowns:

$$f_{vs} = 4.561 \text{ s}^{-1}$$

$$\omega_{6x} = 0.904 \text{ rad/s}$$

$$\omega_{6y} = 0.245 \text{ rad/s}$$

$$\omega_{6z} = 1.159 \text{ rad/s}$$

This gives us the last two angular velocity vectors for the upper and lower damper bodies:

$$\{\omega_6\}_1^T = [0.904 \quad 0.245 \quad 1.159] \text{ rad/s}$$

$$\{\omega_7\}_1^T = [0.904 \quad 0.245 \quad 1.159] \text{ rad/s}$$

From Eqn (4.150) we now have

$$\{V_{C6C7}\}_1 = f_{vs} \{R_{CI}\}_1 = 4.561 \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = \begin{bmatrix} 13.682 \\ -41.045 \\ -1988.378 \end{bmatrix} \text{ mm/s} \quad (4.174)$$

**Table 4.9** Comparison of Angular Velocity Vectors Computed by Theory and MSC ADAMS

Body	Angular Velocity Vectors					
	Theory			MSC ADAMS		
	$\omega_x$ (rad/s)	$\omega_y$ (rad/s)	$\omega_z$ (rad/s)	$\omega_x$ (rad/s)	$\omega_y$ (rad/s)	$\omega_z$ (rad/s)
2	12.266	0.0	0.0	12.266	0.0	0.0
3	14.039	0.0	0.855	14.040	0.0	0.855
4	$-8.774 \times 10^{-3}$	-0.945	$-1.446 \times 10^{-3}$	$-8.774 \times 10^{-3}$	-0.945	$-1.446 \times 10^{-3}$
5	14.121	$3.881 \times 10^{-2}$	1.106	14.121	$5.394 \times 10^{-2}$	1.106
6	0.904	0.245	$1.159 \times 10^{-3}$	0.904	0.245	$1.163 \times 10^{-3}$
7	0.904	0.245	$1.159 \times 10^{-3}$	0.904	0.245	$1.163 \times 10^{-3}$

Since the velocity vector  $\{V_{C6C7}\}_1$  acts along the axis of the strut CI, the magnitude of this vector will be equal to the sliding velocity  $V_s$ .

$$V_s = |V_{C6C7}| = 1988.641 \text{ mm/s} \quad (4.175)$$

At this stage it can be seen that the sliding velocity  $V_s$  is realistic in magnitude. Given knowledge of the damper force–velocity relationship it would be possible to determine the damping forces produced and reacted at points I and C in the system.

A comparison of the angular velocities found from the preceding calculations and those found using an equivalent MSC ADAMS model is shown in [Table 4.9](#).

A comparison of the translational velocities found at points within the suspension system from the preceding calculations and those found using an equivalent MSC ADAMS model is shown in [Table 4.10](#).

**Table 4.10** Comparison of Translational Velocity Vectors Computed by Theory and MSC ADAMS

Point	Translational Velocity Vectors					
	Theory			MSC ADAMS		
	$V_x$ (mm/s)	$V_y$ (mm/s)	$V_z$ (mm/s)	$V_x$ (mm/s)	$V_y$ (mm/s)	$V_z$ (mm/s)
C	-120.555	435.157	1979.499	-120.495	435.224	1979.570
D	-204.345	112.260	3355.321	-204.244	112.316	3355.440
G	0.0	110.394	3373.150	0.0	110.393	3373.130
H	-252.524	112.968	3219.588	-252.210	112.972	3219.588
P	166.468	108.859	3366.0	166.468	108.859	3366.0
$C_6C_7$	13.682	-41.045	-1988.378	13.682	-41.046	-1988.440

### 4.10.3 Acceleration analysis

The approach taken here is as before to initially ignore the damper assembly between points C and I. Solving for the rest of the suspension system will deliver the acceleration  $\{A_C\}_1$  of point C, thus providing a boundary condition allowing a separate analysis of the damper to follow.

Before proceeding with the acceleration analysis it is necessary to identify the unknowns that define the problem. The angular accelerations and angular velocities of the rigid bodies representing suspension components can be used to find the translational accelerations at points within the system. An example of this is shown for the upper wishbone in Figure 4.86.

Referring back to the earlier velocity analysis we can remind ourselves that as the suspension arm is constrained to rotate about the axis AB, ignoring at this stage any possible deflection due to compliance in the suspension bushes, the vector  $\{\alpha_3\}_1$  for the angular acceleration of Body 3 will act along the axis of rotation through AB. The components of this vector would adopt signs consistent with a positive rotation about this axis as shown in Figure 4.86.

When setting up the equations to solve an acceleration analysis it will be, as before, desirable to reduce the number of unknowns based on the knowledge that a particular body is constrained to rotate about a known axis as shown here. The acceleration vector  $\{\alpha_3\}_1$  could for example be represented as follows

$$\{\alpha_3\}_1 = f\alpha_3 \{R_{AB}\}_1 \quad (4.176)$$

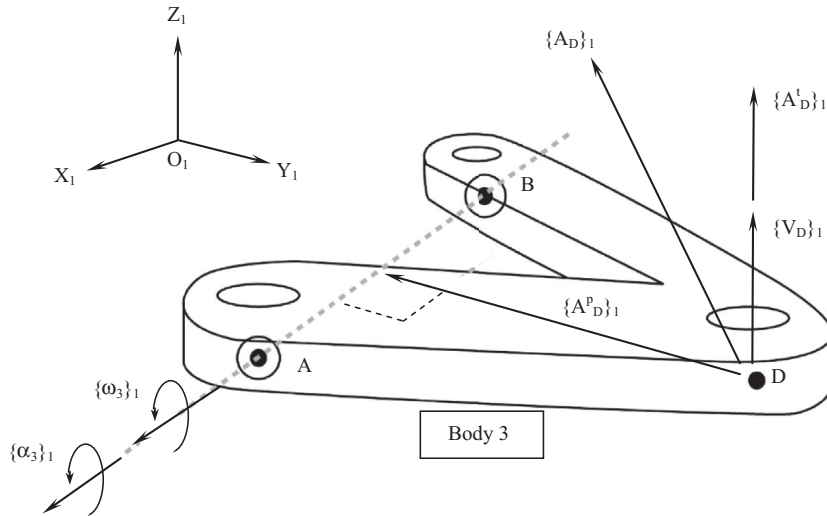


FIGURE 4.86

Angular and translational acceleration and velocity vectors for the upper wishbone.

Since  $\{\alpha_3\}_1$  is parallel to the relative position vector  $\{R_{AB}\}_1$  a scale factor  $f\alpha_3$  can be introduced. This reduces the problem from the three unknown components,  $\alpha_{x3}$ ,  $\alpha_{y3}$  and  $\alpha_{z3}$  of the vector  $\{\alpha_3\}_1$  to a single unknown  $f\alpha_3$ .

It also follows that since point A is considered fixed with an acceleration velocity  $\{A_A\}_1$  equal to zero that the absolute acceleration  $\{A_D\}_1$  of point D can be found from a consideration of the triangle law of vector addition giving

$$\{A_D\}_1 = \{A_{DA}\}_1 \quad (4.177)$$

Once the angular accelerations of Body 3 have been found, together with the known angular velocities, it follows now that the translational acceleration  $\{A_D\}_1$  of for example point D can be found from

$$\{A_D\}_1 = \{A_D^p\}_1 + \{A_D^t\}_1 \quad (4.178)$$

where the centripetal acceleration  $\{A_D^p\}_1$  is given by

$$\{A_D^p\}_1 = \{\omega_3\}_1 \times \{\{\omega_3\}_1 \times \{R_{DA}\}_1\} = \{\omega_3\}_1 \times \{V_D\}_1 \quad (4.179)$$

and the transverse acceleration  $\{A_D^t\}_1$  is given by

$$\{A_D^t\}_1 = \{\alpha_3\}_1 \times \{R_{DA}\}_1 \quad (4.180)$$

As with the velocity analysis, clearly the same approach can be taken with the lower wishbone, Body 2, as with the upper wishbone: using a single scale factor  $f\alpha_2$  to replace the three unknown components,  $\alpha_{x2}$ ,  $\alpha_{y2}$  and  $\alpha_{z2}$  of the vector  $\{\alpha_2\}_1$ . Finally a consideration of the boundaries of this problem again reveals that while points A, B, E, F and J are fixed, the longitudinal acceleration,  $A_{Px}$  and the lateral acceleration,  $A_{Py}$  at the contact point P remain as unknowns.

Thus this analysis can proceed if we can develop 10 equations to solve the 10 unknowns:

$$f\alpha_2, f\alpha_3, \alpha_{x4}, \alpha_{y4}, \alpha_{z4}, \alpha_{x5}, \alpha_{y5}, \alpha_{z5}, A_{Px}, A_{Py}$$

Working through the problem it will be seen that the same strategy used in the velocity analysis can be developed using the triangle law of vector addition to generate sets of equations. As a starting point we will develop a set of three equations using

$$\{A_{DG}\}_1 = \{A_D\}_1 - \{A_G\}_1 \quad (4.181)$$

In this form [Eqn \(4.181\)](#) does not introduce any of the 10 unknowns listed above. It will therefore be necessary to initially define  $\{A_{DG}\}_1$ ,  $\{A_D\}_1$  and  $\{A_G\}_1$  in terms of the angular acceleration vectors that contain unknowns requiring solution. The first step in the analysis can therefore proceed as follows.

Determining an expression for the acceleration  $\{A_G\}_1$  at point G using values for  $\{\omega_2\}_1$  and  $\{V_G\}_1$  found from the earlier velocity analysis gives

$$\{A_G\}_1 = \{A_{GE}\}_1 = \{\omega_2\}_1 \times \{V_G\}_1 + \{\alpha_2\}_1 \times \{R_{GE}\}_1 \quad (4.182)$$

$$\{\alpha_2\}_1 = f_{a2} \{R_{EF}\}_1 = f_{a2} \begin{bmatrix} 230 \\ 0 \\ 0 \end{bmatrix} \text{ rad/s}^2 \quad (4.183)$$

$$\begin{bmatrix} A_{Gx} \\ A_{Gy} \\ A_{Gz} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -12.266 \\ 0 & 12.266 & 0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 110.394 \\ 3373.150 \end{bmatrix} + f_{a2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -230 \\ 0 & 230 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 275 \\ -9 \end{bmatrix} \text{ mm/s}^2 \quad (4.184)$$

$$\begin{bmatrix} A_{Gx} \\ A_{Gy} \\ A_{Gz} \end{bmatrix} = \begin{bmatrix} 0 \\ -41375.058 \\ 1354.093 \end{bmatrix} + \begin{bmatrix} 0 \\ 2070 f_{a2} \\ 63250 f_{a2} \end{bmatrix} \text{ mm/s}^2 \quad (4.185)$$

Determining an expression for the acceleration  $\{A_D\}_1$  at point D using values for  $\{\omega_3\}_1$  and  $\{V_D\}_1$  found from the earlier velocity analysis gives

$$\{A_D\}_1 = \{A_{DA}\}_1 = \{\omega_3\}_1 \times \{V_D\}_1 + \{\alpha_3\}_1 \times \{R_{DA}\}_1 \quad (4.186)$$

$$\{\alpha_3\}_1 = f_{a3} \{R_{AB}\}_1 = f_{a3} \begin{bmatrix} 230 \\ 0 \\ 14 \end{bmatrix} \text{ rad/s}^2 \quad (4.187)$$

$$\begin{bmatrix} A_{Dx} \\ A_{Dy} \\ A_{Dz} \end{bmatrix} = \begin{bmatrix} 0 & -0.855 & 0 \\ 0.855 & 0 & -14.039 \\ 0 & 14.039 & 0 \end{bmatrix} \begin{bmatrix} -203.345 \\ 112.260 \\ 3355.321 \end{bmatrix} + f_{a3} \begin{bmatrix} 0 & -14 & 0 \\ 14 & 0 & -230 \\ 0 & 230 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 239 \\ -15 \end{bmatrix} \text{ mm/s}^2 \quad (4.188)$$

$$\begin{bmatrix} A_{Dx} \\ A_{Dy} \\ A_{Dz} \end{bmatrix} = \begin{bmatrix} -96.030 \\ -47281.651 \\ 1576.804 \end{bmatrix} + \begin{bmatrix} -3346 f_{a3} \\ 1840 f_{a3} \\ 54970 f_{a3} \end{bmatrix} \text{ mm/s}^2 \quad (4.189)$$

Determining an expression for the relative acceleration  $\{A_{DG}\}_1$  of point D relative to point G using values for  $\{\omega_4\}_1$  and  $\{V_{DG}\}_1$  found from the earlier velocity analysis gives

$$\{A_{DG}\}_1 = \{\omega_4\}_1 \times \{V_{DG}\}_1 + \{\alpha_4\}_1 \times \{R_{DG}\}_1 \quad (4.190)$$

$$\begin{bmatrix} A_{DGx} \\ A_{Dgy} \\ A_{DGz} \end{bmatrix} = \begin{bmatrix} 0 & 1.446 \times 10^{-3} & -0.945 \\ -1.446 \times 10^{-3} & 0 & 8.774 \times 10^{-3} \\ 0.945 & -8.774 \times 10^{-3} & 0 \end{bmatrix} \begin{bmatrix} -204.345 \\ 1.866 \\ -17.829 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_{4z} & \alpha_{4y} \\ \alpha_{4z} & 0 & -\alpha_{4x} \\ -\alpha_{4y} & \alpha_{4x} & 0 \end{bmatrix} \begin{bmatrix} -19 \\ -31 \\ 216 \end{bmatrix} \text{ mm/s}^2 \quad (4.191)$$

$$\begin{bmatrix} A_{DGx} \\ A_{Dgy} \\ A_{DGz} \end{bmatrix} = \begin{bmatrix} 16.851 \\ 0.139 \\ -193.122 \end{bmatrix} + \begin{bmatrix} 31\alpha_{4z} + 216\alpha_{4y} \\ -19\alpha_{4z} - 216\alpha_{4x} \\ 19\alpha_{4y} - 31\alpha_{4x} \end{bmatrix} \text{ mm/s}^2 \quad (4.192)$$

We can now apply the triangle law of vector addition to equate the expression for  $\{A_{DG}\}_1$  in Eqn (4.192) with  $\{A_D\}_1$  in Eqn (4.189) and  $\{A_G\}_1$  in Eqn (4.185).

$$\{A_{DG}\}_1 = \{A_D\}_1 - \{A_G\}_1 \quad (4.193)$$

$$\begin{bmatrix} 16.851 \\ 0.139 \\ -193.122 \end{bmatrix} + \begin{bmatrix} 31\alpha_{4z} + 216\alpha_{4y} \\ -19\alpha_{4z} - 216\alpha_{4x} \\ 19\alpha_{4y} - 31\alpha_{4x} \end{bmatrix} = \begin{bmatrix} -96.030 \\ -47281.651 \\ 1576.804 \end{bmatrix} + \begin{bmatrix} -3346f_{a3} \\ 1840f_{a3} \\ 54970f_{a3} \end{bmatrix} - \begin{bmatrix} 0 \\ -41375.058 \\ 1354.093 \end{bmatrix} - \begin{bmatrix} 0 \\ 2070f_{a2} \\ 63250f_{a2} \end{bmatrix} \text{ mm/s}^2 \quad (4.194)$$

Rearranging Eqn (4.194) yields the first three equations required to solve the analysis.

$$\text{Equation 1} \quad 3346 f_{a3} + 216 \alpha_{4y} + 31 \alpha_{4z} = -112.881 \quad (4.195)$$

$$\text{Equation 2} \quad 2070 f_{a2} - 1840 f_{a3} - 216 \alpha_{4x} - 19 \alpha_{4z} = -5906.732 \quad (4.196)$$

$$\text{Equation 3} \quad 63250 f_{a2} - 54970 f_{a3} - 31 \alpha_{4x} + 19 \alpha_{4y} = 415.833 \quad (4.197)$$

We can now proceed to set up the next set of three equations working from point H to point D and using the triangle law of vector addition.

$$\{A_{DH}\}_1 = \{A_D\}_1 - \{A_H\}_1 \quad (4.198)$$

Determining an expression for the velocity  $\{A_H\}_1$  at point H using values for  $\{\omega_5\}_1$  and  $\{V_H\}_1$  found from the earlier velocity analysis gives



$$\{A_H\}_1 = \{A_{HJ}\}_1 = \{\omega_5\}_1 \times \{V_H\}_1 + \{\alpha_5\}_1 \times \{R_{HJ}\}_1 \quad (4.199)$$

$$\begin{bmatrix} A_{Hx} \\ A_{Hy} \\ A_{Hz} \end{bmatrix} = \begin{bmatrix} 0 & -1.106 & 3.881 \times 10^{-2} \\ 1.106 & 0 & -14.121 \\ -3.881 \times 10^{-2} & 14.121 & 0 \end{bmatrix} \begin{bmatrix} -252.524 \\ 112.968 \\ 3219.588 \end{bmatrix} \\ + \begin{bmatrix} 0 & -\alpha_{5z} & \alpha_{5y} \\ \alpha_{5z} & 0 & -\alpha_{5x} \\ -\alpha_{5y} & \alpha_{5x} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 228 \\ -8 \end{bmatrix} \text{mm/s}^2 \quad (4.200)$$

$$\begin{bmatrix} A_{Hx} \\ A_{Hy} \\ A_{Hz} \end{bmatrix} = \begin{bmatrix} 9.602 \times 10^{-3} \\ -45743.094 \\ 1605.022 \end{bmatrix} + \begin{bmatrix} -228\alpha_{5z} - 8\alpha_{5y} \\ 8\alpha_{5x} \\ 228\alpha_{5x} \end{bmatrix} \text{mm/s}^2 \quad (4.201)$$

We already have an expression for  $\{A_D\}_1$  in Eqn (4.189) and can determine an expression for the relative velocity  $\{A_{DH}\}_1$  of point D relative to point H using values for  $\{\omega_4\}_1$  and  $\{V_{DH}\}_1$  found from the earlier velocity analysis.

$$\{A_{DH}\}_1 = \{\omega_4\}_1 \times \{V_{DH}\}_1 + \{\alpha_4\}_1 \times \{R_{DH}\}_1 \quad (4.202)$$

$$\begin{bmatrix} A_{DHx} \\ A_{DHy} \\ A_{DHz} \end{bmatrix} = \begin{bmatrix} 0 & 1.446 \times 10^{-3} & -0.945 \\ -1.446 \times 10^{-3} & 0 & 8.774 \times 10^{-3} \\ 0.945 & -8.774 \times 10^{-3} & 0 \end{bmatrix} \begin{bmatrix} 48.179 \\ -0.708 \\ 135.733 \end{bmatrix} \\ + \begin{bmatrix} 0 & -\alpha_{4z} & \alpha_{4y} \\ \alpha_{4z} & 0 & -\alpha_{4x} \\ -\alpha_{4y} & \alpha_{4x} & 0 \end{bmatrix} \begin{bmatrix} 144 \\ 44 \\ -51 \end{bmatrix} \text{mm/s}^2 \quad (4.203)$$

$$\begin{bmatrix} A_{DHx} \\ A_{DHy} \\ A_{DHz} \end{bmatrix} = \begin{bmatrix} -128.269 \\ 1.121 \\ 45.535 \end{bmatrix} + \begin{bmatrix} -44\alpha_{4z} - 51\alpha_{4y} \\ 144\alpha_{4z} + 51\alpha_{4x} \\ -144\alpha_{4y} + 44\alpha_{4x} \end{bmatrix} \text{mm/s}^2 \quad (4.204)$$

We can now apply the triangle law of vector addition to equate the expression for  $\{A_{DH}\}_1$  in Eqn (4.204) with  $\{A_D\}_1$  in Eqn (4.189) and  $\{A_H\}_1$  in Eqn (4.201).

$$\{A_{DH}\}_1 = \{A_D\}_1 - \{A_H\}_1 \quad (4.205)$$

$$\begin{aligned}
 \begin{bmatrix} -128.269 \\ 1.121 \\ 45.535 \end{bmatrix} + \begin{bmatrix} -44\alpha_{4z} - 51\alpha_{4y} \\ 144\alpha_{4z} + 51\alpha_{4x} \\ -144\alpha_{4y} + 44\alpha_{4x} \end{bmatrix} &= \begin{bmatrix} -96.030 \\ -47281.651 \\ 1576.804 \end{bmatrix} + \begin{bmatrix} -3346f_{a3} \\ 1840f_{a3} \\ 54970f_{a3} \end{bmatrix} \\
 &- \begin{bmatrix} 9.602 \times 10^{-3} \\ -45743.094 \\ 1605.022 \end{bmatrix} + \begin{bmatrix} -228\alpha_{5z} - 8\alpha_{5y} \\ 8\alpha_{5x} \\ 228\alpha_{5x} \end{bmatrix} \text{ mm/s}^2
 \end{aligned} \quad (4.206)$$

Rearranging Eqn (4.206) yields the next three equations required to solve the analysis.

$$\text{Equation 4} \quad 3346 f_{a3} - 51 \alpha_{4y} - 44 \alpha_{4z} - 8 \alpha_{5y} - 228 \alpha_{5z} = 32.229 \quad (4.207)$$

$$\text{Equation 5} \quad -1840 f_{a3} + 51 \alpha_{4x} + 144 \alpha_{4z} + 8 \alpha_{5x} = -1539.678 \quad (4.208)$$

$$\text{Equation 6} \quad -54970 f_{a3} + 44 \alpha_{4x} - 144 \alpha_{4y} + 228 \alpha_{5x} = -73.753 \quad (4.209)$$

We can now proceed to set up the next set of three equations working from point G to point P and using the triangle law of vector addition.

$$\{A_{PG}\}_1 = \{A_P\}_1 - \{A_G\}_1 \quad (4.210)$$

Determining an expression for the relative velocity  $\{A_{PG}\}_1$  of point P relative to point G using values for  $\{\omega_4\}_1$  and  $\{V_{PG}\}_1$  found from the earlier velocity analysis gives

$$\{A_{PG}\}_1 = \{\omega_4\}_1 \times \{V_{PG}\}_1 + \{\alpha_4\}_1 \times \{R_{PG}\}_1 \quad (4.211)$$

$$\begin{aligned}
 \begin{bmatrix} A_{PGx} \\ A_{PGy} \\ A_{PGz} \end{bmatrix} &= \begin{bmatrix} 0 & 1.446 \times 10^{-3} & -0.945 \\ -1.446 \times 10^{-3} & 0 & 8.774 \times 10^{-3} \\ 0.945 & -8.774 \times 10^{-3} & 0 \end{bmatrix} \begin{bmatrix} 166.468 \\ -1.535 \\ -7.150 \end{bmatrix} \\
 &+ \begin{bmatrix} 0 & -\alpha_{4z} & \alpha_{4y} \\ \alpha_{4z} & 0 & -\alpha_{4x} \\ -\alpha_{4y} & \alpha_{4x} & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 58 \\ -176 \end{bmatrix} \text{ mm/s}^2
 \end{aligned} \quad (4.212)$$

$$\begin{bmatrix} A_{PGx} \\ A_{PGy} \\ A_{PGz} \end{bmatrix} = \begin{bmatrix} 6.755 \\ -0.303 \\ 157.326 \end{bmatrix} + \begin{bmatrix} -58\alpha_{4z} - 176\alpha_{4y} \\ -7\alpha_{4z} + 176\alpha_{4x} \\ 7\alpha_{4y} + 58\alpha_{4x} \end{bmatrix} \text{ mm/s}^2 \quad (4.213)$$

We already have an expression for  $\{A_G\}_1$  in Eqn (4.185) and we can define the vector  $\{A_P\}_1$  in terms of the known vertical velocity component,  $A_{Pz}$  and the unknown components,  $A_{Px}$  and  $A_{Py}$ .

$$\{A_P\}_1 = \begin{bmatrix} A_{Px} \\ A_{Py} \\ 0.0 \end{bmatrix} \text{ mm/s}^2 \quad (4.214)$$

We can now apply the triangle law of vector addition to equate the expression for  $\{A_{PG}\}_1$  in Eqn (4.213) with  $\{A_P\}_1$  in Eqn (4.214) and  $\{A_G\}_1$  in Eqn (4.185).

$$\{A_{PG}\}_1 = \{A_P\}_1 - \{A_G\}_1 \quad (4.215)$$

$$\begin{bmatrix} 6.755 \\ -0.303 \\ 157.326 \end{bmatrix} + \begin{bmatrix} -58\alpha_{4z} - 176\alpha_{4y} \\ -7\alpha_{4z} + 176\alpha_{4x} \\ 7\alpha_{4y} + 58\alpha_{4x} \end{bmatrix} = \begin{bmatrix} A_{Px} \\ A_{Py} \\ 0.0 \end{bmatrix} - \begin{bmatrix} 0 \\ -41375.058 \\ 1354.093 \end{bmatrix} - \begin{bmatrix} 0 \\ 2070f_{a2} \\ 63250f_{a2} \end{bmatrix} \text{ mm/s}^2 \quad (4.216)$$

Rearranging Eqn (4.216) yields the next set of three equations required to solve the analysis.

$$\text{Equation 7} \quad 176\alpha_{4y} + 58\alpha_{4z} + A_{Px} = 6.755 \quad (4.217)$$

$$\text{Equation 8} \quad -2070f_{a2} - 176\alpha_{4x} + 7\alpha_{4z} + A_{Py} = -41375.361 \quad (4.218)$$

$$\text{Equation 9} \quad -63250f_{a2} - 58\alpha_{4x} - 7\alpha_{4y} = 1511.419 \quad (4.219)$$

As with the velocity analysis this leaves us with nine equations and ten unknowns. The last equation is again obtained by constraining the rotation of the tie rod (Body 5) to prevent spin about its own axis.

$$\{\alpha_5\}_1 \bullet \{R_{HJ}\}_1 = 0 \quad (4.220)$$

$$\begin{bmatrix} \alpha_{5x} & \alpha_{5y} & \alpha_{5z} \end{bmatrix} \begin{bmatrix} 0 \\ 228 \\ -8 \end{bmatrix} = 0 \text{ mm/s} \quad (4.221)$$

$$\text{Equation 10} \quad 228\alpha_{5y} - 8\alpha_{5z} = 0 \quad (4.222)$$

The 10 equations can now be set up in matrix form ready for solution.

$$\begin{bmatrix} 0 & 3346 & 0 & 216 & 31 & 0 & 0 & 0 & 0 & 0 \\ 2070 & -1840 & -216 & 0 & -19 & 0 & 0 & 0 & 0 & 0 \\ 63250 & -54970 & -31 & 19 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3346 & 0 & -51 & -44 & 0 & -8 & -228 & 0 & 0 \\ 0 & -1840 & 51 & 0 & 144 & 8 & 0 & 0 & 0 & 0 \\ 0 & -54970 & 44 & -144 & 0 & 228 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 176 & 58 & 0 & 0 & 0 & 1 & 0 \\ -2070 & 0 & -176 & 0 & 7 & 0 & 0 & 0 & 0 & 1 \\ -63250 & 0 & -58 & -7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 228 & -8 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{a2} \\ f_{a3} \\ \alpha_{4x} \\ \alpha_{4y} \\ \alpha_{4z} \\ \alpha_{5x} \\ \alpha_{5y} \\ \alpha_{5z} \\ A_{Px} \\ A_{Py} \end{bmatrix} = \begin{bmatrix} -112.881 \\ -5906.732 \\ 415.833 \\ 32.229 \\ -1539.678 \\ -73.753 \\ 6.755 \\ -41375.361 \\ 1511.419 \\ 0 \end{bmatrix} \quad (4.223)$$

Solving Eqn (4.223) yields the following answers for the 10 unknowns:

$$f_{a2} = -5.126 \times 10^{-2} \text{ rad/mm.s}^2$$

$$f_{a3} = -8.182 \times 10^{-2} \text{ rad/mm.s}^2$$

$$\alpha_{4x} = 29.386 \text{ rad/s}^2$$

$$\alpha_{4y} = 3.737 \text{ rad/s}^2$$

$$\alpha_{4z} = -20.847 \text{ rad/s}^2$$

$$\alpha_{5x} = -23.361 \text{ rad/s}^2$$

$$\alpha_{5y} = 6.466 \times 10^{-2} \text{ rad/s}^2$$

$$\alpha_{5z} = 1.843 \text{ rad/s}^2$$

$$A_{Px} = 558.211 \text{ mm/s}^2$$

$$A_{Py} = -36163.671 \text{ mm/s}^2$$

It is now possible to use the two scale factors,  $f_{a2}$  and  $f_{a3}$ , to calculate the angular acceleration vectors,  $\{\alpha_2\}_1$  and  $\{\alpha_3\}_1$ .

$$\{\alpha_2\}_1 = f_{a2} \{R_{EF}\}_1 = -5.126 \times 10^{-2} \begin{bmatrix} 230 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -11.790 \\ 0 \\ 0 \end{bmatrix} \text{ rad/s}^2 \quad (4.224)$$

$$\{\alpha_3\}_1 = f_{\alpha_3} \{R_{AB}\}_1 = -8.182 \times 10^{-2} \begin{bmatrix} 230 \\ 0 \\ 14 \end{bmatrix} = \begin{bmatrix} -18.819 \\ 0 \\ -1.145 \end{bmatrix} \text{rad/s}^2 \quad (4.225)$$

In summary the angular velocity vectors for the rigid bodies are as follows:

$$\begin{aligned} \{\alpha_2\}_1^T &= [-11.790 \ 0 \ 0] \text{rad/s}^2 \\ \{\alpha_3\}_1^T &= [-18.819 \ 0 \ -1.145] \text{rad/s}^2 \\ \{\alpha_4\}_1^T &= [29.386 \ 3.737 \ -20.847] \text{rad/s}^2 \\ \{\alpha_5\}_1^T &= [-23.361 \ 6.466 \times 10^{-2} \ 1.843] \text{rad/s}^2 \end{aligned}$$

We can now proceed to calculate the translational accelerations at all the moving points, C, D, G, H and P, within this part of the model.

$$\{A_C\}_1 = \{A_{CA}\}_1 = \{\omega_3\}_1 \times \{V_C\}_1 + \{\alpha_3\}_1 \times \{R_{CA}\}_1 \quad (4.226)$$

$$\begin{aligned} \begin{bmatrix} A_{Cx} \\ A_{Cy} \\ A_{Cz} \end{bmatrix} &= \begin{bmatrix} 0 & -0.855 & 0 \\ 0.855 & 0 & -14.039 \\ 0 & 14.039 & 0 \end{bmatrix} \begin{bmatrix} -120.555 \\ 435.157 \\ 1979.499 \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 1.146 & 0 \\ -1.146 & 0 & 18.819 \\ 0 & -18.819 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 141 \\ -38 \end{bmatrix} \text{mm/s}^2 \end{aligned} \quad (4.227)$$

$$\begin{bmatrix} A_{Cx} \\ A_{Cy} \\ A_{Cz} \end{bmatrix} = \begin{bmatrix} -372.059 \\ -27893.260 \\ 6109.169 \end{bmatrix} + \begin{bmatrix} 161.445 \\ -583.447 \\ -2653.479 \end{bmatrix} = \begin{bmatrix} -210.614 \\ -28476.707 \\ 3455.690 \end{bmatrix} \text{mm/s}^2 \quad (4.228)$$

$$\{A_D\}_1 = \{A_{DA}\}_1 = \{\omega_3\}_1 \times \{V_D\}_1 + \{\alpha_3\}_1 \times \{R_{DA}\}_1 \quad (4.229)$$

$$\begin{aligned} \begin{bmatrix} A_{Dx} \\ A_{Dy} \\ A_{Dz} \end{bmatrix} &= \begin{bmatrix} -96.030 \\ -47281.651 \\ 1576.804 \end{bmatrix} + \begin{bmatrix} 0 & 1.145 & 0 \\ -1.145 & 0 & 18.819 \\ 0 & -18.819 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 239 \\ -15 \end{bmatrix} = \begin{bmatrix} 177.625 \\ -47432.261 \\ -2920.973 \end{bmatrix} \text{mm/s}^2 \end{aligned} \quad (4.230)$$

$$\{A_G\}_1 = \{A_{GE}\}_1 = \{\omega_2\}_1 \times \{V_G\}_1 + \{\alpha_2\}_1 \times \{R_{GE}\}_1 \quad (4.231)$$

$$\begin{bmatrix} A_{Gx} \\ A_{Gy} \\ A_{Gz} \end{bmatrix} = \begin{bmatrix} 0 \\ -41375.058 \\ 1354.093 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 11.791 \\ 0 & -11.791 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 275 \\ -9 \end{bmatrix} = \begin{bmatrix} 0 \\ -41481.177 \\ -1888.432 \end{bmatrix} \text{mm/s}^2 \quad (4.232)$$

$$\{A_H\}_1 = \{A_{HJ}\}_1 = \{\omega_5\}_1 \times \{V_H\}_1 + \{\alpha_5\}_1 \times \{R_{HJ}\}_1 \quad (4.233)$$

$$\begin{bmatrix} A_{Hx} \\ A_{Hy} \\ A_{Hz} \end{bmatrix} = \begin{bmatrix} 9.602 \times 10^{-3} \\ -45743.094 \\ 1605.022 \end{bmatrix} + \begin{bmatrix} 0 & -1.843 & 0.06466 \\ 1.843 & 0 & -23.361 \\ -0.06466 & 23.361 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 229 \\ -9 \end{bmatrix} = \begin{bmatrix} -422.619 \\ -45953.343 \\ -3744.647 \end{bmatrix} \text{mm/s}^2 \quad (4.234)$$

The acceleration vector  $\{A_P\}_1$  is already available from the initial bump analysis and the solution of Eqn (4.223). In summary the acceleration vectors for the moving points are as follows:

$$\{A_C\}_1^T = [-210.614 \quad -28476.707 \quad 3455.690] \text{mm/s}^2$$

$$\{A_D\}_1^T = [177.625 \quad -47432.261 \quad -2920.973] \text{mm/s}^2$$

$$\{A_G\}_1^T = [0.0 \quad -41481.177 \quad -1888.432] \text{mm/s}^2$$

$$\{A_H\}_1^T = [-422.619 \quad 45953.343 \quad -3744.647] \text{mm/s}^2$$

$$\{A_P\}_1^T = [558.211 \quad -36163.671 \quad 0.0] \text{mm/s}^2$$

Having found the acceleration  $\{A_C\}_1$  at the bottom of the damper unit we can now proceed to carry out a separate analysis of the unit to find the components of acceleration acting between bodies 6 and 7.

As with the velocity analysis, this phase of the acceleration analysis can be facilitated by the modelling of three coincident points,  $C_3$  on Body 3,  $C_6$  on Body 6 and  $C_7$  on Body 7, all located at point C. Note that we already have

$$\{A_{C3}\}_1 = \{A_{C7}\}_1 = \{A_C\}_1 = \begin{bmatrix} -210.614 \\ -28476.707 \\ 3455.690 \end{bmatrix} \text{mm/s}^2 \quad (4.235)$$

We can also calculate the acceleration  $\{A_{C6}\}_1$  from

$$\{A_{C6}\}_1 = \{A_{C6I}\}_1 = \{\omega_6\}_1 \times \{V_{C6I}\}_1 + \{\alpha_6\}_1 \times \{R_{CI}\}_1 \quad (4.236)$$

where

$$\{V_{C6I}\}_1 = \{V_{C6}\}_1 = \{V_{C6C7}\}_1 - \{V_{C7}\}_1 \quad (4.237)$$

$$\{V_{C6I}\}_1 = \begin{bmatrix} 13.628 \\ -41.045 \\ -1988.378 \end{bmatrix} - \begin{bmatrix} -120.555 \\ 435.157 \\ 1979.499 \end{bmatrix} = \begin{bmatrix} 134.183 \\ -476.202 \\ -3967.877 \end{bmatrix} \text{ mm/s} \quad (4.238)$$

therefore  $\{A_{C6}\}_1$  is given by

$$\begin{bmatrix} A_{C6x} \\ A_{C6y} \\ A_{C6z} \end{bmatrix} = \begin{bmatrix} 0 & -1.159 & 0.245 \\ 1.159 & 0 & -0.904 \\ -0.245 & 0.904 & 0 \end{bmatrix} \begin{bmatrix} 134.183 \\ -476.202 \\ -3967.877 \end{bmatrix} + \begin{bmatrix} 0 & -\alpha_{6z} & \alpha_{6y} \\ \alpha_{6z} & 0 & -\alpha_{6x} \\ -\alpha_{6y} & \alpha_{6x} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} \text{ mm/s}^2 \quad (4.239)$$

$$\begin{bmatrix} A_{C6x} \\ A_{C6y} \\ A_{C6z} \end{bmatrix} = \begin{bmatrix} -420.212 \\ 3742.479 \\ -463.361 \end{bmatrix} + \begin{bmatrix} 9\alpha_{6z} - 436\alpha_{6y} \\ 3\alpha_{6z} + 436\alpha_{6x} \\ -3\alpha_{6y} - 9\alpha_{6x} \end{bmatrix} \text{ mm/s}^2 \quad (4.240)$$

If we now consider the relative acceleration vector  $\{A_{C6C7}\}_1$  we can see that this involves the relative acceleration between points on two bodies where relative rotation and sliding occurs. Referring back to Chapter 2 we can now identify the four components of acceleration associated with the combined rotation and sliding motion as the centripetal acceleration  $\{A^p_{C6C7}\}_1$ , the transverse acceleration  $\{A^t_{C6C7}\}_1$ , the Coriolis acceleration  $\{A^c_{C6C7}\}_1$  and the sliding acceleration  $\{A^s_{C6C7}\}_1$ .

$$\{A^p_{C6C7}\}_1 = \{\omega_6\}_1 \times \{\omega_6\}_1 \times \{R_{C6C7}\}_1 \quad (4.241)$$

$$\{A^t_{C6C7}\}_1 = \{\alpha_6\}_1 \times \{R_{C6C7}\}_1 \quad (4.242)$$

$$\{A^c_{C6C7}\}_1 = 2\{\omega_6\}_1 \times \{V_s\}_1 \quad (4.243)$$

$$\{A^s_{C6C7}\}_1 = |A^s_{C6C7}| \{l_{CI}\}_1 \quad (4.244)$$

Since the  $C_6$  and  $C_7$  are coincident points it follows that  $\{A^p_{C6C7}\}_1$  and  $\{A^t_{C6C7}\}_1$  are zero. It also follows that the sliding velocity  $\{V_s\}_1$  is equal to  $\{V_{C6C7}\}_1$ . We can also introduce a scale factor,  $A_s$ , to simplify the sliding acceleration calculation giving

$$\{A^c_{C6C7}\}_1 = 2\{\omega_6\}_1 \times \{V_{C6C7}\}_1 \quad (4.245)$$

$$\{A^s_{C6C7}\}_1 = A_s \{R_{CI}\}_1 \quad (4.246)$$

Combining these components of acceleration gives  $\{A_{C6C7}\}_1$  as

$$\{A_{C6C7}\}_1 = 2\{\omega_6\}_1 \times \{V_{C6C7}\}_1 + A_s \{R_{CI}\}_1 \quad (4.247)$$

$$\begin{bmatrix} A_{C6C7x} \\ A_{C6C7y} \\ A_{C6C7z} \end{bmatrix} = 2 \begin{bmatrix} 0 & -1.159 & 0.245 \\ 1.159 & 0 & -0.904 \\ -0.245 & 0.904 & 0 \end{bmatrix} \begin{bmatrix} 13.682 \\ -41.045 \\ -1988.378 \end{bmatrix} + A_s \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} \text{ mm/s}^2 \quad (4.248)$$

$$\begin{bmatrix} A_{C6C7x} \\ A_{C6C7y} \\ A_{C6C7z} \end{bmatrix} = \begin{bmatrix} -879.162 \\ 3626.702 \\ -80.457 \end{bmatrix} + A_s \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} \text{ mm/s}^2 \quad (4.249)$$

Applying the triangle law of vector addition yields

$$\{A_{C6C7}\}_1 = \{A_{C6}\}_1 - \{A_{C7}\}_1 \quad (4.250)$$

$$\begin{bmatrix} -879.162 \\ 3626.702 \\ -80.457 \end{bmatrix} + A_s \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = \begin{bmatrix} -420.212 \\ 3742.479 \\ -463.361 \end{bmatrix} + \begin{bmatrix} 9\alpha_{6z} - 436\alpha_{6y} \\ 3\alpha_{6z} + 436\alpha_{6x} \\ -3\alpha_{6y} - 9\alpha_{6x} \end{bmatrix} - \begin{bmatrix} -210.614 \\ -28476.707 \\ 3455.690 \end{bmatrix} \text{ mm/s}^2 \quad (4.251)$$

Rearranging Eqn (4.251) yields three equations that can be used to solve this part of the analysis.

$$\text{Equation 1} \quad 3 A_s + 436 \alpha_{6y} - 9 \alpha_{6z} = 669.564 \quad (4.252)$$

$$\text{Equation 2} \quad -9 A_s - 436 \alpha_{6x} - 3 \alpha_{6z} = 28592.484 \quad (4.253)$$

$$\text{Equation 3} \quad -436 A_s + 9 \alpha_{6x} + 3 \alpha_{6y} = -3838.594 \quad (4.254)$$

This leaves us with four unknowns,  $\alpha_{6x}$ ,  $\alpha_{6y}$ ,  $\alpha_{6z}$  and  $A_s$  but only three equations. We can use the same approach here as used in the preceding velocity analysis. Since the spin degree of freedom of Body 6 about the axis CI has no bearing on the overall solution we can again use the vector dot product to enforce perpendicularity of  $\{\alpha_6\}_1$  to  $\{R_{CI}\}_1$ . This will yield the fourth equation as follows.

$$\{\alpha_6\}_1 \bullet \{R_{CI}\}_1 = 0 \quad (4.255)$$

$$\begin{bmatrix} \alpha_{6x} & \alpha_{6y} & \alpha_{6z} \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = 0 \text{ mm/s} \quad (4.256)$$

$$\text{Equation 4} \quad 3 \alpha_{6x} - 9 \alpha_{6y} - 436 \alpha_{6z} = 0 \quad (4.257)$$



The four equations can now be set up in matrix form ready for solution.

$$\begin{bmatrix} 3 & 0 & 436 & -9 \\ -9 & -436 & 0 & -3 \\ -436 & 9 & 3 & 0 \\ 0 & 3 & -9 & -436 \end{bmatrix} \begin{bmatrix} A_s \\ \alpha_{6x} \\ \alpha_{6y} \\ \alpha_{6z} \end{bmatrix} = \begin{bmatrix} 669.564 \\ 28592.484 \\ -3838.594 \\ 0 \end{bmatrix} \quad (4.258)$$

Solving Eqn (4.258) yields the following answers for the four unknowns:

$$A_s = 7.457 \text{ s}^{-2}$$

$$\alpha_{6x} = -65.730 \text{ rad/s}^2$$

$$\alpha_{6y} = 1.147 \text{ rad/s}^2$$

$$\alpha_{6z} = -0.483 \text{ rad/s}^2$$

This gives us the last two angular acceleration vectors for the upper and lower damper bodies:

$$\{\alpha_6\}_1^T = [-65.730 \quad 1.147 \quad -0.483] \text{ rad/s}^2$$

$$\{\alpha_7\}_1^T = [-65.730 \quad 1.147 \quad -0.483] \text{ rad/s}^2$$

From Eqn (4.247) we now have

$$\{A_{C6C7}\}_1 = 2\{\omega_6\}_1 \mathbf{X} \{V_{C6C7}\}_1 + A_s \{R_{CI}\}_1 \quad (4.259)$$

$$\{A_{C6C7}\}_1 = \begin{bmatrix} -879.162 \\ 3626.702 \\ -80.457 \end{bmatrix} + 7.457 \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = \begin{bmatrix} -856.791 \\ 3559.589 \\ -3331.709 \end{bmatrix} \text{ mm/s}^2 \quad (4.260)$$

A comparison of the angular accelerations found from the preceding calculations and those found using an equivalent MSC ADAMS model is shown in Table 4.11.

A comparison of the translational accelerations found, at points within the suspension system from the preceding calculations and those found using an equivalent MSC ADAMS model is shown in Table 4.12.

#### 4.10.4 Static analysis

As discussed in this chapter, a starting point for suspension component loading studies is to use equivalent static forces to represent the loads acting through the road wheel, associated with real-world driving conditions. In this example the vector analysis method is used to carry out a static analysis where a vertical load of 10,000 N is applied at the tyre contact patch, this being representative in magnitude of the loads used for a 3G bump case on a typical vehicle of this size.

**Table 4.11** Comparison of Angular Acceleration Vectors Computed by Theory and MSC ADAMS

Body	Angular Acceleration Vectors					
	Theory			MSC ADAMS		
	$\alpha_x$ (rad/s <sup>2</sup> )	$\alpha_y$ (rad/s <sup>2</sup> )	$\alpha_z$ (rad/s <sup>2</sup> )	$\alpha_x$ (rad/s <sup>2</sup> )	$\alpha_y$ (rad/s <sup>2</sup> )	$\alpha_z$ (rad/s <sup>2</sup> )
2	-11.790	0.0	0.0	-11.791	0.0	0.0
3	-18.819	0.0	-1.145	-18.823	0.0	-1.146
4	29.386	3.737	-20.847	29.395	3.737	-20.840
5	-23.361	$6.466 \times 10^{-2}$	1.843	-23.384	0.100	2.053
6	-65.730	1.147	-0.483	-57.204	2.663	-0.449
7	-65.730	1.147	-0.483	-57.204	2.663	-0.449

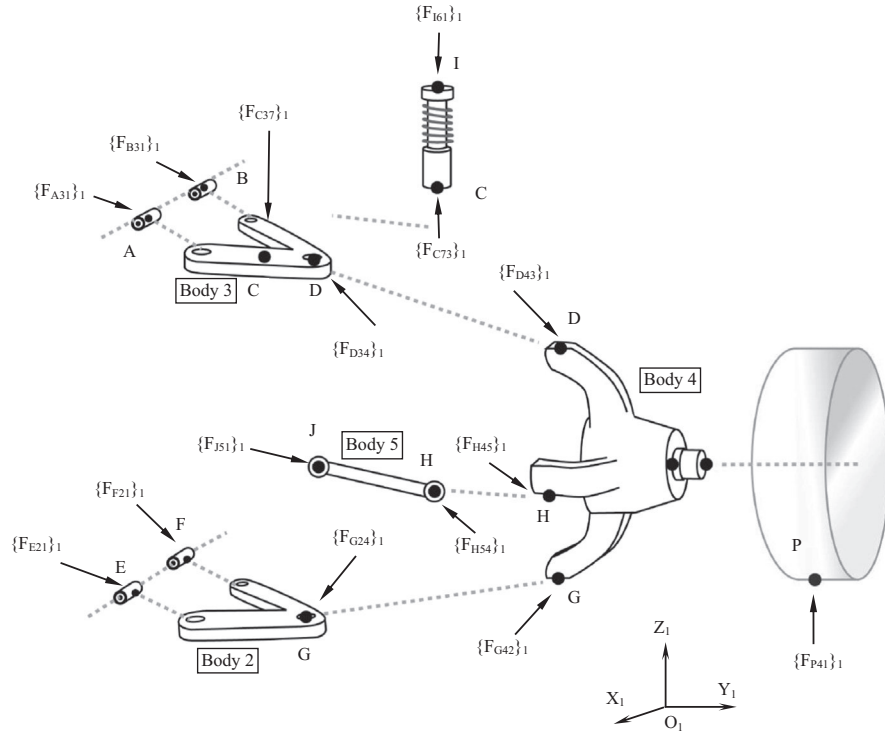
**Table 4.12** Comparison of Translational Acceleration Vectors Computed by Theory and MSC ADAMS

Point	Translational Acceleration Vectors					
	Theory			MSC ADAMS		
	$A_x$ (rad/s <sup>2</sup> )	$A_y$ (rad/s <sup>2</sup> )	$A_z$ (rad/s <sup>2</sup> )	$A_x$ (rad/s <sup>2</sup> )	$A_y$ (rad/s <sup>2</sup> )	$A_z$ (rad/s <sup>2</sup> )
C	-210.614	-28,476.707	3455.690	-210.385	-28,478.60	3456.320
D	177.625	-47,432.261	-2920.973	177.846	-47,433.70	-2921.760
G	0.0	-41,481.177	-1888.432	0.0	-41,480.60	-1888.430
H	-422.619	-45,953.343	-3744.647	-420.215	-45,933.00	-3722.660
P	558.211	-36,163.671	0.0	557.780	-36,161.60	0.0
C <sub>6</sub> C <sub>7</sub>	-856.791	3559.589	-3331.709	-957.170	3544.09	-2566.940

In this analysis we are ignoring gravity and the self-weight of the suspension components as this contribution tends to be minor compared with overall vehicle loads reacted at the tyre contact patch and diffused into the suspension system. For completeness the effects of self-weight will be included in a follow-on demonstration of a dynamic analysis.

Before attempting any vector analysis to determine the distribution of forces it is necessary to prepare a free body diagram and label the bodies and forces in an appropriate manner as shown in [Figure 4.87](#).

For the action–reaction forces shown acting between the bodies in [Figure 4.87](#) Newton’s third law would apply. The interaction, for example, at point D between Body 3 and Body 4 requires  $\{F_{D43}\}_1$  and  $\{F_{D34}\}_1$  to be equal and opposite equal. Thus instead of including the six unknowns  $F_{D43x}$ ,  $F_{D43y}$ ,  $F_{D43z}$ ,  $F_{D34x}$ ,  $F_{D34y}$  and  $F_{D34z}$  we can reduce this to three unknowns  $F_{D43x}$ ,  $F_{D43y}$ ,  $F_{D43z}$ . In a similar manner,



**FIGURE 4.87**

Free body diagram for double wishbone suspension system static force analysis.

looking at the connections at points G and H we can see for all connections to Body 4 that the following applies:

$$\{F_{D43}\}_1 = -\{F_{D34}\}_1 \quad (4.261)$$

$$\{F_{G42}\}_1 = -\{F_{G24}\}_1 \quad (4.262)$$

$$\{F_{H45}\}_1 = -\{F_{H54}\}_1 \quad (4.263)$$

In this model we are treating the connections and mounts as pin-jointed, or as the equivalent spherical joints in an MBS model. For the track rod, Body 5, both ends of the linkage are pin-jointed and the force by definition must, if we allow ourselves the assumption to ignore gravity for this study, act along the axis HJ. In a similar manner the force acting on Body 7 at the base of the strut at point C must be equal and opposite to the force acting at the top on Body 6 at point I.

$$\{F_{J51}\}_1 = -\{F_{H54}\}_1 \quad (4.264)$$

$$\{F_{C37}\}_1 = -\{F_{C73}\}_1 = \{F_{I61}\}_1 \quad (4.265)$$

The number of unknowns can be reduced even further, by using scale factors to exploit the knowledge that the lines of action of the forces are known.

$$\{F_{H54}\}_1 = f_{S1} \{R_{JH}\}_1 \quad (4.266)$$

$$\{F_{C37}\}_1 = f_{S2} \{R_{CI}\}_1 \quad (4.267)$$

This results in the following set of 20 unknowns that must be found to solve for static equilibrium:

$$\begin{aligned} &F_{A31x}, F_{A31y}, F_{A31z} \\ &F_{B31x}, F_{B31y}, F_{B31z} \\ &F_{D43x}, F_{D43y}, F_{D43z} \\ &F_{E21x}, F_{E21y}, F_{E21z} \\ &F_{F21x}, F_{F21y}, F_{F21z} \\ &F_{G24x}, F_{G24y}, F_{G24z} \\ &f_{S1}, f_{S2} \end{aligned}$$

The problem can be solved by setting up the equations of equilibrium for Bodies 2, 3 and 4. The use of scale factors to model the forces acting along Body 5 and the strut, Bodies 6 and 7, means that these bodies cannot be used to generate any useful equations to solve the problem. Thus we could generate 18 equations as follows.

For Body 2 summing forces and taking moments about point G gives

$$\sum \{F_2\}_1 = \{0\}_1 \quad (4.268)$$

$$\sum \{M_{G2}\}_1 = \{0\}_1 \quad (4.269)$$

For Body 3 summing forces and taking moments about point D gives

$$\sum \{F_3\}_1 = \{0\}_1 \quad (4.270)$$

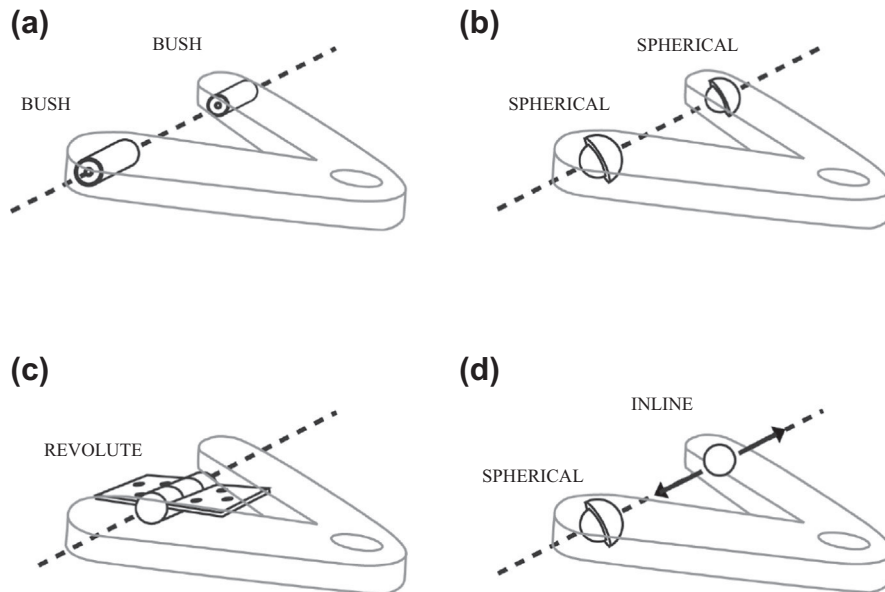
$$\sum \{M_{D3}\}_1 = \{0\}_1 \quad (4.271)$$

For Body 4 summing forces and taking moments about point G gives

$$\sum \{F_4\}_1 = \{0\}_1 \quad (4.272)$$

$$\sum \{M_{G4}\}_1 = \{0\}_1 \quad (4.273)$$

This leaves us with the requirement to generate another two equations for solution. The answer comes from a more considered study of the connections or mounts between the upper and lower wishbones and the ground part. Four possible MBS modelling solutions are shown in [Figure 4.88](#).

**FIGURE 4.88**

Wishbone mount modelling strategies. (a) Wishbone mounted by two bushes. (b) Wishbone mounted by two spherical joints. (c) Wishbone mounted by a single revolute joint. (d) Wishbone mounted by a spherical joint and inline joint primitive

In [Figure 4.88\(a\)](#) the wishbone is mounted using two bush force elements. Using this configuration the wishbone is mounted on an elastic foundation and the body has six rigid body degrees of freedom relative to the part on which it is mounted, which for this example is a nonmoving ground part. If the actual wishbone is mounted on the vehicle in this way this would be the MBS modelling solution of choice if as discussed earlier the simulation aimed to produce accurate predictions of the mount reaction forces. The movement of the wishbone relative to the part on which it is mounted is controlled by the compliance in the bushes. This typically would allow relatively little resistance to rotation about an axis through the bushes, while strongly resisting motion in the other five degrees of freedom.

In [Figure 4.88\(b\)](#) the wishbone is constrained by a spherical joint at each bush location. Each spherical joint constrains three degrees of freedom. This is in fact equivalent to our vector-based model shown as a free body diagram in [Figure 4.88](#) where we currently have three constraint reaction forces at each of our mount locations A, B, E and F. The problem with this approach is that the wishbone initially has six degrees of freedom and the two spherical joints remove three, each leaving for the wishbone body a local balance of zero degrees of freedom. This is clearly not valid as, in the absence of friction or other forces, the wishbone is not physically constrained from rotating about an axis through the two spherical joints.

This is a classic MBS modelling problem where we have introduced a redundant constraint or overconstrained the model. It should also be noted that this is the root of our requirement for two more equations for the manual analysis, each equation being related to the local overconstraint of each wishbone. Early versions of MBS programs such as MSC ADAMS were rather unforgiving in these circumstances and any attempt to solve such a model would cause the solver to fail with the appropriate error messages. More modern versions are able to identify and remove redundant constraints allowing a solution to proceed. While this undoubtedly adds to the convenience of model construction it does isolate less-experienced users from the underlying theory and modelling issues we are currently discussing. In any event, if the required outcome is to predict loads at the mount points, the removal of the redundant constraints, although not affecting the kinematics, cannot be relied on to distribute correctly the forces to the mounts.

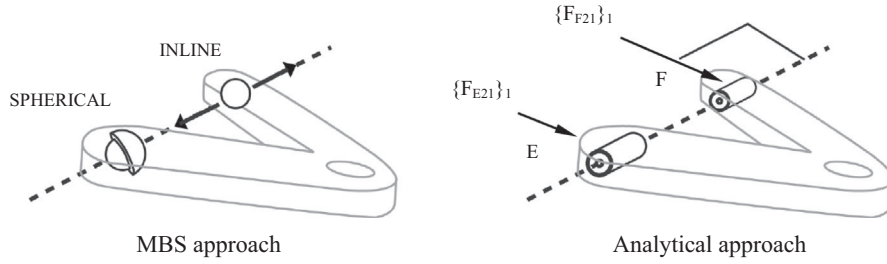
In [Figure 4.88\(c\)](#) the two wishbone mount connections are represented by a single revolute joint. This is the method suggested earlier as a suitable start for predicting the suspension kinematics but will again not be useful for predicting the mount reaction forces. In this model the single revolute joint will carry the combined translational reaction forces at both mounts with additional moment reactions that would not exist in the real system.

The final representation shown in [Figure 4.88\(d\)](#) allows a model that uses rigid constraint elements and can predict reaction forces at each mount without using the ‘as is’ approach of including the bush compliances or introducing redundant constraints. This is achieved by modelling one mount with a spherical joint and the other mount with an inline joint primitive, as described earlier in Chapter 3. The inline primitive constrains two degrees of freedom to maintain the mount position on the axis through the two mount locations. This constraint does not prevent translation along the axis through the mounts, this ‘thrust’ being reacted by the single spherical joint. Thus this selection of rigid constraints provides us with a solution that is not overconstrained. Although the MBS approach would best utilise the model with two bushes to predict the mount reaction forces, the model in [Figure 4.88\(d\)](#) provides us with an understanding of the overconstraint problem and a methodology we can adapt to progress the vector-based analytical solution.

If we return now to the analytical solution and consider the lower wishbone Body 2, we can see in [Figure 4.89](#) that a comparable approach to the use of the MBS inline joint primitive constraint is to ensure that the line of action of one of the mount reaction forces, say  $\{F_{F21}\}_1$ , is perpendicular to the axis EF through the two wishbone mounts.

Thus we can derive the final two equations needed to progress the analytical solution using the familiar approach with the vector dot product to constrain the reaction force at the mount to be perpendicular to an axis through the mounts at say, point B for the upper wishbone and point F for the lower wishbone.

$$\{F_{F21}\}_1 \bullet \{R_{EF}\}_1 = 0 \quad (4.274)$$


**FIGURE 4.89**

Comparable multibody systems (MBS) and analytical wishbone mounting models.

$$\{F_{B31}\}_1 \bullet \{R_{AB}\}_1 = 0 \quad (4.275)$$

Having established the 20 equations required for solution, it is possible to set up the equations starting with the force equilibrium of Body 2.

$$\sum \{F_2\}_1 = \{0\}_1 \quad (4.276)$$

$$\{F_{E21}\}_1 + \{F_{F21}\}_1 + \{F_{G24}\}_1 = \{0\}_1 \quad (4.277)$$

$$\begin{bmatrix} F_{E21x} \\ F_{E21y} \\ F_{E21z} \end{bmatrix} + \begin{bmatrix} F_{F21x} \\ F_{F21y} \\ F_{F21z} \end{bmatrix} + \begin{bmatrix} F_{G24x} \\ F_{G24y} \\ F_{G24z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ N} \quad (4.278)$$

The summation of forces in Eqn (4.278) leads to the first set of three equations:

$$\text{Equation 1} \quad F_{E21x} + F_{F21x} + F_{G24x} = 0 \quad (4.279)$$

$$\text{Equation 2} \quad F_{E21y} + F_{F21y} + F_{G24y} = 0 \quad (4.280)$$

$$\text{Equation 3} \quad F_{E21z} + F_{F21z} + F_{G24z} = 0 \quad (4.281)$$

Taking moments about point G for the forces acting on Body 2 gives

$$\sum \{M_{G2}\}_1 = \{0\}_1 \quad (4.282)$$

$$\{R_{EG}\}_1 \times \{F_{E21}\}_1 + \{R_{FG}\}_1 \times \{F_{F21}\}_1 = \{0\}_1 \quad (4.283)$$

$$\begin{bmatrix} 0 & -9 & -275 \\ 9 & 0 & -115 \\ 275 & 115 & 0 \end{bmatrix} \begin{bmatrix} F_{E21x} \\ F_{E21y} \\ F_{E21z} \end{bmatrix} + \begin{bmatrix} 0 & -9 & -275 \\ 9 & 0 & 115 \\ 275 & -115 & 0 \end{bmatrix} \begin{bmatrix} F_{F21x} \\ F_{F21y} \\ F_{F21z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ Nmm} \quad (4.284)$$

Multiplying out the matrices in Eqn (4.284) yields the next set of three equations:

$$\text{Equation 4} \quad -9 F_{E21y} - 275 F_{E21z} - 9 F_{F21y} - 275 F_{F21z} = 0 \quad (4.285)$$

$$\text{Equation 5} \quad 9 F_{E21x} - 115 F_{E21z} + 9 F_{F21x} + 115 F_{F21z} = 0 \quad (4.286)$$

$$\text{Equation 6} \quad 275 F_{E21x} + 115 F_{E21y} + 275 F_{F21x} - 115 F_{F21y} = 0 \quad (4.287)$$

Consider next Body 3 and the equations required for force equilibrium.

$$\sum \{F_3\}_1 = \{0\}_1 \quad (4.288)$$

$$\{F_{A31}\}_1 + \{F_{B31}\}_1 + f_{S2} \{R_{CI}\}_1 + \{F_{D34}\}_1 = \{0\}_1 \quad (4.289)$$

$$\begin{bmatrix} F_{A31x} \\ F_{A31y} \\ F_{A31z} \end{bmatrix} + \begin{bmatrix} F_{B31x} \\ F_{B31y} \\ F_{B31z} \end{bmatrix} + f_{S2} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} + \begin{bmatrix} F_{D34x} \\ F_{D34y} \\ F_{D34z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ N} \quad (4.290)$$

The summation of forces in Eqn (4.290) leads to the next set of three equations:

$$\text{Equation 7} \quad F_{A31x} + F_{B31x} + 3 f_{S2} + F_{D34x} = 0 \quad (4.291)$$

$$\text{Equation 8} \quad F_{A31y} + F_{B31y} - 9 f_{S2} + F_{D34y} = 0 \quad (4.292)$$

$$\text{Equation 9} \quad F_{A31z} + F_{B31z} - 436 f_{S2} + F_{D34z} = 0 \quad (4.293)$$

Taking moments about point D for the forces acting on Body 3 gives

$$\sum \{M_{D3}\}_1 = \{0\}_1 \quad (4.294)$$

$$\{R_{AD}\}_1 \times \{F_{A31}\}_1 + \{R_{BD}\}_1 \times \{F_{B31}\}_1 + \{R_{CD}\}_1 \times f_{S2} \{R_{CI}\}_1 = \{0\}_1 \quad (4.295)$$

$$\begin{bmatrix} 0 & -15 & -239 \\ 15 & 0 & -115 \\ 239 & 115 & 0 \end{bmatrix} \begin{bmatrix} F_{A31x} \\ F_{A31y} \\ F_{A31z} \end{bmatrix} + \begin{bmatrix} 0 & -1 & -239 \\ 1 & 0 & 115 \\ 239 & -115 & 0 \end{bmatrix} \begin{bmatrix} F_{B31x} \\ F_{B31y} \\ F_{B31z} \end{bmatrix} + f_{S2} \begin{bmatrix} 0 & 23 & -98 \\ -23 & 0 & 0 \\ 98 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ Nmm} \quad (4.296)$$

Multiplying out the matrices in Eqn (4.296) yields the next set of three equations:

$$\text{Equation 10} \quad -15 F_{A31y} - 239 F_{A31z} - 1 F_{B31y} - 239 F_{B31z} + 42521 f_{S2} = 0 \quad (4.297)$$

$$\text{Equation 11} \quad 15 F_{A31x} - 115 F_{A31z} + 1 F_{B31x} + 115 F_{B31z} - 69 f_{S2} = 0 \quad (4.298)$$



$$\text{Equation 12} \quad 239 F_{A31x} + 115 F_{A31y} + 239 F_{B31x} - 115 F_{B31y} + 294 f_{S2} = 0 \quad (4.299)$$

Consider last Body 4 and the equations required for force equilibrium.

$$\sum \{F_4\}_1 = \{0\}_1 \quad (4.300)$$

$$\{F_{P41}\}_1 - \{F_{D34}\}_1 - \{F_{G24}\}_1 - f_{S1} \{R_{JH}\}_1 = \{0\}_1 \quad (4.301)$$

$$\begin{bmatrix} 0 \\ 0 \\ 10000 \end{bmatrix} - \begin{bmatrix} F_{D34x} \\ F_{D34y} \\ F_{D34z} \end{bmatrix} - \begin{bmatrix} F_{G24x} \\ F_{G24y} \\ F_{G24z} \end{bmatrix} - f_{S1} \begin{bmatrix} 0 \\ -228 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ N} \quad (4.302)$$

The summation of forces in Eqn (4.302) leads to the next set of three equations:

$$\text{Equation 13} \quad -F_{D34x} - F_{G24x} = 0 \quad (4.303)$$

$$\text{Equation 14} \quad -F_{D34y} - F_{G24y} + 228 f_{S1} = 0 \quad (4.304)$$

$$\text{Equation 14} \quad -F_{D34z} - F_{G24z} - 8 f_{S1} = -10000 \quad (4.305)$$

Taking moments about point G for the forces acting on Body 4 gives

$$\sum \{M_{G4}\}_1 = \{0\}_1 \quad (4.306)$$

$$\{R_{PG}\}_1 \times \{F_{P41}\}_1 - \{R_{DG}\}_1 \times \{F_{D34}\}_1 - \{R_{HG}\}_1 \times f_{S1} \{R_{JH}\}_1 = \{0\}_1 \quad (4.307)$$

$$\begin{bmatrix} 0 & 176 & 58 \\ -176 & 0 & 7 \\ -58 & -7 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10000 \end{bmatrix} - \begin{bmatrix} 0 & -216 & -31 \\ 216 & 0 & 19 \\ 31 & -19 & 0 \end{bmatrix} \begin{bmatrix} F_{D34x} \\ F_{D34y} \\ F_{D34z} \end{bmatrix} - f_{S1} \begin{bmatrix} 0 & -267 & -75 \\ 267 & 0 & 163 \\ 75 & -163 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -228 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ Nmm} \quad (4.308)$$

Multiplying out the matrices in Eqn (4.308) yields the next set of three equations:

$$\text{Equation 16} \quad 216 F_{D34y} + 31 F_{D34z} - 60276 f_{S1} = -580000 \quad (4.309)$$

$$\text{Equation 17} \quad -216 F_{D34x} - 19 F_{D34z} - 1304 f_{S1} = 0 \quad = -70000 \quad (4.310)$$

$$\text{Equation 18} \quad -31 F_{D34x} + 19 F_{D34y} - 37164 f_{S1} = 0 \quad (4.311)$$

Finally applying the vector dot product to ensure that no thrust for the force  $\{F_{F21}\}_1$  acts along the axis EF gives

$$\{F_{F21}\}_1 \bullet \{R_{EF}\}_1 = 0 \quad (4.312)$$

$$\begin{bmatrix} F_{F21x} & F_{F21y} & F_{F21z} \end{bmatrix} \begin{bmatrix} 230 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ Nmm} \quad (4.313)$$

$$230 F_{F21x} = 0 \quad (4.314)$$

For this particular suspension system the line EF is parallel to the model x-axis yielding the trivial result  $F_{F21x}$  being equal to zero. In this case we can therefore ignore  $F_{F21x}$  in the following matrix solution of the system equations.

The axis AB for the upper wishbone is not parallel to a model axis and therefore applying the vector dot product to ensure that  $\{F_{B31}\}_1$  is perpendicular to the line AB yields the final equation needed to solve the remaining 19 unknowns.

$$\{F_{B31}\}_1 \bullet \{R_{AB}\}_1 = 0 \quad (4.315)$$

$$\begin{bmatrix} F_{B31x} & F_{B31y} & F_{B31z} \end{bmatrix} \begin{bmatrix} 230 \\ 0 \\ 14 \end{bmatrix} = 0 \text{ Nmm} \quad (4.316)$$

$$\text{Equation 19} \quad 230 F_{B31x} + 14 F_{B31z} = 0 \quad (4.317)$$

The 19 equations can now be set up in matrix form ready for solution.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 & -275 & -9 & -275 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & -115 & 0 & 115 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 275 & 115 & 0 & -115 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -436 \\ 0 & -15 & -239 & 0 & -1 & -239 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 42521 \\ 15 & 0 & -115 & 1 & 0 & 115 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -69 \\ 239 & 115 & 0 & 239 & -115 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 294 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 228 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 216 & 31 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -60276 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -216 & 0 & -19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1304 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -31 & 19 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -37164 & 0 \\ 0 & 0 & 0 & 230 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} F_{A31x} \\ F_{A31y} \\ F_{A31z} \\ F_{B31x} \\ F_{B31y} \\ F_{B31z} \\ F_{D34x} \\ F_{D34y} \\ F_{D34z} \\ F_{E21x} \\ F_{E21y} \\ F_{E21z} \\ F_{F21y} \\ F_{F21z} \\ F_{G21x} \\ F_{G24y} \\ F_{G24z} \\ \delta S1 \\ \delta S2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -10000 \\ -580000 \\ -70000 \\ 0 \\ 0 \end{bmatrix} \quad (4.318)$$

Examination of the square matrix in Eqn (4.318) indicates a large number of zero terms, hence the matrix is referred to as sparse. As discussed in Chapter 3 this is a typical characteristic of the matrices generated in MBS and is one of the reasons that fast and efficient matrix inversion techniques can be deployed. The overall result is that MBS programs appear to solve quite complex engineering problems with a

much lower requirement for computational effort than comparable other CAE methods such as nonlinear FE analysis. Solving Eqn (4.318) yields the following answers for the 20 unknowns:

$$\begin{aligned}
 F_{A31x} &= 645.173 \text{ N} & F_{E21x} &= -557.482 \text{ N} \\
 F_{A31y} &= 2006.948 \text{ N} & F_{E21y} &= -1453.911 \text{ N} \\
 F_{A31z} &= 3412.360 \text{ N} & F_{E21z} &= 47.583 \text{ N} \\
 F_{B31x} &= -204.112 \text{ N} & F_{F21x} &= 0 \text{ N} \\
 F_{B31y} &= 3022.800 \text{ N} & F_{F21y} &= -2787.021 \text{ N} \\
 F_{B31z} &= 3353.266 \text{ N} & F_{F21z} &= 91.212 \text{ N} \\
 F_{D34x} &= -557.482 \text{ N} & F_{G24x} &= 557.482 \text{ N} \\
 F_{D34y} &= -4680.486 \text{ N} & F_{G24y} &= 4240.9322 \text{ N} \\
 F_{D34z} &= 10154.217 \text{ N} & F_{G24z} &= -138.979 \text{ N} \\
 f_{S1} &= -1.92786767 \text{ N/mm} & f_{S2} &= 38.8069785 \text{ N/mm}
 \end{aligned}$$

It is now possible to use the two scale factors found,  $f_{S1}$  and  $f_{S2}$ , to calculate the force vectors  $\{F_{H54}\}_1$  and  $\{F_{C37}\}_1$ .

$$\{F_{H54}\}_1 = f_{S1} \{R_{JH}\}_1 = -1.92786767 \begin{bmatrix} 0 \\ -228 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 439.554 \\ -15.942 \end{bmatrix} \text{ N} \quad (4.319)$$

$$\{F_{C37}\}_1 = f_{S2} \{R_{CI}\}_1 = 38.8069785 \begin{bmatrix} 3 \\ -9 \\ -436 \end{bmatrix} = \begin{bmatrix} 116.421 \\ -349.263 \\ -16919.843 \end{bmatrix} \text{ N} \quad (4.320)$$

In summary the force vectors are as follows:

$$\begin{aligned}
 \{F_{A31}\}_1^T &= [ 645.173 \quad 2006.948 \quad 3412.360 ] \text{ N} \\
 \{F_{B31}\}_1^T &= [ -204.112 \quad 3022.800 \quad 3353.266 ] \text{ N} \\
 \{F_{D34}\}_1^T &= [ -557.482 \quad -4680.486 \quad 10154.217 ] \text{ N} \\
 \{F_{E21}\}_1^T &= [ -557.482 \quad -1453.911 \quad 47.583 ] \text{ N} \\
 \{F_{F21}\}_1^T &= [ 0 \quad -2787.021 \quad 91.212 ] \text{ N} \\
 \{F_{G24}\}_1^T &= [ 557.482 \quad 4240.932 \quad -138.979 ] \text{ N} \\
 \{F_{H54}\}_1^T &= [ 0 \quad 439.554 \quad -15.942 ] \text{ N} \\
 \{F_{C37}\}_1^T &= [ 116.421 \quad -349.263 \quad -16919.843 ] \text{ N}
 \end{aligned}$$

**Table 4.13** Comparison of Force Vectors Computed by Theory and MSC ADAMS

Force	Force Vectors					
	Theory			MSC ADAMS		
	$F_x$ (N)	$F_y$ (N)	$F_z$ (N)	$F_x$ (N)	$F_y$ (N)	$F_z$ (N)
$F_{A31}$	645.173	2006.948	3412.360	645.173	2006.950	3412.360
$F_{B31}$	-204.112	3022.800	3353.266	-204.112	3022.800	3353.270
$F_{C37}$	116.421	-349.263	-16919.843	116.421	-349.263	-16919.800
$F_{D34}$	-557.482	-4680.486	10154.217	-557.482	-4680.490	10154.200
$F_{E21}$	-557.482	-1453.911	47.583	-557.482	-1453.910	47.582
$F_{F21}$	0.0	-2787.021	91.212	0.0	-2787.020	91.212
$F_{G24}$	557.482	4240.932	-138.979	557.482	4240.930	-138.794
$F_{H54}$	0.0	439.554	-15.942	0.0	439.554	-15.423

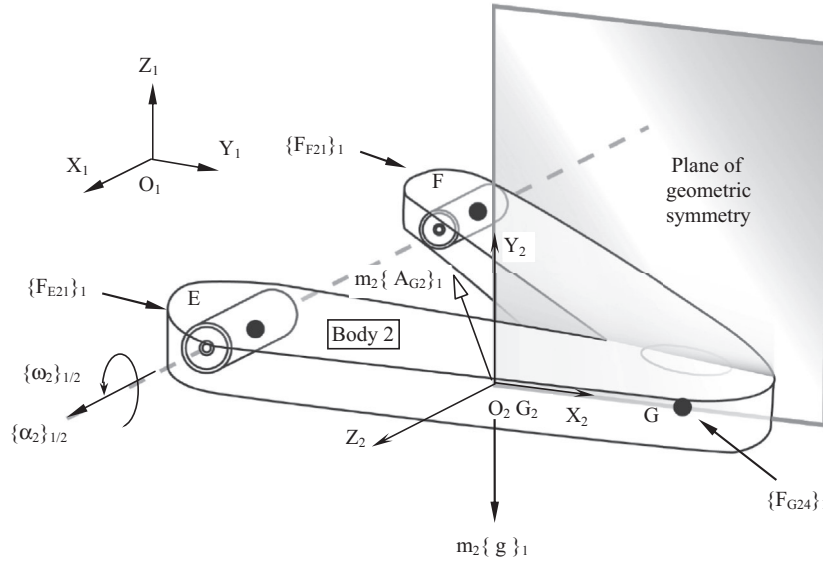
A comparison of the forces found, at points within the suspension system, from the preceding calculations and those found using an equivalent MSC ADAMS model is shown in [Table 4.13](#).

#### 4.10.5 Dynamic analysis

Having carried out a static analysis it is possible to progress to a full dynamic analysis of the system. For the suspension system considered here a theoretical solution could be formulated on the basis of the same set of 20 unknown constraint forces as used in the previous static analysis. Referring back to Chapter 2 however, the reader will realise that the addition of inertial forces and the use of a local body-centred coordinate system for the moment balance will add to the complexity of the solution. For brevity a full theoretical solution will not be performed here but rather the six equations of motion for Body 2 will be set up using, by way of example, the velocities and accelerations found earlier. The process of setting up the equations of motion for the other bodies would follow in a similar manner. Body 2 can be considered in isolation as illustrated with the free body diagram shown in [Figure 4.90](#).

For the dynamic analysis we can take it that the physical properties of the suspension component; mass, mass moments of inertia, centre of mass location and orientation of the body principal axis system are all known.

The coordinate data provided with this example have only provided definitions so far for the locations of points such as those defining suspension mounts and joints connecting the linkages. Mass centre positions have not been provided. For a dynamic analysis the mass centre locations of all moving bodies are required in order to set up the equations of motion. For this example using Body 2 the position of the mass centre,  $G_2$ , relative to the inertial reference frame,  $O_1$ , is defined by the position vector  $\{R_{G2O1}\}_1$  and assumed to be


**FIGURE 4.90**

Free Body diagram for suspension lower wishbone body 2

$$\{R_{G2O1}\}_1^T = [7 \ 500 \ -85] \text{ mm}$$

The mass of Body 2,  $m_2$ , is taken to be 3.5 kg. It should also be noted from Figure 4.90 that the principal axes of Body 2 are located at the mass centre,  $G_2$ , and are defined by the reference frame,  $O_2$ . The transformation from reference frame  $O_1$  to  $O_2$  is obtained through a set of three Euler angle rotations as shown in Figure 4.91.

The mass moments of inertia for Body 2, measured about the principal axes of the body  $O_2$ , are taken to be for this example:

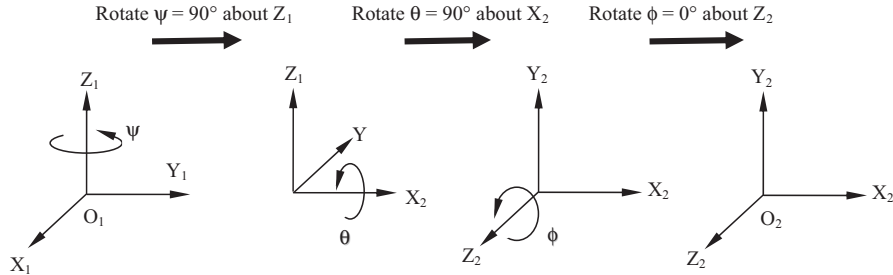
$$I_{21} = I_{2xx} = 1.5 \times 10^3 \text{ kgmm}^2$$

$$I_{22} = I_{2yy} = 38 \times 10^3 \text{ kgmm}^2$$

$$I_{23} = I_{2zz} = 38 \times 10^3 \text{ kgmm}^2$$

The  $X_2Y_2$  plane of  $O_2$  is taken to be a plane of geometric symmetry for the part so that all cross products of inertia are zero. The inertia matrix for Body 2  $[I_2]_{2/2}$  measured from and referred to reference frame  $O_2$  is therefore

$$[I_2]_{2/2} = \begin{bmatrix} 1.5 \times 10^3 & 0 & 0 \\ 0 & 38 \times 10^3 & 0 \\ 0 & 0 & 38 \times 10^3 \end{bmatrix} \text{ kgmm}^2 \quad (4.321)$$


**FIGURE 4.91**

Definition of body principal axis system using Euler angle rotations.

From the previous velocity and acceleration analysis we also have

$$\{\omega_2\}_1^T = [12.266 \ 0 \ 0] \text{ rad/s}$$

$$\{\alpha_2\}_1^T = [-10.642 \ 0 \ 0] \text{ rad/s}^2$$

Before progressing to set up the equations of motion we need to do one more calculation to find the acceleration  $\{A_{G2}\}_1$  of the mass centre for Body 2.

$$\{A_{G2}\}_1 = \{A_{G2E}\}_1 = \{\omega_2\}_1 \times \{V_{G2}\}_1 + \{\alpha_2\}_1 \times \{R_{G2E}\}_1 \quad (4.322)$$

where

$$\{V_{G2}\}_1 = \{V_{G2E}\}_1 = \{\omega_2\}_1 \times \{R_{G2E}\}_1 \quad (4.323)$$

$$\begin{bmatrix} V_{G2x} \\ V_{G2y} \\ V_{G2z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -12.266 \\ 0 & 12.266 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 155 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 61.33 \\ 1901.23 \end{bmatrix} \text{ mm/s} \quad (4.324)$$

therefore

$$\begin{bmatrix} A_{G2x} \\ A_{G2y} \\ A_{G2z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -12.266 \\ 0 & 12.266 & 0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 61.33 \\ 1901.23 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 10.642 \\ 0 & -10.642 & 0 \end{bmatrix} \begin{bmatrix} -115 \\ 155 \\ -5 \end{bmatrix} \text{ mm/s}^2 \quad (4.325)$$

$$\begin{bmatrix} A_{G2x} \\ A_{G2y} \\ A_{G2z} \end{bmatrix} = \begin{bmatrix} 0 \\ -23372.797 \\ -869.336 \end{bmatrix} \text{ mm/s}^2 \quad (4.326)$$

Before progressing further it is important to refer back to Chapter 3 and state that Newton's Second Law is only applicable for a consistent set of units. In effect

this means either converting from millimetres to metres before carrying out the dynamic analysis or incorporating a units consistency factor (UCF) in the equations of motion

$$\sum \{F_2\}_1 = \frac{m_2 \{A_{G2}\}_1}{UCF} \quad (4.327)$$

Since our current dimensions for length are in millimetres and we need to work in SI, our UCF value here is 1000. So converting  $\{A_{G2}\}_1$  to metres per square second gives

$$\begin{bmatrix} A_{G2x} \\ A_{G2y} \\ A_{G2z} \end{bmatrix} = \begin{bmatrix} 0 \\ -23.373 \\ -0.869 \end{bmatrix} \text{m/s}^2 \quad (4.328)$$

We also need to define a vector  $\{g\}_1$  for gravitational acceleration, which for the reference frame  $O_1$  used here would be

$$\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} \text{m/s}^2 \quad (4.329)$$

For Body 2 summing forces and applying Newton's Second Law gives

$$\sum \{F_2\}_1 = m_2 \{A_{G2}\}_1 \quad (4.330)$$

$$\{F_{E21}\}_1 + \{F_{F21}\}_1 + \{F_{G24}\}_1 + m_2 \{g\}_1 = m_2 \{A_{G2}\}_1 \quad (4.331)$$

$$\begin{bmatrix} F_{E21x} \\ F_{E21y} \\ F_{E21z} \end{bmatrix} + \begin{bmatrix} F_{F21x} \\ F_{F21y} \\ F_{F21z} \end{bmatrix} + \begin{bmatrix} F_{G24x} \\ F_{G24y} \\ F_{G24z} \end{bmatrix} + 3.5 \begin{bmatrix} 0 \\ 0 \\ -9.81 \end{bmatrix} = 3.5 \begin{bmatrix} 0 \\ -23.373 \\ -0.869 \end{bmatrix} \text{N} \quad (4.332)$$

The summation of forces in Eqn (4.332) leads to the first set of three equations:

$$\text{Equation 1} \quad F_{E21x} + F_{F21x} + F_{G24x} = 0 \quad (4.333)$$

$$\text{Equation 2} \quad F_{E21y} + F_{F21y} + F_{G24y} = -81.806 \quad (4.334)$$

$$\text{Equation 3} \quad F_{E21z} + F_{F21z} + F_{G24z} = 31.294 \quad (4.335)$$

For the rotational equations it is convenient to refer the vectors to the reference frame  $O_2$  fixed in and rotating with Body 2. The rotational equations of motion for Body 2 may be written as Euler's equations of motion in vector form as

$$\sum \{M_{G2}\}_{1/2} = [I_2]_{\%2} \{\alpha_2\}_{\%2} + [\omega_2]_{\%2} [I_2]_{\%2} \{\omega_2\}_{\%2} \quad (4.336)$$

Before progressing this the angular velocity vector  $\{\omega_2\}_1$  and angular acceleration vector  $\{\alpha_2\}_1$  need to be transformed from reference frame  $O_1$  to  $O_2$  to give  $\{\omega_2\}_{1/2}$  and  $\{\alpha_2\}_{1/2}$ . By inspection it can be seen from [Figure 4.91](#) that the transformation is trivial and that due to the wishbone geometry and constraints  $\omega_{2x}$  and  $\alpha_{2x}$  in frame  $O_1$  simply become  $\omega_{2z}$  and  $\alpha_{2z}$  when referenced to frame  $O_2$ . The process of vector transformation described in Chapter 2 will however be applied to illustrate the process for more general geometries. In this case we have only two rotations to account for, the first being  $90^\circ$   $\Psi$  about the z-axis, followed by a  $90^\circ$  rotation  $\theta$  about the x-axis. Thus for the angular velocity vector we have

$$\{\omega_2\}_{1/2} = \begin{bmatrix} \omega_{2x2} \\ \omega_{2y2} \\ \omega_{2z2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_{2x1} \\ \omega_{2y1} \\ \omega_{2z1} \end{bmatrix} \text{ rad/s} \quad (4.337)$$

$$\{\omega_2\}_{1/2} = \begin{bmatrix} \omega_{2x2} \\ \omega_{2y2} \\ \omega_{2z2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 12.266 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12.266 \end{bmatrix} \text{ rad/s} \quad (4.338)$$

The transformation of the angular acceleration vector takes place in a similar manner so that we have

$$\begin{aligned} \{\omega_2\}_{1/2}^T &= [0 \ 0 \ 12.266] \text{ rad/s} \\ \{\alpha_2\}_{1/2}^T &= [0 \ 0 \ -10.642] \text{ rad/s}^2 \end{aligned}$$

Referring back to [Eqn \(4.328\)](#) and the free body diagram in [Figure 4.90](#) we can see that it is convenient to sum moments of forces acting on Body 2 about the mass centre  $G_2$  to eliminate the inertial force  $m_2\{A_2\}_1$  acting through the mass centre. In order to carry out the moment balance we will need to establish new relative position vectors  $\{R_{EG2}\}_{1/2}$ ,  $\{R_{FG2}\}_{1/2}$  and  $\{R_{GG2}\}_{1/2}$ . We will also need to define the vector components in metres for consistency. Working first in frame  $O_1$  we have:

$$\begin{aligned} \{R_{EG2}\}_1^T &= [0.115 \ -0.155 \ 0.005] \text{ m} \\ \{R_{FG2}\}_1^T &= [-0.115 \ -0.155 \ 0.005] \text{ m} \\ \{R_{GG2}\}_1^T &= [0.0 \ 0.120 \ -0.004] \text{ m} \end{aligned}$$

Applying a vector transformation for the vector  $\{R_{EG2}\}_1$  from frame  $O_1$  to  $O_2$  gives

$$\{R_{EG2}\}_{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.115 \\ -0.155 \\ 0.005 \end{bmatrix} = \begin{bmatrix} -0.155 \\ 0.005 \\ 0.115 \end{bmatrix} \text{ m} \quad (4.339)$$



Applying the same vector transformation to  $\{\mathbf{R}_{FG2}\}_{1/2}$  and  $\{\mathbf{R}_{GG2}\}_{1/2}$  gives us the three relative position vectors, referenced to the correct frame  $O_2$  and in consistent units, needed for the moment balance:

$$\{\mathbf{R}_{EG2}\}_{1/2}^T = 10^{-3} \begin{bmatrix} -155 & 5 & 115 \end{bmatrix} \text{m}$$

$$\{\mathbf{R}_{FG2}\}_{1/2}^T = 10^{-3} \begin{bmatrix} -155 & 5 & -115 \end{bmatrix} \text{m}$$

$$\{\mathbf{R}_{GG2}\}_{1/2}^T = 10^{-3} \begin{bmatrix} 120 & 4 & 0 \end{bmatrix} \text{m}$$

Before writing the rotational equations of motion we can first determine the moment balance of the constraint forces acting at E, F and G.

$$\begin{aligned} \sum \{\mathbf{M}_{G2}\}_{1/2} &= \{\mathbf{R}_{EG2}\}_{1/2} \times \{\mathbf{F}_{E21}\}_{1/2} + \{\mathbf{R}_{FG2}\}_{1/2} \times \{\mathbf{F}_{F21}\}_{1/2} + \{\mathbf{R}_{GG2}\}_{1/2} \times \{\mathbf{F}_{G24}\}_{1/2} \\ &= 10^{-3} \begin{bmatrix} 0 & -115 & 5 \\ 115 & 0 & 155 \\ -5 & -155 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{E21x2} \\ \mathbf{F}_{E21y2} \\ \mathbf{F}_{E21z2} \end{bmatrix} + 10^{-3} \begin{bmatrix} 0 & 115 & 5 \\ -115 & 0 & 155 \\ -5 & -155 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{F21x2} \\ \mathbf{F}_{F21y2} \\ \mathbf{F}_{F21z2} \end{bmatrix} + 10^{-3} \begin{bmatrix} 0 & -4 & 0 \\ 4 & 0 & -120 \\ 0 & 120 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{F}_{G24x2} \\ \mathbf{F}_{G24y2} \\ \mathbf{F}_{G24z2} \end{bmatrix} \text{Nm} \end{aligned} \quad (4.340)$$

Considering next the rotational inertial terms we have

$$\sum \{\mathbf{M}_{G2}\}_{1/2} = [\mathbf{I}_2]_{\frac{1}{2}} \{\alpha_2\}_{\frac{1}{2}} + [\omega_2]_{\frac{1}{2}} [\mathbf{I}_2]_{\frac{1}{2}} \{\omega_2\}_{\frac{1}{2}} \quad (4.341)$$

$$\begin{aligned} \sum \{\mathbf{M}_{G2}\}_{1/2} &= \begin{bmatrix} I_{2xx} & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & I_{2zz} \end{bmatrix} \begin{bmatrix} \alpha_{2x2} \\ \alpha_{2y2} \\ \alpha_{2z2} \end{bmatrix} + \begin{bmatrix} 0 & -\omega_{2z2} & \omega_{2y2} \\ \omega_{2z2} & 0 & -\omega_{2x2} \\ -\omega_{2y2} & \omega_{2x2} & 0 \end{bmatrix} \begin{bmatrix} I_{2xx} & 0 & 0 \\ 0 & I_{2yy} & 0 \\ 0 & 0 & I_{2zz} \end{bmatrix} \begin{bmatrix} \omega_{2x2} \\ \omega_{2y2} \\ \omega_{2z2} \end{bmatrix} \text{Nm} \\ &= \begin{bmatrix} 1.5 \times 10^{-3} & 0 & 0 \\ 0 & 38 \times 10^{-3} & 0 \\ 0 & 0 & 38 \times 10^{-3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -10.642 \end{bmatrix} + \begin{bmatrix} 0 & -12.266 & 0 \\ 12.266 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1.5 \times 10^{-3} & 0 & 0 \\ 0 & 38 \times 10^{-3} & 0 \\ 0 & 0 & 38 \times 10^{-3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 12.266 \end{bmatrix} \text{Nm} \end{aligned} \quad (4.342)$$

Equating Eqn (4.340) with Eqn (4.342) yields the rotational equations of motion for Body 2:

$$\text{Equation 4} \quad (-115F_{E21y2} + 5F_{E21z2} + 115F_{F21y2} + 5F_{F21z2} - 4F_{G24y2}) \times 10^{-3} = 0 \quad (4.343)$$

$$\text{Equation 5} \quad (115F_{E21x2} + 155F_{E21z2} - 115F_{F21x2} + 155F_{F21z2} + F_{G24x2} - 0.120F_{G24z2}) \times 10^{-3} = 0 \quad (4.344)$$

$$\text{Equation 6} \quad (-5F_{E21x2} - 155F_{E21y2} - 5F_{F21x2} - 155F_{F21y2} + 120F_{G24y2}) \times 10^{-3} = -404.396 \times 10^{-3} \quad (4.345)$$

At this stage the observant reader will note that the inertial terms in Eqn (4.342) have only yielded a numerical value for the moment balance about the principal  $Z_2$  axis. This makes sense as the  $Z_2$  axis has been chosen to be parallel to the fixed axis of body rotation through points E and F. In the absence of components of angular velocity or acceleration about  $X_2$  and  $Y_2$  Eqns (4.343) and (4.344) above simplify to a static moment balance. It should also be noted that when rotation is constrained about a single principal axis, the right-hand part of Eqn (4.341)  $[\omega_2]_{1/2} [I_2]_{1/2} \{\omega_2\}_{1/2}$ , is entirely zero to indicate a lack of gyroscopic terms in the absence of rotational coupling. Choosing a body-centred axis system for the upper wishbone Body 3, with an axis parallel to an axis through points A and B, would yield a similar formulation. This would not however be the case, for example, if we continued to set up the equations for Body 4 where there is no single fixed axis of rotation.

Before leaving the area of formulating equations of motion for dynamic analysis we should also ensure that the impression is not given that the use of a two-force body type scale factor, as used for the static analysis with the tie rod Body 5, can be employed here. Figure 4.92 shows free body diagrams for both a static and dynamic analysis of the tie rod. For the static analysis it can be seen that, with the assumption that gravity is ignored, the reaction forces at J and H act along the axis of the tie rod allowing a scale factor to be used. For the dynamic analysis it can be seen that the inertial forces do not allow such an assumption and that a set of six equations of motion for Body 5 will be required for the solution.

If at this stage we ignore the mass effects of the damper assembly we can represent the force  $\{F_{C37}\}_1$  acting on Body 3 at point C using as before a scalar. Since the line of action of  $\{F_{C37}\}_1$  is known to act along the line CI it is possible to define the force using the magnitude of  $|F_{C37}|$  factored with the unit vector  $\{l_{CI}\}_1$ , acting along the line from I to C, as follows:

$$\{F_{C37}\}_1 = F_s \{l_{CI}\}_1 \quad (4.346)$$

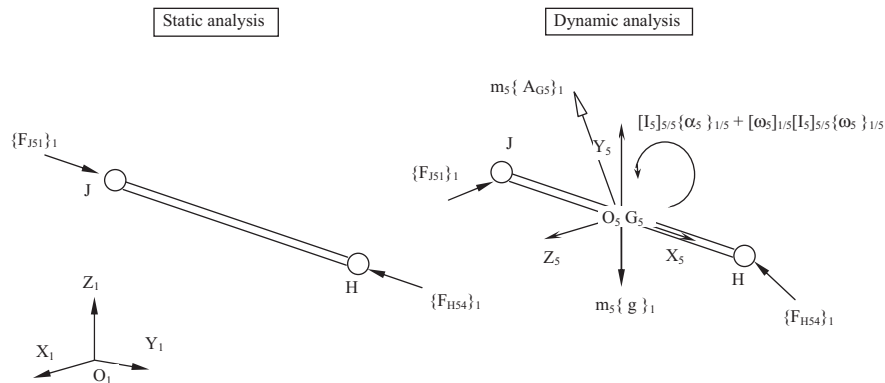


FIGURE 4.92

Free body diagrams for static and dynamic analysis of the tie rod.

where  $F_s$  is the magnitude of the force  $|F_{C37}|$  with a sign assigned that is positive if the force acts towards point C from I. In this analysis, and under normal driving conditions,  $F_s$  will be positive.

A consideration of the complete suspension system indicates that the following set of 25 unknowns must be found to solve for dynamic forces:

$$F_{A31x}, F_{A31y}, F_{A31z}$$

$$F_{B31x}, F_{B31y}, F_{B31z}$$

$$F_{D43x}, F_{D43y}, F_{D43z}$$

$$F_{E21x}, F_{E21y}, F_{E21z}$$

$$F_{F21x}, F_{F21y}, F_{F21z}$$

$$F_{G24x}, F_{G24y}, F_{G24z}$$

$$F_{H54x}, F_{H54y}, F_{H54z}$$

$$F_{J51x}, F_{J51y}, F_{J51z}$$

$$F_s$$

Each moving body, Bodies 2, 3, 4 and 5, yields six equations of motion that can be used to solve the dynamic analysis. Using the same approach as demonstrated with Body 2 the following equations would be generated for all the bodies

$$\sum \{F_2\}_1 = m_2 \{A_{G2}\}_1 \quad (4.347)$$

$$\sum \{M_{G2}\}_{1/2} = [I_2]_{\frac{1}{2}} \{\alpha_2\}_{\frac{1}{2}} + [\omega_2]_{\frac{1}{2}} [I_2]_{\frac{1}{2}} \{\omega_2\}_{\frac{1}{2}} \quad (4.348)$$

$$\sum \{F_3\}_1 = m_3 \{A_{G3}\}_1 \quad (4.349)$$

$$\sum \{M_{G3}\}_{1/3} = [I_3]_{\frac{1}{3}} \{\alpha_3\}_{\frac{1}{3}} + [\omega_3]_{\frac{1}{3}} [I_3]_{\frac{1}{3}} \{\omega_3\}_{\frac{1}{3}} \quad (4.350)$$

$$\sum \{F_4\}_1 = m_4 \{A_{G4}\}_1 \quad (4.351)$$

$$\sum \{M_{G4}\}_{1/4} = [I_4]_{\frac{1}{4}} \{\alpha_4\}_{\frac{1}{4}} + [\omega_4]_{\frac{1}{4}} [I_4]_{\frac{1}{4}} \{\omega_4\}_{\frac{1}{4}} \quad (4.352)$$

$$\sum \{F_5\}_1 = m_5 \{A_{G5}\}_1 \quad (4.353)$$

$$\sum \{M_{G5}\}_{1/5} = [I_5]_{\frac{1}{5}} \{\alpha_5\}_{\frac{1}{5}} + [\omega_5]_{\frac{1}{5}} [I_5]_{\frac{1}{5}} \{\omega_5\}_{\frac{1}{5}} \quad (4.354)$$

The equations of motion above yield 24 equations leaving one further equation to be derived to solve the 25 unknowns. The final equation allows us to formulate the scalar  $F_s$  with the appropriate magnitude and sign to represent the force acting along the

strut. Referring back to the discussion of spring and damper forces in Chapter 3 we are reminded that for a linear formulation based on the spring stiffness,  $k$ , free length,  $L$  and the damping coefficient,  $c$ , of the damper we can formulate the force using

$$F_s = k (L - |R_{CI}|) - c \times VR_{CI} \quad (4.355)$$

The term  $(L - |R_{CI}|)$  represents the deflection of the spring relative to the free length. The term  $VR_{CI}$  represents the radial line of sight velocity. This is effectively the magnitude of the velocity vector  $\{V_{CI}\}_1$  given a sign so that  $VR_{CI}$  is negative when points C and I are approaching each other in bump and is positive when separating in rebound. The result of this is that the component of spring force is positive when the spring is compressed and the damper force component is positive during bump motion.

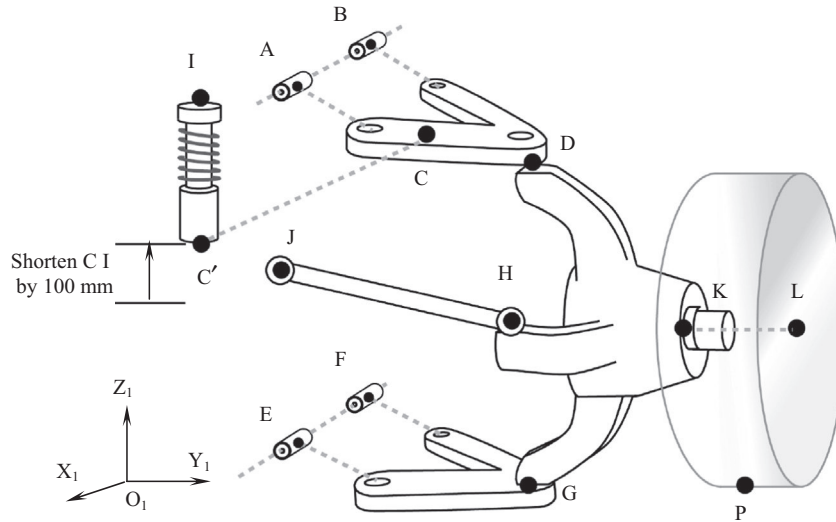
#### 4.10.6 Geometry analysis

The preceding use of vectors to carry out three-dimensional velocity, acceleration, static force and dynamic force analyses of the double wishbone suspension system should have provided the reader with an insight into the computational work performed by an MBS program during the solution phase. An important aspect of this is that in all the preceding analyses the geometry has been assumed fixed throughout the solution. This is in fact not fully representative of the problem. For example for the static analysis the damper acting between C and I is assumed to be locked so that although the reaction force at C can be determined, the suspension does not move despite a considerable vertical load being applied at the tyre contact patch.

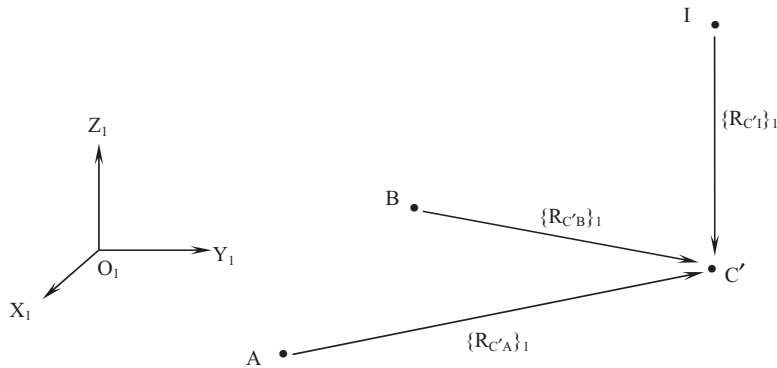
In reality the damper has a sliding degree of freedom that allows the length CI to shorten until the additional compression of the spring produces the force required for the suspension system to be in static equilibrium. This mechanism is the key behind the iterations described in Chapter 3 that take place during a solution step at a given point in time. In effect all the preceding vector analyses can be considered typical of the computations during one of many analysis iterations at a given point in time.

To demonstrate the final phase in this process a vector analysis will now be performed to determine the new position of the movable points throughout the suspension system due to a deflection in the suspension spring unit. In this case we will shorten the line CI by 100 mm taking this to be representative of the movement for this suspension with typical spring and damper properties. We can consider that we are looking here at the suspension moving between the defined or model input position, to the full bump position. During a typical analysis iteration the movement would in fact be far less than this but the following calculations will illustrate the process and complete our treatment of vector analysis in this chapter.

Before proceeding with the analysis, [Figure 4.93](#) is provided to remind us of the suspension configuration, the point labelling system and to illustrate the shortening of the damper unit.


**FIGURE 4.93**

Shortening of damper unit for double wishbone suspension geometry analysis.


**FIGURE 4.94**

Locating new point  $C'$  by triangulation.

In order to establish the position of any point that has moved in the suspension system we must work from three points for which the coordinates are already established. To begin the analysis we can consider finding the new position of point  $C'$  working from three points  $A$ ,  $B$  and  $I$  that are fixed and cannot move as shown in Figure 4.94.

In this example the positions of  $A$ ,  $B$  and  $I$  are known, as are the lengths  $AC$ ,  $BC$  and  $IC'$ . The length  $IC'$  takes into account the shortening of the strut by 100 mm but the lengths  $AC$  and  $BC$  are unchanged. The new position of  $C'$  is unknown

and must be solved. In terms of vectors this can be expressed using the following known inputs:

$$\{\mathbf{R}_A\}_1^T = [A_x \ A_y \ A_z]$$

$$\{\mathbf{R}_B\}_1^T = [B_x \ B_y \ B_z]$$

$$\{\mathbf{R}_I\}_1^T = [I_x \ I_y \ I_z]$$

$$|\mathbf{R}_{CA}|$$

$$|\mathbf{R}_{CB}|$$

$$|\mathbf{R}_{CI}|$$

In order to solve the three unknowns  $C'_x$ ,  $C'_y$  and  $C'_z$ , which are the components of the position vector  $\{\mathbf{R}_C\}_1$ , it is necessary to set up three equations as follows:

$$|\mathbf{R}_{CA}|^2 = (C'_x - A_x)^2 + (C'_y - A_y)^2 + (C'_z - A_z)^2 \quad (4.356)$$

$$|\mathbf{R}_{CB}|^2 = (C'_x - B_x)^2 + (C'_y - B_y)^2 + (C'_z - B_z)^2 \quad (4.357)$$

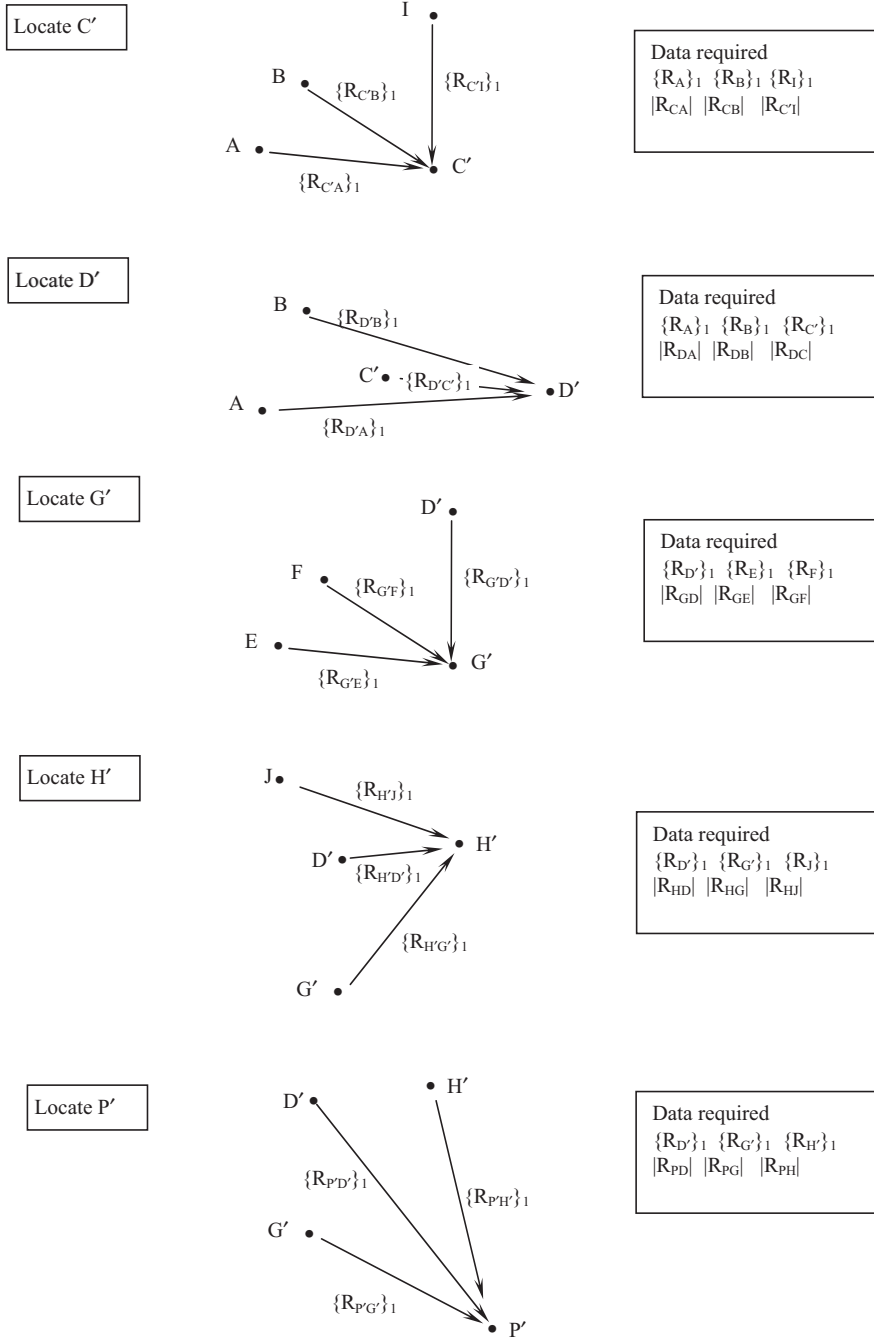
$$|\mathbf{R}_{CI}|^2 = (C'_x - I_x)^2 + (C'_y - I_y)^2 + (C'_z - I_z)^2 \quad (4.358)$$

In Chapter 2 it was demonstrated that the simultaneous solution of [Eqns \(4.356\)–\(4.358\)](#) results in a quadratic with two solutions, one of which will be correct, for  $C'_x$ ,  $C'_y$  and  $C'_z$ . Having demonstrated in Chapter 2 the manipulations required to solve a set of three such equations we will content ourselves here to show the process followed to set up all the equations for this suspension system but use a computer program written in BASIC to solve them.

[Figure 4.95](#) illustrates the process that would be followed to solve the coordinates of all movable points where at each stage the positions of the three reference points must be either fixed or previously found if movable. The length between each of the three reference points and the movable point must also be fixed and known. This will only work if the movable point lies on the same rigid body as each of the reference points.

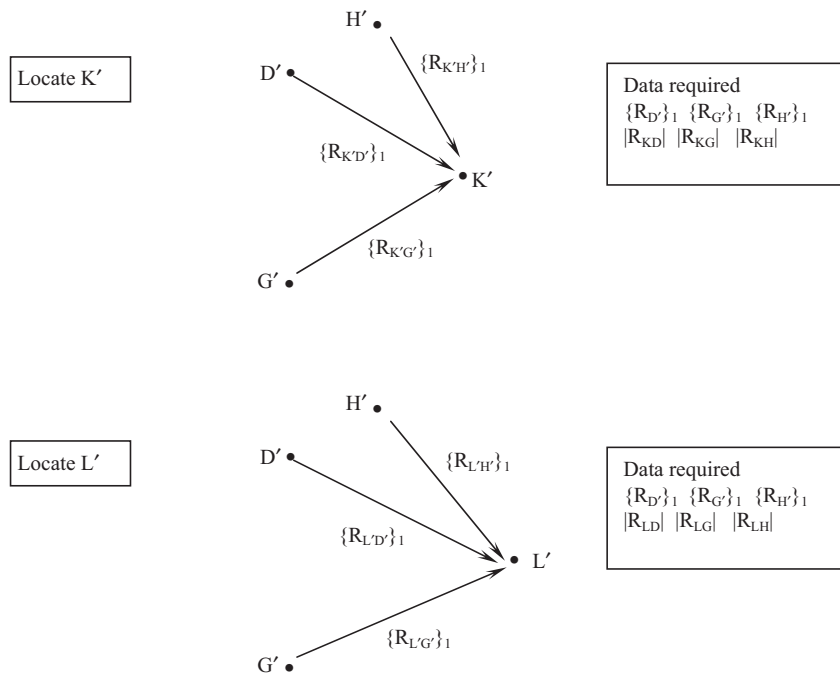
In addition to locating the movable points just described we will also need to determine the new positions  $K'$  and  $L'$  of the two points located on the wheel spin axis as shown in [Figure 4.96](#). These two positions will be used with the starting locations  $K$  and  $L$  to determine the change in steer and camber angle between the two suspension configurations.

Having followed the process outlined here we obtain the new positions shown in [Table 4.14](#). The results obtained using vector theory are compared with those from the equivalent MSC ADAMS model where a motion input has been used to shorten the strut by 100 mm.



**FIGURE 4.95**

Calculation sequence to solve double wishbone suspension geometry.



**FIGURE 4.96**

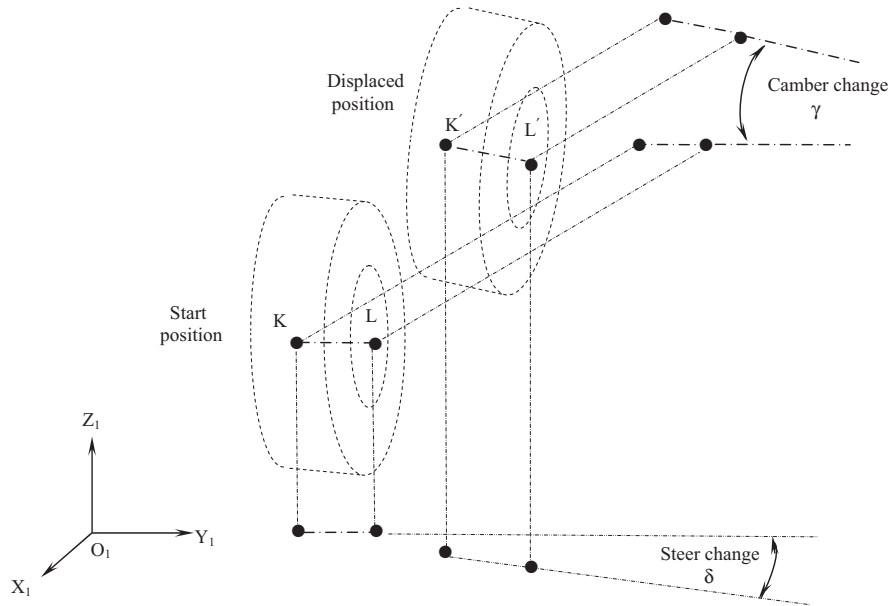
Location of points K' and L' on the wheel spin axis.

**Table 4.14** Comparison of Movable Point Locations Computed by Theory and MSC ADAMS

Point	Suspension Position Vectors					
	Theory			MSC ADAMS		
	R <sub>x</sub> (mm)	R <sub>y</sub> (mm)	R <sub>z</sub> (mm)	R <sub>x</sub> (mm)	R <sub>y</sub> (mm)	R <sub>z</sub> (mm)
C'	−18.133	476.250	204.752	−18.133	476.250	204.753
D'	−21.719	534.601	286.699	−21.721	534.604	286.696
G'	7.0	573.627	73.087	7.0	573.629	73.084
H'	−168.982	493.368	330.131	−168.984	493.371	330.127
P'	8.978	638.864	−100.491	8.979	638.867	−100.494
K'	−4.232	550.301	160.835	−4.233	550.305	160.832
L'	−1.781	628.197	164.076	−1.783	628.199	164.073

Having calculated the new positions of all the movable nodes, the movement of the tyre contact patch, in this case taken to be point P, could be used to establish for example the lateral movement. Referring back to Chapter 2 we can also use the methods described there to determine the bump steer as shown in Figure 4.97.




**FIGURE 4.97**

Using vectors to determine camber and steer angle change.

The change in steer angle or bump steer can be determined by finding the angle  $\delta$  between the projection of  $KL$  and  $K'L'$  onto the global  $X_1Y_1$  plane. The projection is achieved after setting the  $z$ -coordinates of all four position vectors to zero and then applying the vector dot product as shown in Eqn (4.359).

$$\cos \delta = \{R_{KL}\}_1 \bullet \{R_{K'L'}\} / |R_{KL}| |R_{K'L'}| \quad (4.359)$$

The change in camber angle,  $\gamma$ , is obtained in a similar manner where the projection this time takes place in the global  $Y_1Z_1$  plane by setting all the  $x$ -coordinates to zero.

$$\cos \gamma = \{R_{KL}\}_2 \bullet \{R_{K'L'}\} / |R_{KL}| |R_{K'L'}| \quad (4.360)$$

Note that in Figure 4.97 the change in steer and camber angles are both shown as positive for this suspension located on the front left-hand side of the vehicle with the  $X_1$ -axis pointing forwards. A comparison of the answers found by theory with those from MSC ADAMS is given in Table 4.15

**Table 4.15** Comparison of Steer and Camber Angle Change Computed by Theory and MSC ADAMS

	Theory Change in Angle (degrees)	MSC ADAMS Change in Angle (degrees)
Steer	1.802	1.802
Camber	-2.383	-2.382

# Tyre Characteristics and Modelling

# 5

*When people say that tyres are round and black, I tell them they have the black bit right. Mostly.*

Jan Prins, Jaguar Land Rover

## 5.1 Introduction

The handling performance and directional response of a vehicle are greatly influenced by the mechanical force and moment generating characteristics of the tyres. In road vehicle dynamics the manner in which a vehicle accelerates, brakes and corners is controlled by the forces generated over four relatively small tyre contact patches. Even motorsport vehicles predominantly use aerodynamic forces to modify the vertical forces on the tyres in order to harvest the mechanical amplification of those forces by the friction coefficient, which typically exceeds unity in motorsport applications. Applications such as rallying with a lower friction coefficient may use aerodynamic side forces directly but they are both obvious (from the predominance of features approximately parallel with the plane of symmetry of the vehicle) and unusual.

If the tread pattern and the road texture is also considered it is clear that the area of frictional contact is reduced even more significantly. [Figure 5.1](#) shows the deflection of a vehicle's tyres under hard cornering and helps to illustrate the significant requirements on the tyre to produce forces that control the relatively large mass of the vehicle.

It is not intended here to discuss the construction of the tyre carcass, materials or tread pattern. This is addressed by more general texts on vehicle dynamics (Gillespie, 1992) or more focussed books on the subject of tyres (French, 1989), (Moore, 1975). Rather this chapter will start by describing the mechanisms required to generate the vertical tyre forces that support the vehicle, the longitudinal forces required for driving and braking and the lateral forces needed for cornering. The distribution of pressure and stress will also generate local moments acting at the tyre contact patch. A good understanding of these force and moment characteristics is essential before introducing the various mathematical tyre models available and describing the methods used to implement these with multibody systems (MBS) vehicle models.

A spectrum of tyre models exists. Because of the subtlety and deeply nonlinear behaviour of the tyre as a whole, they tend to be isolated in a separate subroutine in all but the simplest multibody model schemes. This chapter will proceed to a discussion of the common elements of tyre modelling and a discussion of the currently

**FIGURE 5.1**

Examples of tyre deflection under hard cornering.

popular tyre models. When it comes to functional models of tyres it is worth making a distinction between ‘a set of expressions that produce tyre-like forces and are formulated in a parametric fashion’ — which might perhaps be called the *tyre model architecture* — and ‘such expressions, with all parameters present and calibrated to represent a particular tyre’ — which might be called a *tyre-specific model*.

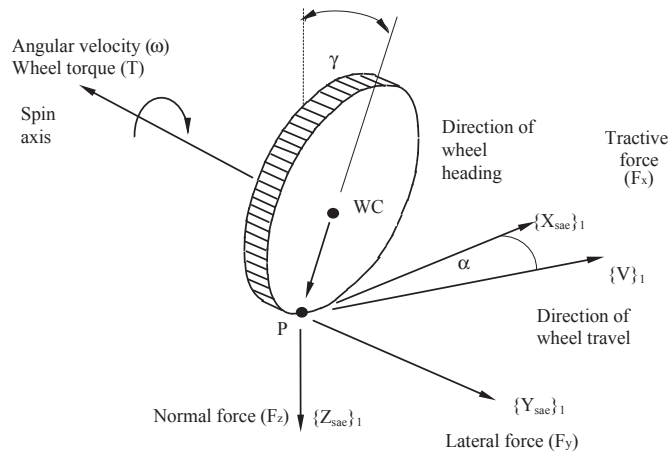
Before a computer simulation can be performed, the actual tyre force and moment characteristics must be estimated or obtained from experimental tests. A traditional approach is to test the tyre using a tyre test machine and to measure the resulting force and moment components for various camber angles, slip angles and values of vertical force. There follows the generation of a number of parameters that must be derived from the measured data before the simulation can proceed. The quality of the tyre-specific model will be a compromise between the accuracy of the fit, relevance of the parameters, and the availability of methods to generate the parameters.

As a final introductory comment it should be noted that what is referred to as a ‘tyre’ model is actually a tyre-road interaction model. Placing the same tyre on a different surface will modify its behaviour substantially, as anyone who has ever driven on ice will have noticed. The most versatile tyre-model architectures allow the tyre-specific model to encounter different surfaces during the course of a single manoeuvre, but require proportionally more data to support this functionality.

## 5.2 Tyre axis frames and geometry

### 5.2.1 The SAE J2047 and ISO 8855 tyre axis frames

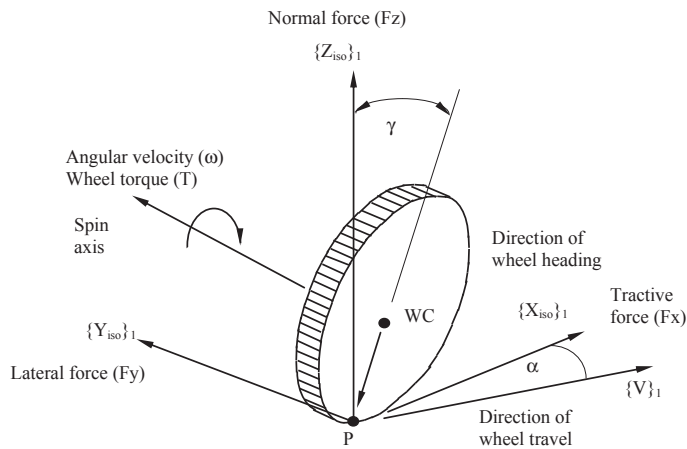
To assist with the description of the forces and moments generated by a tyre, an axis frame shown in simplified form in [Figure 5.2](#) has been defined by the SAE (1976).

**FIGURE 5.2**

SAE tyre axis system.

In this frame the X-axis is the intersection of the wheel plane and the road plane with the positive direction taken for the wheel moving forward. The Z-axis is perpendicular to the road plane with a positive direction assumed to be acting downwards. The Y-axis is in the road plane and its direction dictated by the use of a right-handed orthogonal axis frame. The angles  $\alpha$  and  $\gamma$  represent the slip angle and camber angle respectively. The SAE frame will be used throughout this text unless stated.

It should be noted that not all practitioners adhere rigidly to this frame in their publications and another frame, the ISO 8855 tyre frame, is gaining favour. This frame,

**FIGURE 5.3**

ISO 8855 tyre axis system.

shown simplified in Figure 5.3 has become a de facto standard for new tyre-model architectures.

The ISO 8855 tyre axis system is one of three axis systems described in the Tyre Data Exchange protocol, TYDEX, with the accompanying note that while the ISO definition presumes a horizontal ground plane — that is to say normal to the gravity vector — TYDEX does not enforce this condition. It will be seen later that distortions in the tyre carcass will cause the contact patch to move away from the rigid wheel plane shown in Figures 5.2 and 5.3. Although the components of force are still assumed to act through the contact point P, the distortions will introduce offsets and additional components of moment also acting about the point P.

While it is preferable to use one of the existing tyre axis systems, from time to time organisational convenience or historical compatibility suggests a different axis system be used. There is no conceptual difficulty in this save for a requirement to avoid errors of presumption, which is avoided with repeated and clear declaration of frames of reference.

### 5.2.2 Definition of tyre radii

The definition of tyre radii is important for the formulation of slip in the contact patch. In general we consider a tyre to have an unloaded radius, a loaded radius and an effective rolling radius. The unloaded tyre radius,  $R_u$ , is straightforward to comprehend and is shown in Figure 5.4. For a rigid disc with radius  $R_u$  rolling forward with no sliding (fully geared to the road), during one revolution the disc will move forward a distance  $2\pi R_u$ .

As can be seen from Figure 5.4, due to tyre deflection the distance moved forward will be less than for the rigid disc and can be related to the effective rolling radius giving

$$R_u > R_e > R_l \quad (5.1)$$

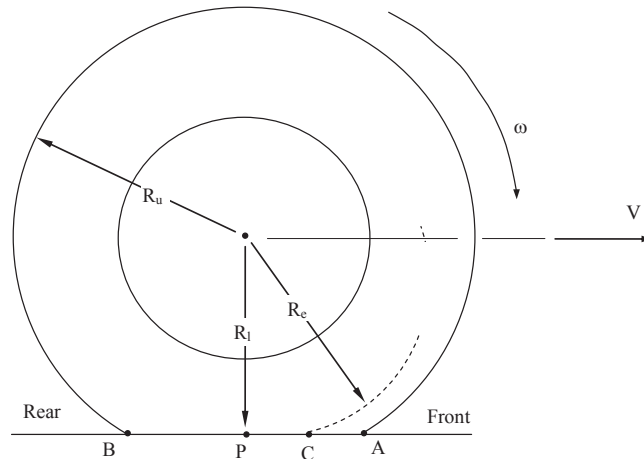


FIGURE 5.4

Definition of tyre radii.

Another definition of effective rolling radius is provided by Moore (1975), this being the distance from the wheel centre to a point C where the distance AC is taken to be one quarter of the total tyre contact patch length AB.

The following are other definitions related to tyre radius provided in SAE J670e (1976):

1. The loaded radius,  $R_l$ , is the distance from the centre of the tyre contact patch to the wheel centre measured in the wheel plane.
2. The static loaded radius is the *loaded radius* of a stationary tyre inflated to the normal recommended pressure.
3. The effective rolling radius,  $R_e$ , is the ratio of the linear velocity of the wheel centre in the  $X_{SAE}$  direction to the angular velocity of the wheel.

While such considerations are easily imagined when the tyre is upright on a smooth surface with a constant load, it may be readily appreciated that discerning the rolling radius for a real tyre with a nonzero inclination angle and on a non-smooth surface is constantly varying and somewhat difficult to discern in anything resembling real time using the above notions.

A more detailed treatment of effective rolling radius is provided by Phillips (2000) based on the representation given in Figure 5.5 that allows the effective rolling radius to be related to the unloaded radius and tyre deflection. For the tyre shown in Figure 5.5, the wheel axle is considered fixed and the road moving such that the relative forward velocity of the wheel is  $V$ . If a number of equidistant radial lines are drawn on the tyre the number passing point A in a given time must be the same as the number passing point P in the contact patch.

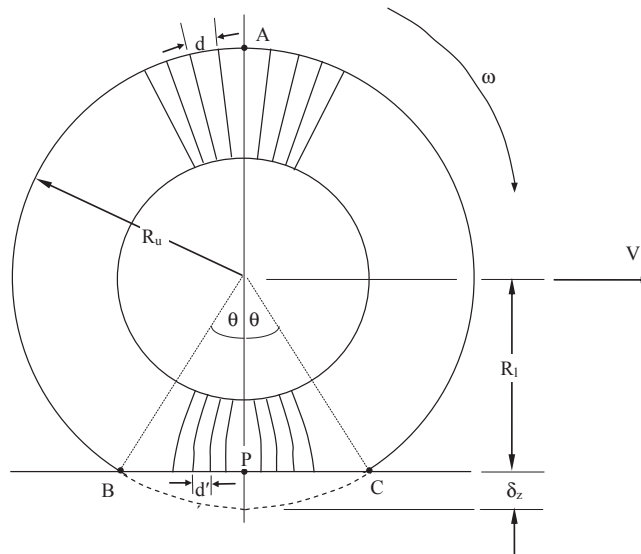


FIGURE 5.5

Deformation of rolling tyre.

If we take

$d$  = the distance between the radial lines at the tyre outer radius near A

$d'$  = the distance between the radial lines at the contact patch near P

then

$$\frac{\omega R_u}{d} = \frac{V}{d'} \quad (5.2)$$

therefore

$$R_e = \frac{V}{\omega} = R_u \frac{d'}{d} \quad (5.3)$$

The tread band is subject to a longitudinal compressive strain within the contact patch  $\varepsilon$  where

$$\varepsilon = \frac{d - d'}{d} \quad (5.4)$$

$$d' = d(1 - \varepsilon) \quad (5.5)$$

therefore

$$R_e = R_u (1 - \varepsilon) \quad (5.6)$$

Assuming that the strain in the contact line is constant we have (assuming

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots \dots)$$

$$1 - \varepsilon = \frac{\text{cordBC}}{\text{arcBC}} = \frac{\sin \theta}{\theta} \approx 1 - \frac{\theta^2}{6} \quad (5.7)$$

From Figure 5.5 we also have (assuming  $\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots \dots$ )

$$\delta_z = R_u (1 - \cos \theta) \approx R_u \frac{\theta^2}{2} \quad (5.8)$$

From Eqns (5.7) and (5.8) we have

$$1 - \varepsilon = 1 - \frac{\delta_z}{3R_u} \quad (5.9)$$

From Eqns (5.6) and (5.9) we have

$$R_e = R_u - \frac{\delta_z}{3} \quad (5.10)$$

If we substitute the loaded radius as  $R_l = R_u - \delta_z$  into (5.10) we get

$$R_e = R_l + \frac{2\delta_z}{3} \quad (5.11)$$

Despite the complexity of the approach, the final approximation in Eqn (5.11) is pleasingly simple and useful in time-varying applications. With some knowledge of vertical stiffness, characteristics, it can be used to estimate rolling radius with useful accuracy in real time applications.

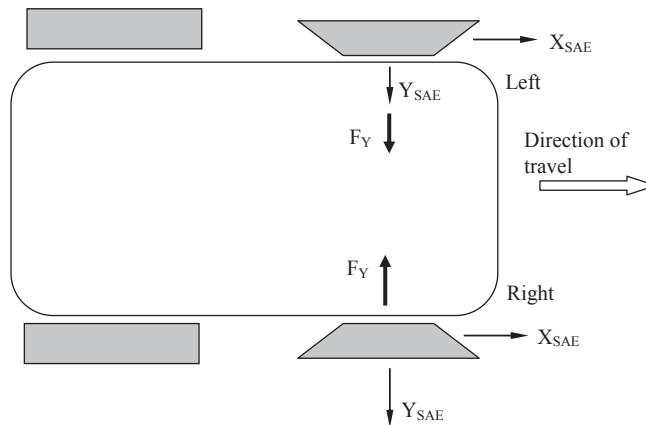
### 5.2.3 Tyre asymmetry

Although it is not intended to address the construction of a tyre in this textbook a brief mention is needed on two types of tyre asymmetry that can occur, these being conicity and plysteer. Both types of asymmetry can occur during tyre fabrication and have the effect of introducing small amounts of lateral force and aligning moment when a tyre is running at zero slip angle. It will be seen later in this chapter that these offsets in lateral force or aligning moment are visible when plotted against slip angle and that representation in a simulation will depend on the sophistication of the tyre model used.

In a modern vehicle these effects become more important when considering refinement and the ‘on-centre’ feel of the vehicle, particularly when driving for long periods at high motorway speeds.

Conicity is an effect that arises due to assuming the tyre to have the shape of a truncated cone as shown in Figure 5.6.

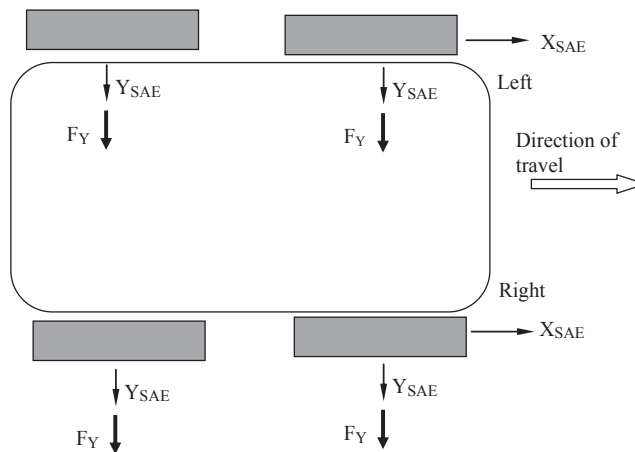
When considering the effect of conicity it must be realised that incorporation in a tyre model must take careful account of the tyre axis system used. In Figure 5.6, for



**FIGURE 5.6**

Generation of tyre lateral forces due to conicity.



**FIGURE 5.7**

Generation of tyre lateral forces due to plysteer.

example, the tyres shown with exaggerated conicity produce a force towards the apex of the cone as the vehicle travels on a straight heading. For the tyre on the left side this is a force that is positive when referred to the  $Y_{SAE}$  axis. If the same tyre is now switched to the right side of the vehicle, reversing the direction of rotation, the force is still towards the apex but is now negative when referred to the  $Y_{SAE}$  axis.

Plysteer is an effect that arises due to a small bias in the positioning of the cords within the tyre belt layers. This is shown in Figure 5.7 where it is evident that the opposite occurs to conicity in that switching a tyre from the left to the right of the vehicle does not reverse the lateral force direction. Thus for a vehicle fitted with tyres all exhibiting the same plysteer there will be a tendency for the vehicle to drift off a straight course without some steering correction. This will correct the course of the vehicle but will cause the rear wheels to ‘track’ to the side of the front wheels so that the vehicle progresses with a crab like motion, albeit imperceptible to the driver.

In general, plysteer and conicity are ‘noise’ effects and not something that can be used to meaningfully modify the behaviour of the vehicle. However, knowledge of them is important when distinguishing between real-world measured data and concept-level symmetrical side-force characteristics. It should be kept in mind, though, that typical roads are constructed to have some kind of side slope — called ‘cross fall’ by highway engineers but popularly and ambiguously referred to as ‘camber’ by everyone else. Cross fall will induce side forces of comparable magnitude to plysteer and conicity; typical slopes for drainage purposes are of the order of 1%, giving side forces around 1% of the vehicle weight by inspection.

It can be seen that since the features of the tyre influencing both conicity and plysteer are not visible for tyres which are non-handed and non-directional then it is a statistical inevitability that some vehicles will end up with an unfavourable distribution of either plysteer or conicity. When tyres are marked with a rotation arrow or the

words ‘inside’ and ‘outside’ then these effects are partially controlled. Tyres which use both a rotation arrow and inside/outside markings are completely controlled and are typically uniquely manufactured for each corner of the vehicle; such tyres may expect to show the best control of plysteer and conicity noise effects at the vehicle level.

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## 5.3 The tyre contact patch

### 5.3.1 Friction

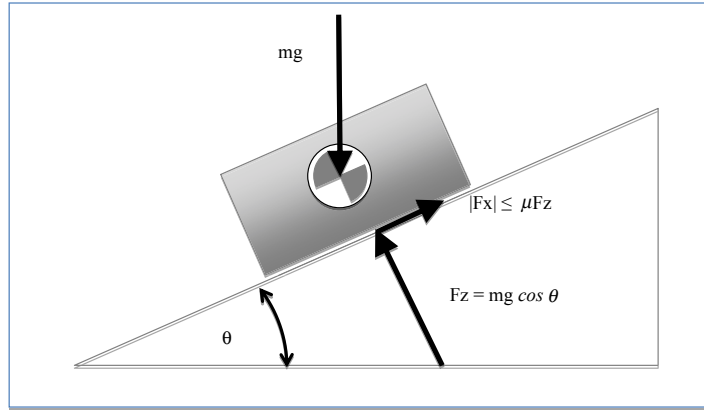
The classical laws of friction as often taught in school can be summarised as:

1. Friction is a property of two contacting surfaces. It does not make sense to discuss friction as if it were a material property.
2. Frictional force is linearly proportional to normal force and can be defined using a coefficient of friction (frictional force/normal force).
3. The coefficient of friction is independent of contact area between the two surfaces.
4. The static coefficient of friction is generally different to (and often greater than) the kinetic (or dynamic or sliding) coefficient of friction.
5. The coefficient of friction is independent of sliding speed.

A detailed treatment of this subject with regard to tyres is given by Moore (1975), where it is shown that the above laws are flawed, or limited in certain conditions such as high tyre pressures. The essential reason for this is the faintly outrageous behaviour of rubber and rubberlike compounds, which were not included in early experiments on friction carried out by, for example, Charles-Augustin de Coulomb, who had materials like glass, wood and leather to experiment with. The concept of a coefficient of friction, which varies with sliding velocity, will however prove useful for describing the tyre models used later in this chapter.

Friction as a force is often poorly understood and it is worth discussing explicitly. Even with the simplified laws of friction as described above, a block of material standing on a surface can be imagined to be subject to frictional forces at the interface of the surface and block. When the surface is level then the block and the surface are pressing on each other with equal and opposite normal forces due to the weight of the block and its reaction. With no other forces acting, there are no frictional forces present, although the *capacity* for them to arise is present. The frictional forces themselves only arise in reaction to an applied external force — an inclination of the surface or an externally applied force parallel to the plane of the surface (Figure 5.8).

The frictional forces calculated by the ‘laws’ of friction are in fact the maximum capacity for reactive forces to be generated, and when the applied forces are lower than this level then the frictional forces rise and fall to preserve equilibrium. Thus the frictional force in the stationary block is time-varying if the applied force is also time-varying. This concept becomes particularly important for modelling clutches, as discussed in Chapter 8.

**FIGURE 5.8**

Frictional forces for a block on an inclined plane.

If the applied force is greater than the maximum capacity for frictional force then the difference between the applied force and the maximum frictional force is a net force available to accelerate the block. This is the simplest possible description of friction, often described as Coulomb friction. It is easy to formulate mathematically

$$\text{For } V_x > 0: \quad \hat{F}_x = -\mu F_z \quad (5.12)$$

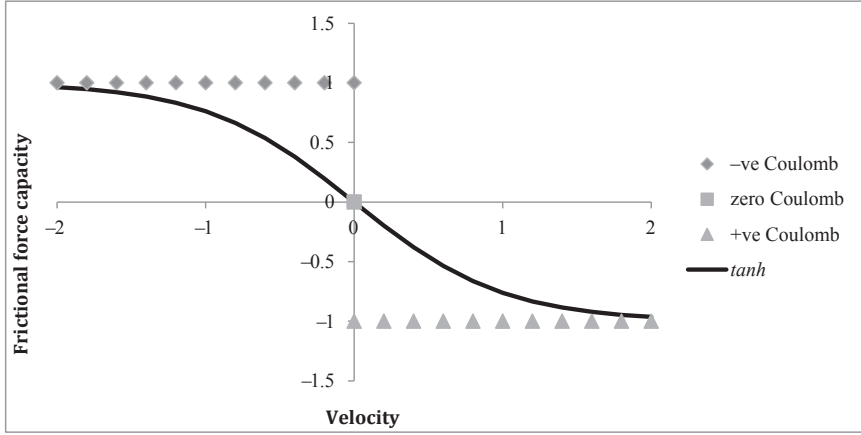
$$\text{For } V_x = 0: \quad \hat{F}_x = 0 \quad (5.13)$$

$$\text{For } V_x < 0: \quad \hat{F}_x = \mu F_z \quad (5.14)$$

However, it can be readily seen that numerically integrating this solution can be challenging owing to the discontinuity present in its formulation. A common work-around is to use the hyperbolic tangent function,  $\tanh$ , as a function of sliding velocity (Figure 5.9).

The  $\tanh$  function has the advantage of being numerically continuous but it does have the disadvantage of having a finite gradient at the origin. This means that, for example, if a block on an inclined plane is modelled then its weight can only be reacted with some nonzero velocity and therefore the block will not stay still. Even compressing the  $\tanh$  formulation using a velocity multiplier cannot make the slope infinite, although it can make it so steep that the block does not move significantly during the simulation period. Students will soon discover, however, that being more and more demanding of the local gradient makes for a very lengthy solution time, requiring as it does a high degree of accuracy.

Sometimes the word ‘stiction’ is used in discussion of friction but its use is not preferred by the authors since many practitioners either fail to define it or use it inconsistently, or both.

**FIGURE 5.9**

A dimensionless comparison between Coulomb friction and a numerically continuous implementation with  $\tanh$ .

Other friction models are available, most notably for the authors a comparatively simple formulation that remains continuously differentiable and captures Coulomb, viscous and so-called Stribeck effects, which produce ‘law 4’ of the friction laws described above. As originally described by Makkar et al. (2005), the model is exceptionally simple, producing a frictional force  $F$  as a function of a displacement  $x$  and its first derivative

$$F = g_1 (\tanh(g_2 \dot{x}) - \tanh(g_3 \dot{x})) + g_4 \tanh(g_5 \dot{x}) + g_6 \dot{x} \quad (5.15)$$

where the terms  $g_i$  are parameters: describing the model. The parameters are not particularly usable as is, and can be defined in terms of more readily assimilated parameters

$$R = \log \left( \frac{g_2}{g_3} \right) \quad (5.16)$$

$$S = -0.099R^6 + 0.7815R^5 - 2.5644R^4 + 4.5815R^3 - 4.9237R^2 + 3.2794R - 1.1697 \quad (5.17)$$

$$g_1 = \frac{\mu_1 - \mu_0}{10^S} \quad (5.18)$$

$$g_2 = \frac{\tanh^{-1}(0.99)}{V_{SPk}} \quad (5.19)$$

$$g_3 = \frac{\tanh^{-1}(0.99)}{V_{Send}} \quad (5.20)$$

$$g_4 = \mu_0 \quad (5.21)$$

$$g_5 = \frac{\tanh^{-1}(0.99)}{V_{C99}} \quad (5.22)$$

$$g_6 = \mu_2 \quad (5.23)$$

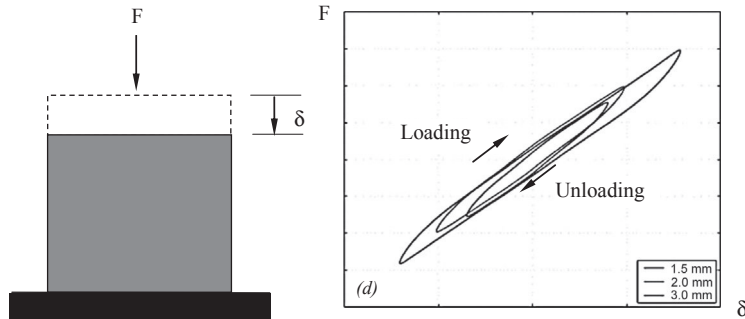
where

- $\mu_0$  is the sliding (Coulomb) friction coefficient
- $V_{C99}$  is the sliding speed at which 99% of Coulomb friction is achieved
- $\mu_1$  is the breakout ('static') friction coefficient
- $V_{SPk}$  is sliding speed at which peak friction is achieved
- $V_{Send}$  is the sliding speed above which Stribeck effects are absent
- $\mu_2$  is the viscous dissipation term

It should be noted that the six-term Coulomb—viscous—Stribeck model described still suffers from the finite gradient problem and cannot hold a block still on an inclined plane. Discussion of yet more elaborate models, such as the Leuven friction model, are beyond the scope of this text but the interested reader is referred to the wider literature for a detailed view (van Geffen, 2009). All these models are essentially empirical, in that they fit some mathematical function to observed behaviour.

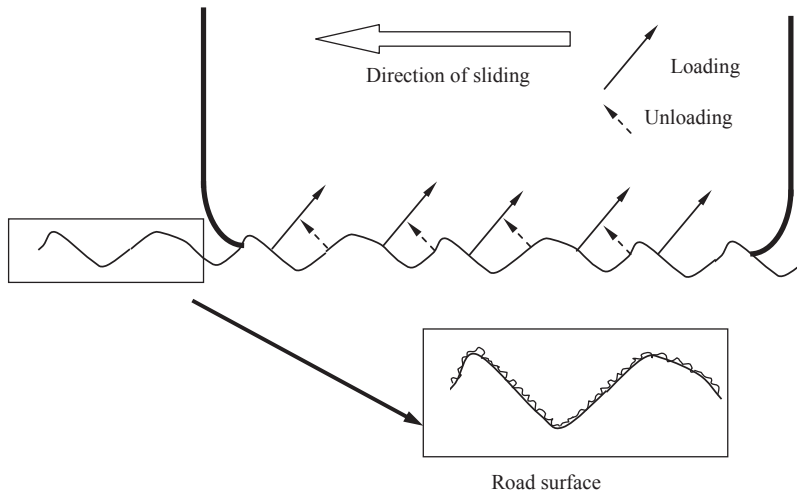
For real tyres the friction generated between the tread rubber and the road surface is generated through two mechanisms, these being hysteresis and adhesion.

As already discussed in Chapter 3, rubber displays substantial hysteresis. In order to understand the influence of hysteresis on tyre/road friction, consider a block of rubber subjected to an increasing and then a decreasing load as shown in Figure 5.10. As the rubber is loaded and unloaded it can be seen that, for a given displacement  $\delta$ , the force  $F$  is greater during the loading phase than the unloading phase.



**FIGURE 5.10**

Hysteresis in rubber.

**FIGURE 5.11**

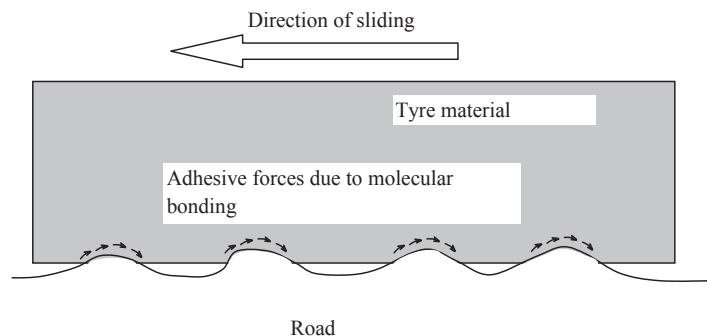
Loading and unloading of a single tread block of tyre rubber in the contact patch.

If we consider the situation where the same block of rubber is sliding over a non-smooth surface, it can be seen from [Figure 5.11](#) that an element of rubber in the contact patch will be subject to continuous compressive loading and unloading.

As the rubber slides over the irregular road surface compressive forces normal to the surface are generated and relieved as the rubber is loaded and unloaded. The resolved vertical sum of all the forces gives the vertical reaction. The resolved horizontal sum of all the forces gives the force due to hysteresis. Due to the hysteresis, the sum of the loading forces is greater than the sum of the unloaded forces, resulting in a force opposing the direction of sliding. It should be emphasised that the scale of [Figure 5.11](#) is frequently misunderstood, being presumed in error as the gaps between aggregate and binder in the road surface; it is rather the texture on the surface of an individual stone that is shown in close-up and is something of a surprisingly intricate ‘fractal style’ effect. It can be seen that higher hysteresis compounds will deliver more of this effect and that it is greatly sensitive to texture; highly polished surfaces will deliver little of this effect. It is, however, broadly insensitive to surface contamination.

The adhesive component, shown in [Figure 5.12](#), results from momentary intermolecular links generated between the exposed surface atoms of rubber and road material in the contact area (Kummer, 1966). This component of friction ‘tops up’ the hysteretic force on dry roads but is greatly reduced when the road surface is contaminated with even a thin film of water or powder. ‘Slick’ tyres, with no tread and increased surface contact area, are used for racing on dry roads to maximise this force.

It is perfectly possible to manufacture a rubber compound in which the affinity for the road surface exceeds the ability of the tread compound to hold itself together, in which case the rubber at the road surface will remain there while the bulk of the

**FIGURE 5.12**

Frictional force component due to adhesion.

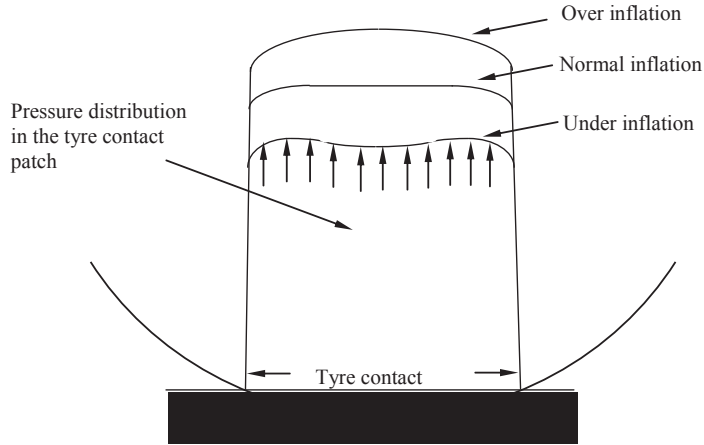
rubber is torn from it and moves on. This will typically leave rubber on the road surface and also results in a characteristic pattern on the tyre surface. Many tyre compounds can generate large forces at the road/tyre interface, which leads to a great deal of energy being applied to the interface; in some cases this raises the temperature sufficiently not only to reduce the shear strength of the rubber compound, but also to locally liquefy a bituminous binder in the road surface, floating it to the top. Thus it is often true that black marks on the road surface are some combination of rubber and bitumen (where such an ingredient is present in the road surface, of course; on concrete this simply cannot be so and the mark is all rubber).

The interested reader is referred to 'The Racing and High Performance Tire' (Haney, 2003) for an excellent empirical discourse on interpreting tyre behaviour in performance-limit conditions by referring to after-the-fact evidence such as tyre surface markings. For a discussion on the liquefaction of road binder and its resulting effects, see (Bullas, 2006) and (Bullas, 2008).

### 5.3.2 Pressure distribution in the tyre contact patch

In order to understand the manner by which forces and moments are generated in the contact patch of a rolling tyre, an initial appreciation of the stresses acting on an element of tread rubber in the contact patch is required. This pressure distribution interacts with the frictional mechanisms described above in order to produce the aggregate forces and moments within the contact patch. Each element will be subject to a normal pressure  $p$  and a shear stress  $\tau$  acting in the road surface. The element will not slip on the road if  $\tau < \mu p$  where  $\mu$  is the coefficient of friction between the tread rubber and the road surface.

The pressure distribution depends on tyre load and whether the tyre is stationary, rolling, driven or braked. The pressure distribution is not uniform and will vary both along and across the contact patch. In order to understand the mechanics involved with the generation of forces and moments in the contact patch, some simplification of the pressure distribution will be adopted here starting with [Figure 5.13](#) where

**FIGURE 5.13**

Pressure distribution in a stationary tyre contact patch.

typical pressure distributions in the tyre contact patch for a stationary tyre and the effects of inflation pressure are considered.

Generally the pressure rises steeply at the front and rear of the contact patch to a value that is approximately equal to the tyre inflation pressure. Overinflation causes an area of higher pressure in the centre of the contact patch while underinflation leads to an area of reduced pressure in the centre of the patch.

When the tyre is rolling it will be shown later that pressure distribution in the contact patch is not symmetric and is greater towards the front of the contact patch.

While it can be asserted with some certainty that the integrated vertical forces throughout the contact patch must sum to the applied load, it is an error to assert that the contact patch dimensions can be simply predicted by, for example, load, width and inflation pressure. It can be generally observed for conventional tyres that vertical stiffness is broadly linear until rim contact occurs (op het Veld, 2006) and (Reimpell and Sponagel, 1988). A moment's consideration suggests that the growth of contact patch area is nonlinear with vertical stiffness and therefore the resulting average contact pressure cannot scale linearly.

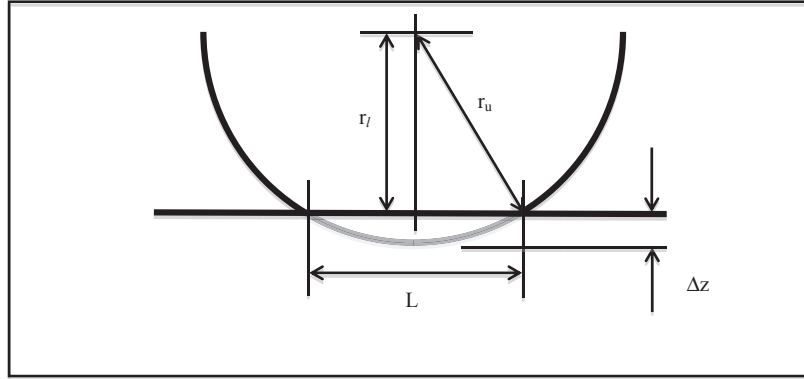
In [Figure 5.14](#), it can be seen that the contact patch length can be found using Pythagoras, as given in [Eqn \(5.24\)](#)

$$L = 2 \sqrt{r_u^2 - r_l^2} = 2 \sqrt{2 r_u^2 \Delta z - \Delta z^2} \quad (5.24)$$

Presuming the width of the contact patch is constant, for a linear stiffness we can say the average contact pressure  $P$  must follow a form as follows

$$P = \frac{k \Delta z}{2W \sqrt{2 r_u \Delta z - \Delta z^2}} \quad (5.25)$$



**FIGURE 5.14**

Contact patch length.

where  $k$  is the tyre vertical stiffness and  $W$  is the tyre width. Considering the tyre as an isothermal volume, we might estimate the change in nominal pressure  $P_{\Delta z}$  from initial pressure  $P_0$  by calculating a change in volume compared to initial volume  $V_0$

$$P_{\Delta z} = P_0 \frac{V_0}{V_{\Delta z}} \quad (5.26)$$

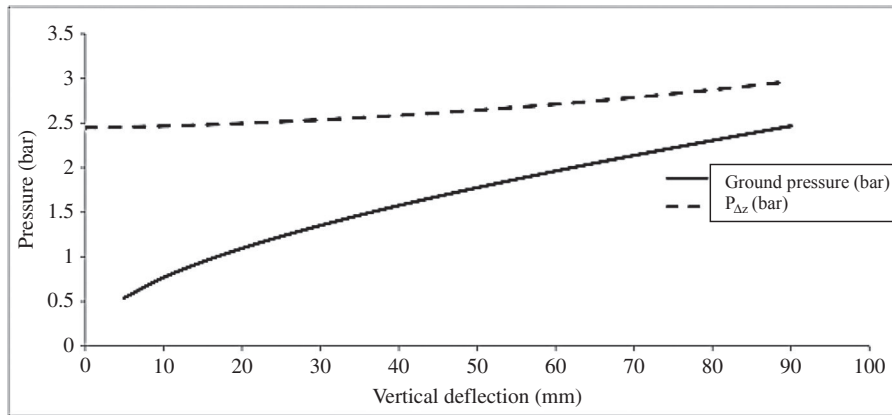
$$V_0 = \pi (r_u^2 - r_w^2) W \quad (5.27)$$

$$V_{\Delta z} = V_0 - \Delta V \quad (5.28)$$

$$\Delta V = \cos^{-1} \left( \frac{n}{r_u} \right) r_u^2 - \frac{L - n}{2} \quad (5.29)$$

where  $r_w$  is the radius of the wheel rim. Considering some typical values of  $k = 250$  N/mm,  $W = 205$  mm,  $r_u = 320$  mm,  $P_0 = 2.45$  bar and  $r_w = 228$  mm, Figure 5.15 can be plotted.

It can be seen that the average contact pressure never reaches the tyre pressure. The only reasonable conclusion to draw from this is that inflation pressure is a poor surrogate for ground pressure; the tyre is not a balloon in any meaningful sense but rather a structure whose properties are manipulated by the presence of air. The existence of run-flat tyres with a braced sidewall suggests this is so, and recent advances in contact pressure distribution measurements have resulted in a proliferation of data supporting the assertion. Despite this, it is still regarded by some as a matter for debate.

**FIGURE 5.15**

A comparison between inflation pressure as given in [Eqn \(5.26\)](#) and load divided by contact patch area, referred to as 'ground pressure'.

## 5.4 Tyre force and moment characteristics

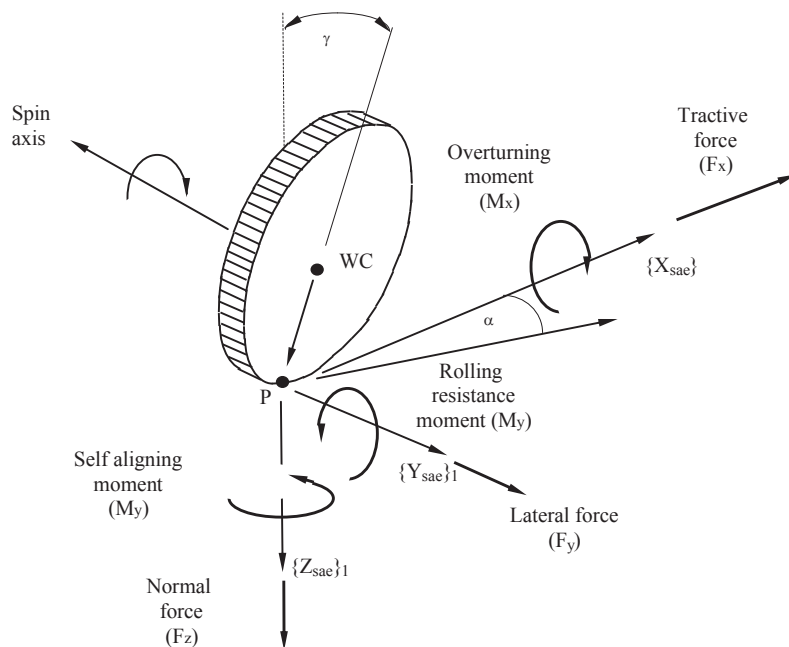
### 5.4.1 Components of tyre force and stiffness

The local pressures and stresses distributed over the tyre contact patch can be integrated to produce forces and moments referenced to a local coordinate system within the contact patch. Using the SAE tyre axis system the full set of forces and moments are as shown in [Figure 5.16](#).

The following section will explain the mechanical characteristics of each force and moment component. The order in which these components are described will be that which most facilitates an understanding of the mechanisms and dependencies rather than following the local order of the SAE tyre axis system. The tractive force  $F_x$  and lateral force  $F_y$  depend on the magnitude of the normal force component  $F_z$ . Hence the normal force is described first.

It should also be noted that more than one mechanism will be involved in the generation of each component. The tractive force has formulations involving driving, braking and rolling resistance. The lateral force is dependent on both slip and camber angle. It is also not possible to treat components of force and moment in isolation. It is, for example, necessary to provide a single explanation as to how the self aligning moment and lateral force resulting from slip angle arise due to stress distributions within the tyre contact patch.

Finally the reader should also be reminded that vehicle dynamics is traditionally a subject where various terms are used to describe the same thing. For example, vertical force, normal force and tyre load may be used to mean the same thing by various authors. Other examples where confusion may arise include the use of aligning torque, aligning moment or self aligning moment, longitudinal or tractive force and lateral or cornering force.

**FIGURE 5.16**

Tyre forces and moments shown acting in the SAE tyre axis system.

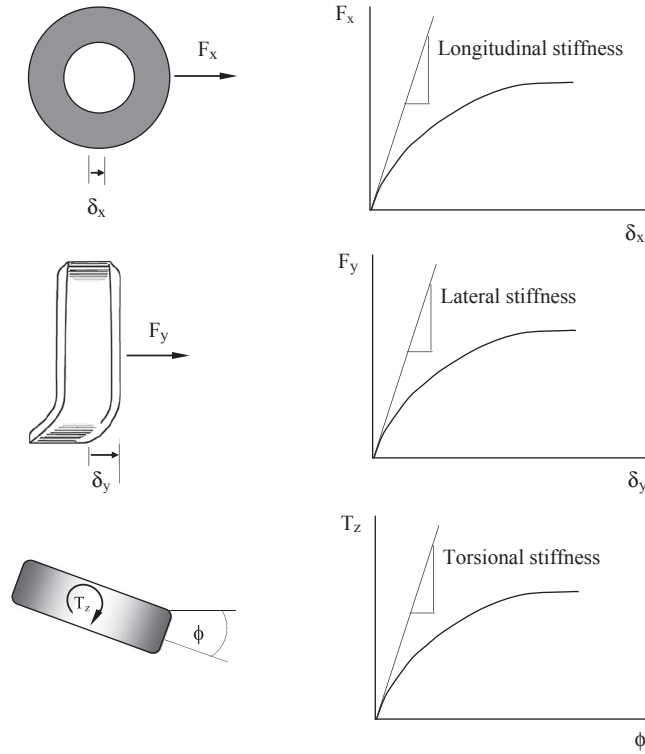
The use of the term stiffness can also add confusion to newcomers to the subject area. A traditional static force/displacement approach is used by Moore (1975) to define longitudinal, lateral and torsional stiffness of a tyre. In each case a non-rolling tyre is mounted on a plate and incrementally loaded as indicated in Figure 5.17 until complete sliding occurs. Plotting graphs of force or moment against displacement or rotation allows the stiffness parameters to be obtained from the slopes at the origin.

We will see later that terms such as cornering stiffness and aligning moment stiffness are associated with a rolling tyre and should not be confused with the lateral and torsional stiffness defined here. The term longitudinal stiffness can be particularly misleading as another definition is commonly used when longitudinal tractive forces due to driving and braking are discussed.

The explanations that follow will initially deal with each force mechanism in isolation, for example lateral forces arising due to slip angle and camber angle are considered separately with no simultaneous longitudinal tractive force. Following this, a more complex treatment involving combinations of the various force components will be addressed.

### 5.4.2 Normal (vertical) force calculations

The calculation of normal force in the tyre is relatively straightforward compared with the calculation of longitudinal or lateral forces. The normal force will however



**FIGURE 5.17**

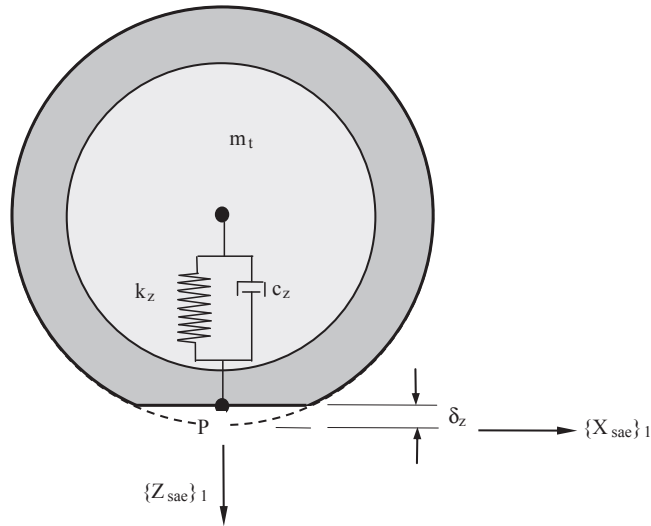
Measurement of stiffness in a non-rolling tyre.

always be negative when computed using the SAE tyre axis system. This is not particularly elegant when presenting the dependencies of other force components on the normal force. To overcome this, a positive value of this force component is often referred to as vertical force or tyre load. In SAE J670e (1976) vertical load is taken as the negative of normal force.

It is generally sufficient to treat the tyre as a linear spring and damper when computing the vertical force component, notwithstanding the reservations expressed in Chapter 4 about hysteretic damping and ways to model it. The tyre is quite lightly damped and in any case in the running vehicle its motions are dominated by wheel hop, so the use of an equivalent viscous damping term with the equivalence point taken at wheel hop is satisfactory for simple handling models. The calculation of the vertical force  $F_z$  acting at point P in the tyre contact patch has a contribution due to stiffness  $F_{zk}$  and a contribution due to damping  $F_{zc}$ . These forces act in the direction of the  $\{Z_{sae}\}_1$  vector shown in Figure 5.18

$$F_z = F_{zk} + F_{zc} \quad (5.30)$$

$$F_{zk} = -k_z \delta z \quad (5.31)$$

**FIGURE 5.18**

Vertical tyre force model based on a linear spring damper.

$$F_{zc} = -c_z V_z \quad (5.32)$$

where

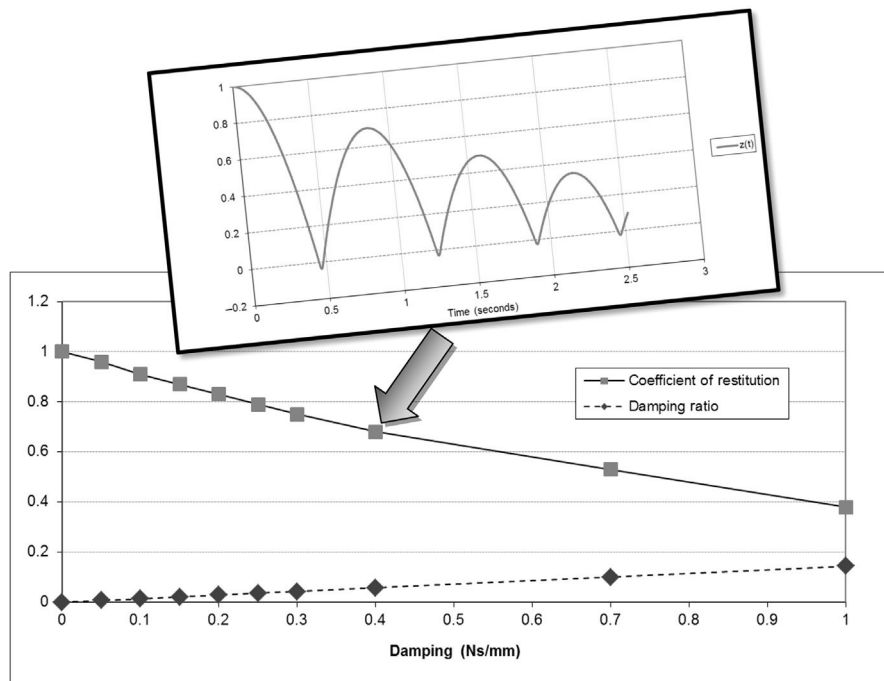
$$c_z = 2.0 \zeta \sqrt{m_t \cdot k_z} \quad (5.33)$$

and

- $m_t$  = mass of tyre
- $k_z$  = radial tyre stiffness
- $\zeta$  = radial damping ratio
- $\delta_z$  = tyre penetration
- $V_z$  = rate of change of tyre penetration

A linear model of tyre vertical force may need to be extended to a nonlinear model for applications involving very heavy vehicles or studies where the tyre encounters obstacles in the road or terrain of a similar size to the contact patch or smaller. This could also be applicable for parallel work in the aircraft industry where established tyre models have been formulated to simulate the behaviour of the aircraft on the runway, particularly on landing, and potential problems with wheel shimmy (Smiley, 1957; Smiley and Horne, 1960). Where a nonlinear model of vertical tyre force is required, the most straightforward approach would be to represent the stiffness-based component of the force by a cubic spline interpolation of measured static force–displacement data.

Damping data are often difficult to come by and a simple test used with adequate success by the author (Harty) has been to force fit the results of a drop

**FIGURE 5.19**

A simply predicted drop-and-bounce test result with resulting damping.

test to the chosen model. A wheel and tyre assembly, with the tyre at the pressure of interest, is held at some useful distance above the ground, say 1 m. The assembly is dropped — that is to say cleanly released with no initial vertical velocity — and observed. The observation can conveniently be carried out with a video camera suitably mounted so as to largely remove perspective effects. The resulting motion of the wheel — a series of bounces successively decreasing in height — can be captured in a frame-by-frame examination of the video trace where only the highest points are digitised. A simple vertical model, of the kind already described, can be integrated through time and force-fitted characteristics to replicate the observed results. An example is shown in [Figure 5.19](#).

Although somewhat approximate, the influence of tyre damping on the overall solution is relatively small and its measurement in this fashion captures it with adequate resolution in terms of vehicle-level behaviour.

### 5.4.3 Longitudinal force in a free rolling tyre (rolling resistance)

Under normal driving conditions a tyre is continually subject to a wide range of tractive driving and braking forces. This section discusses the formulation of driving and braking forces under pure slip conditions, i.e. straight-line motion only. The more

complex situation of combined slip, for example simultaneous braking and cornering, is addressed later in this chapter.

As a starting point it can be shown that slip will always be present in the tyre contact patch even in the absence of tractive driving and braking forces. Consider first the free rolling tyre shown in Figure 5.20 and the mechanism that leads to the generation of longitudinal slip. The model used in Figure 5.20 has simplifications but will help to develop an initial understanding. As the tyre rolls forward the radius reduces as tread material approaches point A at the front of the tyre contact patch. At this point we can say that the forward velocity  $V$  of the wheel relative to the road surface is given by

$$V = \omega R_e \quad (5.34)$$

The tread material approaching the front of the contact patch will have a tangential velocity  $V^t$  relative to the wheel centre O given by

$$V^t = \omega R_u \quad (5.35)$$

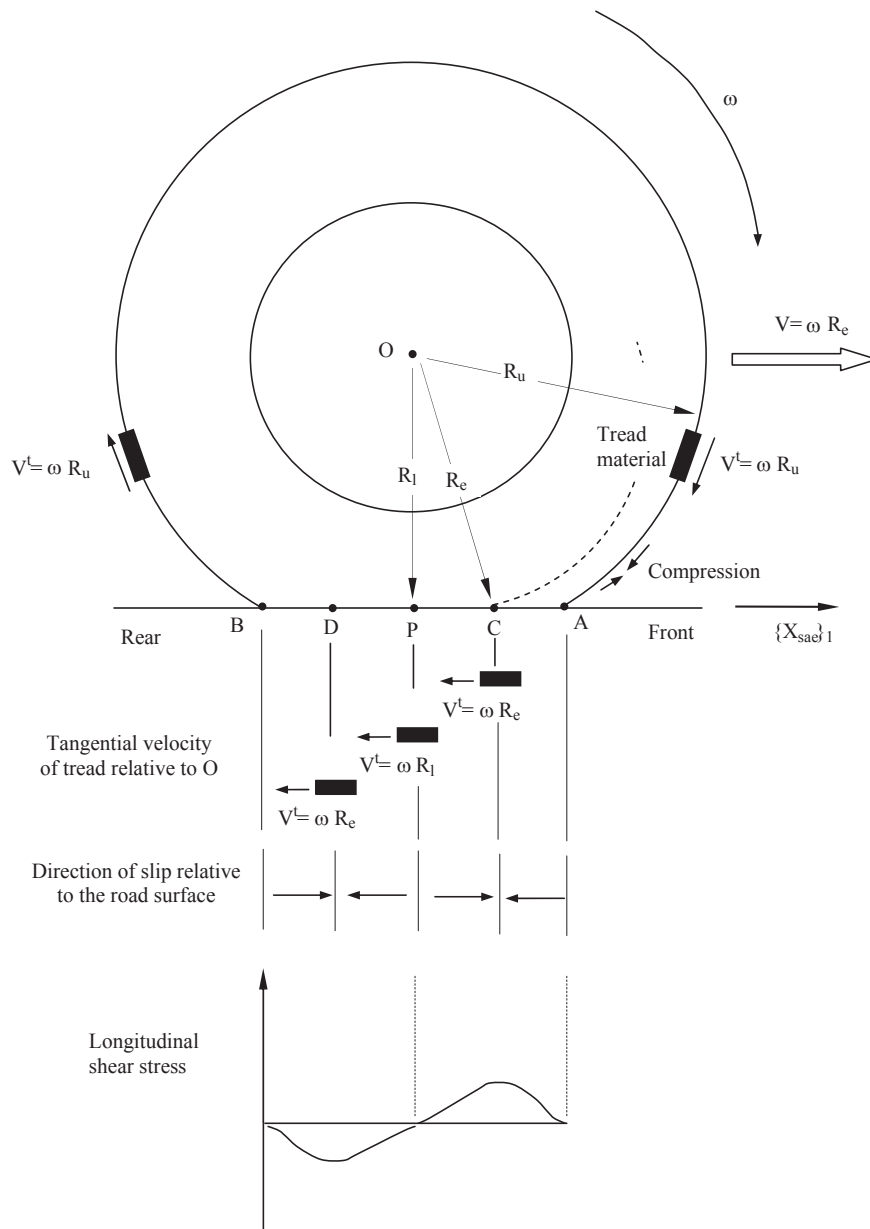
As the tread material gets close to the start of the contact patch, the tyre radius decreases causing the tangential velocity of the tread material to decrease causing circumferential compression of tread material just before it enters the contact patch.

As the tread material enters the contact patch at point A the rearward tangential velocity relative to the wheel centre is just slightly greater than the forward velocity of the vehicle. This results in initial rearward slip of tread material relative to the road surface between point A and C. At point C it is assumed that the radius has reduced to a value equivalent to the effective rolling radius  $R_e$  resulting in the rearward tangential velocity matching the forward vehicle velocity and theoretically producing a point of zero slip in the tyre. Over the central region of the contact patch between C and D the radius reduces to a value below the effective rolling radius reversing the slip in the tyre to the forward direction. At the centre of the patch P the radius reduces to the loaded radius  $R_l$ . In theory this point would produce the lowest tangential velocity and the highest forward slip, although experimental observations (Moore, 1975) indicate that the tangential speed does not reduce to this level. Between point D and B the radius recovers to a value greater than the effective rolling radius causing the direction of slip to reverse again to a forward direction.

It is clear that the direction of slip changes several times as tread moves through the contact patch resulting in the distribution of longitudinal shear stress of the type shown at the bottom of Figure 5.20. The shear stress is plotted to be consistent with the SAE reference frame and is not symmetric; the net effect being to produce an overall force, the rolling resistance, acting in the negative  $X_{SAE}$  direction.

It should be noted that the two-dimensional model presented is not fully representative as components of lateral slip are also introduced in a free rolling tyre due to deformation of the side walls as shown in Figure 5.21.

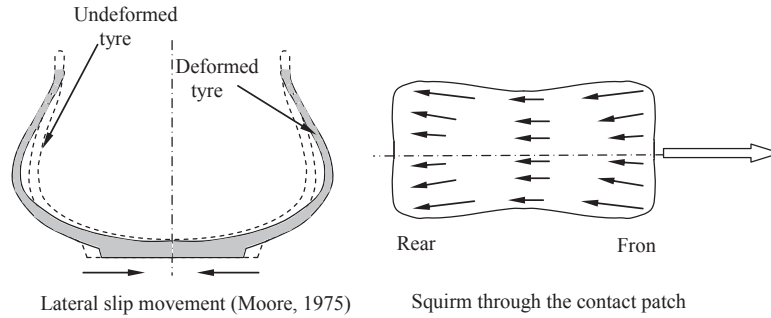
As the tyre carcass deforms in the vicinity of the contact patch the deformation of the side walls creates additional inwards movement of the tread material (Moore, 1975).



**FIGURE 5.20**

Generation of slip in a free rolling tyre.



**FIGURE 5.21**

Lateral distortion of the contact patch for a free rolling tyre.

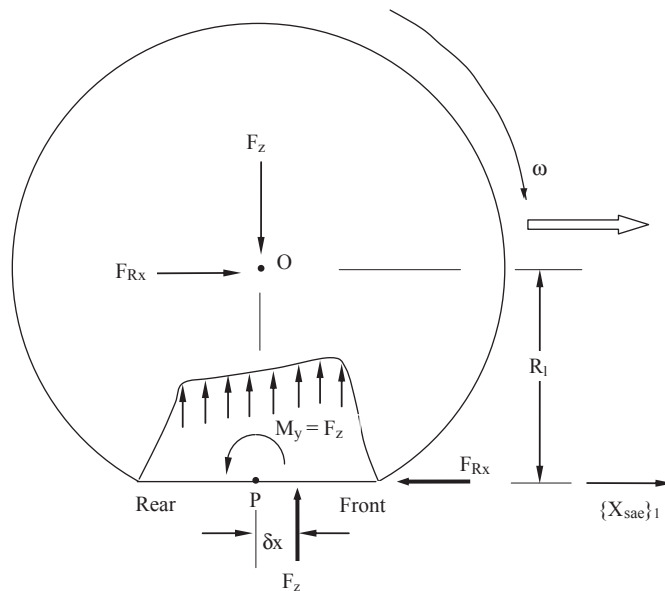
This causes the contact patch to assume an hourglass shape creating an effect referred to as 'squirm' (Gillespie, 1992) as the tread material moves through the contact patch.

Before moving on to consider the driven or braked tyre we will now consider the rolling resistance forces generated in a free rolling tyre. Rolling resistance results from energy losses in the tread rubber and side walls. Energy loss in the tread rubber is produced by hysteresis. If we refer again to Figure 5.9 it is clear for a block of rubber, or tread material, there is more force required at any given displacement during the loading phase than the unloading phase. As tread material moves through the contact patch it will be loaded until it reaches the midpoint of the contact patch and unloaded as it moves to the rear of the contact patch. This and the additional losses due to hysteresis in the side walls leads to a pressure distribution that is not symmetrical as shown for the stationary tyre in Figure 5.13 and has a greater pressure distribution in the front half of the contact patch as shown Figure 5.22.

The pressure distribution implies that the resultant tyre load  $F_z$  acts through the centre of pressure, a distance  $\delta x$  forward of the wheel centre. For equilibrium, a couple exists that must oppose the tyre load and its reaction acting down through the wheel centre. The couple that reacts the wheel load couple results from the rolling resistance force  $F_{Rx}$  acting longitudinally in the negative  $X_{SAE}$  axis and reacted at the wheel centre where

$$F_{Rx} = \frac{F_z \delta x}{R_1} \quad (5.36)$$

The rolling resistance may also be referenced by a rolling resistance coefficient, this being the rolling resistance force  $F_{Rx}$  divided by the tyre load  $F_z$ . By definition therefore the rolling resistance moment  $M_y$  is  $F_z \delta x$  and the rolling resistance moment coefficient is  $\delta x$ . Rigorous adherence to the sign convention associated with the tyre reference frame is essential when implementing these formulations in a tyre model. In Figure 5.22, to assist understanding,  $F_z$  is represented as the vertical force acting on the tyre rather than the negative normal force computed in the  $Z_{SAE}$  direction.

**FIGURE 5.22**

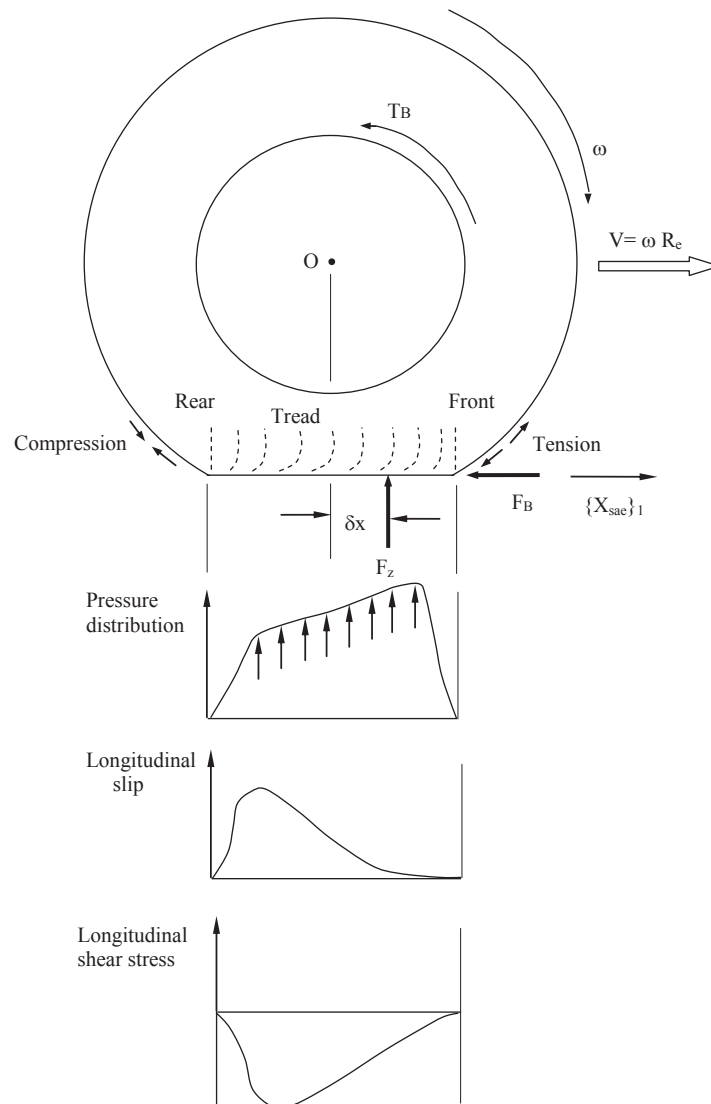
Generation of rolling resistance in a free rolling tyre.

The rolling resistance force is very small in comparison with other forces acting at the contact patch, a rolling resistance of the order of 1% of vehicle weight being typical for a car tyre. This and the fact that the rolling resistance force may vary by up to 30% of the average value during one revolution (Phillips, 2000) make accurate measurement difficult.

#### 5.4.4 Braking force

During braking the activation of the brake mechanism will apply forces to the rotating wheel that at this stage may be treated as a brake torque  $T_B$  acting about the wheel centre and opposing the rolling motion of the wheel. During this process the tread material in the tyre will begin to slide relative to the road, giving rise to slip. As the angular velocity of the wheel reduces, a braking force is generated that tends to move the contact patch rearwards relative to the wheel centre. This effect will introduce circumferential tension in the tread just before entering the contact patch as opposed to the compression noted earlier in this area for a free rolling tyre. As the contact patch distorts rearward during braking, compression will instead be generated in the tread just leaving the patch as shown in Figure 5.23.

During braking, as for free rolling, as tread material approaches the contact patch the radius will reduce with a consequent reduction in the tangential speed of the tread material relative to the wheel centre. For moderate braking, the tread initially entering the contact patch will initially bend rearward under the action of shear stresses for a short distance before the tangential velocity of the tread material slows

**FIGURE 5.23**

Generation of force in a braked tyre.

to the forward velocity of the tyre  $\omega R_e$  and slip begins to progressively develop as the tread material moves back through the contact patch.

As the tread approaches the rear of the tread, the pressure begins to unload, releasing the deformation in the tread. The tangential velocity of the tread material begins to increase and the frictional braking force reduces rapidly to zero at the rear of the patch. These actions produce distributions of pressure, slip and longitudinal

shear stress of the type shown at the bottom of Figure 5.23 where the shear stress is plotted as negative in accordance with the resultant braking force resolved in the SAE tyre coordinate system. It should also be noted that when braking, the pressure distribution tends to differ from that in a free rolling tyre as the peak, and hence resultant tyre load, move further forward in the tyre contact patch.

If the resulting braking force is maintained the angular velocity of the tyre will reduce from its free rolling value and will eventually become zero when the wheel is fully locked. A measure of the slip generated can be defined by a slip ratio or percentage slip. In this text the term slip ratio  $S$  will be used

$$S = \frac{\omega_0 - \omega_B}{\omega_0} \quad (5.37)$$

where

$\omega_0$  is the angular velocity of the free rolling wheel

$\omega_B$  is the angular velocity of the braked wheel

If we consider the case of a braked wheel it can be seen from Eqn (5.18) that for the free rolling state the slip ratio will be zero and that for the fully locked and skidding wheel the slip ratio will be 1.0. Multiplying slip ratio by 100 gives the term percentage slip.

In SAE J670e (1976) the sign is reversed to produce a slip ratio of  $-1.0$  for the fully locked wheel, convenient when plotting braking force that is negative in the SAE system against slip ratio

$$S = \frac{\omega_B - \omega_0}{\omega_0} \quad (5.38)$$

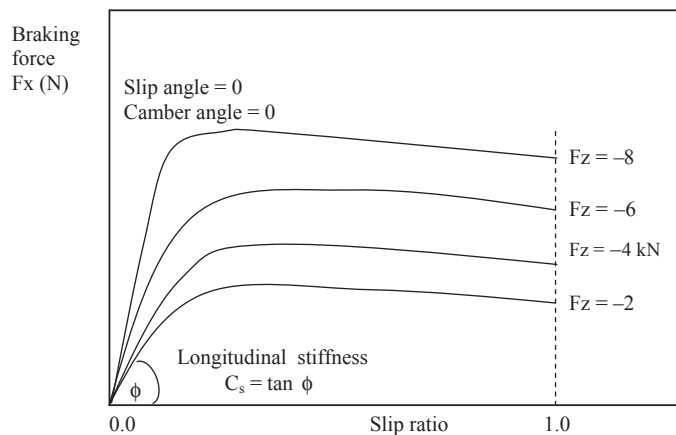
A further definition that is popular is to substitute  $V = \omega_0 R_e$  into Eqn (5.37) giving

$$S = \frac{V - \omega_B R_e}{V} \quad (5.39)$$

It should be noted that slip ratio has been subject to various definitions by researchers and research groups in tyre companies, several are listed in Milliken and Milliken (1995). Some tyre models use  $R_l$  instead of  $R_e$  when formulating slip ratio, which may require careful consideration when using a general purpose MBS program to model anti-lock braking system (ABS) (Ozdalyan and Blundell, 1998).

Plotting curves of braking force, for convenience shown positive here, against slip ratio for a range of tyre loads will generally produce curves of the type shown in Figure 5.24.

Examination of the curves in Figure 5.24 reveals that at each vertical load the braking force increases rapidly in a linear manner to reach a peak value that,

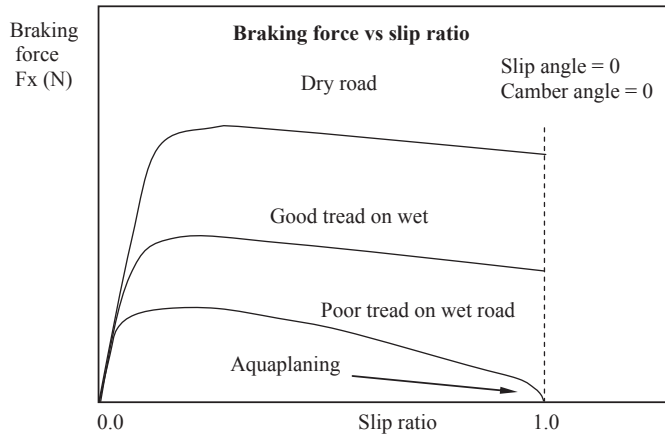
**FIGURE 5.24**

Braking force versus slip ratio.

depending on tyre design and road conditions, would typically occur at a slip ratio anywhere between 0.15 and 0.3. After this point the braking force will level out or reduce as the wheel approaches the fully locked situation. Examination of curves such as these, sometimes called ‘mu-slip’ curves, facilitates an understanding of ABS operation where cycling the brake pressure maintains a slip ratio near the peak braking force position for each wheel on the vehicle. This is desirable not only to maximise braking effort but also to maintain a rolling wheel for cornering and directional stability.

An important property of each curve is the slope at the origin, referred to as the longitudinal stiffness,  $C_s$ . It can be seen that this is not a constant but increases with load, which is significant when considering the capability of any tyre model to be used in braking simulations. In Figure 5.24 the curves are shown to pass through the origin. In practice a small vertical offset in longitudinal force will be apparent for a free rolling tyre, this being the rolling resistance discussed earlier.

It is important to reiterate that frictional forces are not the property of the tyre alone. The effects of road material and texture, or contamination with water and ice, are also significant. Figure 5.25 demonstrates typical curves of braking force against slip ratio, at a given tyre load for various road conditions (Phillips, 2000). These curves demonstrate that on wet roads peak values of braking force as expected reduce and that for a locked wheel with poor tread a dangerous situation known as hydroplaning or aquaplaning can arise where the tyre runs on a film of water and traction is effectively lost. The curves of braking force can also be categorised by two coefficients of friction associated with the peak braking force and that associated with total sliding at a slip ratio of 1.0. On dry roads it is possible to obtain a coefficient of friction for good tyres substantially in excess of 1.0, on wet roads this could typically reduce to about 0.5 or lower for tyres with poor tread while a road contaminated with ice may only achieve a peak value of 0.1.

**FIGURE 5.25**

The effect of road contamination on braking.

It can be seen when examining the curves in Figure 5.25 that the longitudinal stiffness is relatively unaffected by surface contamination. This is particularly dangerous for a road with ice or the poor tyre on a wet road. In these conditions the peak braking force occurs rapidly at low slip ratio causing the vehicle to skid before any possible corrective action from the average driver.

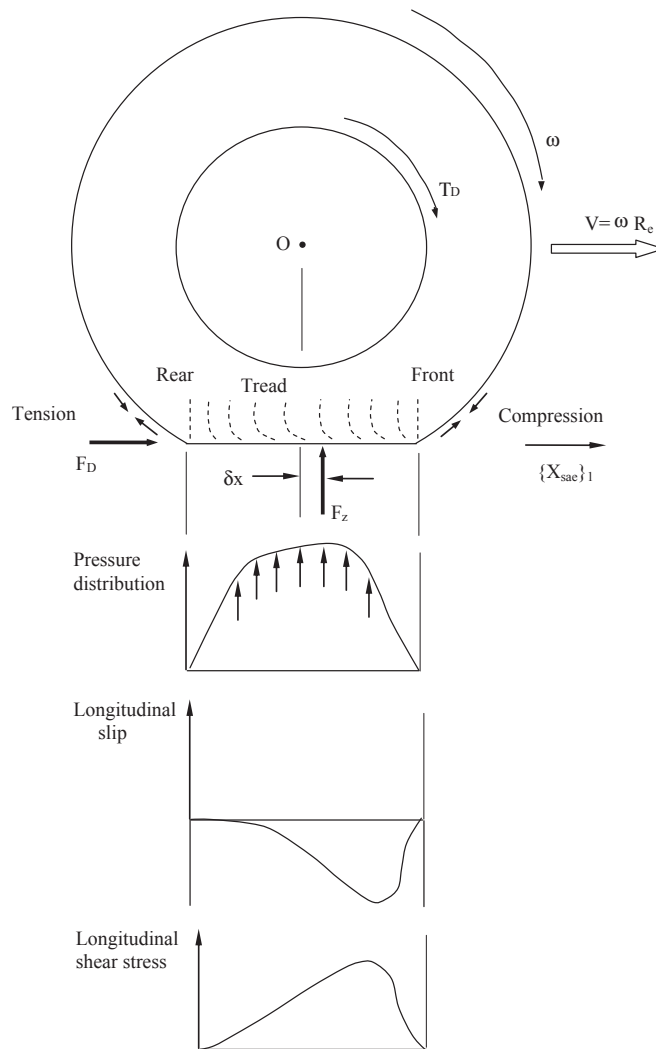
In addition to the above it is also known that an increase in vehicle speed will reduce peak values of braking force and that other parameters such as tyre inflation pressure will have an effect, a more detailed treatment of which is given by Pacejka (2012).

### 5.4.5 Driving force

During driving the transmission will impart a driving torque  $T_D$  to the rotating wheel as shown in Figure 5.26. As the angular velocity of the wheel increases a driving force is generated that tends to move the contact patch forward relative to the wheel centre. This effect will introduce circumferential compression in the tread just before entering the contact patch and tension on leaving as opposed to braking.

As the wheel is driven the tread initially entering the contact patch will initially bend forward under the action of shear stresses for a short distance. As the tread approaches the rear of the tread the pressure begins to unload releasing the deformation in the tread and progressive sliding develops. These actions produce distributions of pressure, slip and longitudinal shear stress of the type shown at the bottom of Figure 5.26. It can also be seen that when driving, the pressure distribution tends to differ from that in a braked tyre as the peak, and hence resultant tyre load, move further to the rear in the tyre contact patch reducing the offset  $\delta x$ .

If the resulting driving force is maintained the angular velocity of the tyre will increase from its free rolling value and will eventually begin to spin. A measure



**FIGURE 5.26**

Generation of force in a driven tyre.

of the slip generated for the driven tyre can be defined by a further modification to the slip ratio  $S$

$$S = \frac{\omega_D - \omega_0}{\omega_0} \quad (5.40)$$

where

$\omega_0$  is the angular velocity of the free rolling wheel

$\omega_D$  is the angular velocity of the driven wheel

For driving a slip ratio of 1.0 is sometimes taken to define the onset of wheel spin. From Eqn (5.40) this will occur when the angular velocity of the driven wheel reaches a value of twice that for free rolling. Unlike braking, the slip ratio in driving can exceed 1.0 as the wheel angular velocity continues to increase. This definition of 'spin' is somewhat arbitrary. For both tractive and braking cases the relationship between longitudinal force and slip ratio is such that the wheel behaviour converges for slip ratios smaller than those at which peak force is produced. However for larger slip ratios the wheel behaviour diverges rapidly. For spin in particular, angular velocity increases very quickly until torque is reduced.

#### 5.4.6 Generation of lateral force and aligning moment

The generation of lateral force and aligning moment in the tyre result from combinations of the same mechanisms and are thus treated together here. As a starting point it is helpful to consider Figure 5.27, which is adapted from the sketches for forces and torques provided by Olley (1945). Figure 5.27 is particularly useful for relating the sign convention for the lateral forces and aligning moments plotted and discussed throughout this chapter.

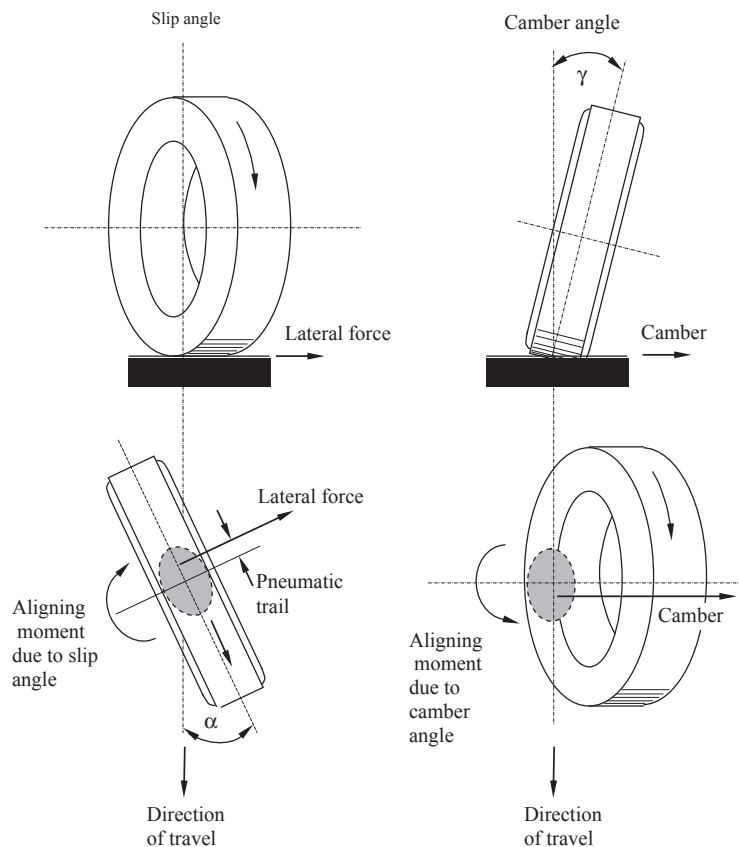
From Figure 5.27 it can be seen that for a tyre rolling with a slip angle  $\alpha$  at zero camber angle the lateral force generated due to the distribution of shear stress in the contact patch acts to the rear of the contact patch centre creating a lever arm known as the pneumatic trail. This mechanism introduces the aligning moment and has a stabilising or 'centring' effect on the road wheel. This is an important aspect of the steering 'feel' that is fed back to the driver through the steering system.

Similarly it can be seen from Figure 5.27 that for a tyre rolling with a camber angle  $\gamma$  at zero slip angle, the lateral force generated is called camber thrust. Due to the conditions in the contact patch the camber thrust acts in front of the contact patch centre creating a mechanism that creates a moment. Although this is referred to here as an aligning moment it has the opposite effect of the aligning moment resulting from slip angle and is sometimes called the camber torque as there is no resultant aligning action on the road wheel. The importance of the camber-induced moment is small for passenger cars but can be large for motorcycles.

#### 5.4.7 The effect of slip angle

In order to understand the mechanisms that lead to the generation of lateral force and aligning moment resulting due to slip angle it is useful to start with Figure 5.28 showing the distribution of pressure  $p$ , and the lateral stress in the contact patch. The upper part of the figure provides a side view and the lower part is a top view looking down on to the contact patch. The lateral stress boundary,  $\mu p$ , represents the limit available between the tread rubber and the road surface. If the lateral stress is below this limit no sliding will occur but once the lateral stress reaches this limit the tread rubber will commence sliding.

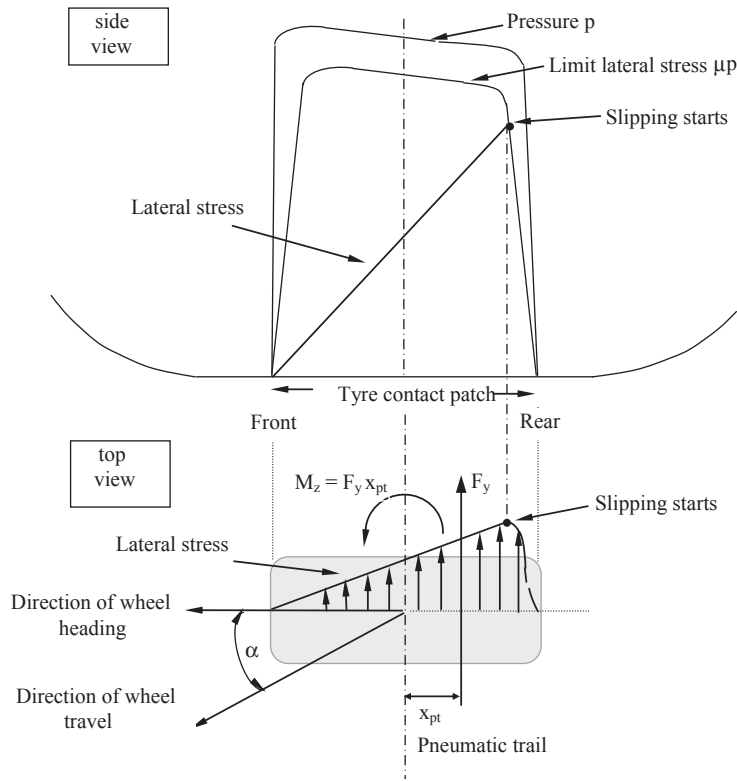


**FIGURE 5.27**

Forces and moments due to slip and camber angle.

When the tyre rolls at a slip angle  $\alpha$ , tread rubber that is put down on the road surface at the front of the contact patch moves back through the patch at the same slip angle, deforming the sidewalls of the tyre, so that the lateral stress in the tread rubber steadily increases as shown. At a certain point in the contact patch the lateral stress reaches the limit boundary after which sliding takes place until the tread rubber leaves the rear of the contact patch and the lateral stress returns to zero.

As the slip angle increases, the rate at which lateral stress is generated as the tread rubber moves back through the contact patch increases so that the point at which slippage commences moves forward in the contact patch. It can also be seen that as the slip angle increases, the area under the lateral stress curve increases. This area is a measure of the resulting lateral force  $F_y$  generated by integrating the stress over the contact patch. At low slip angles, when the lateral stress shape is substantially triangular, there is a nearly linear relationship between lateral force and slip angle. In general this linearity only extends to one or two degrees of slip angle.



**FIGURE 5.28**

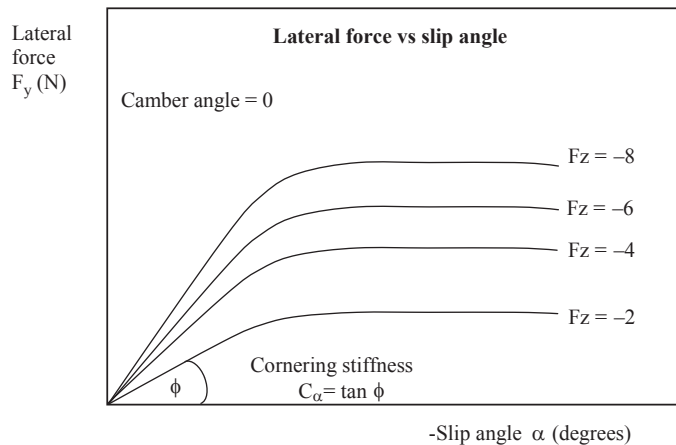
Generation of lateral force and aligning moment due to slip angle.

As the slip angle increases, the amount of rubber involved in sliding gradually extends from the rear of the tyre contact patch until all the rubber is sliding and the lateral stress follows the boundary limit,  $\mu p$ , distribution.

Since the form of the pressure distribution is a measure of the tyre load ( $-F_z$ ), it follows that the maximum lateral force  $F_{y \max}$  is found from

$$F_{y \max} = -\mu F_z \quad (5.41)$$

In practice this maximum is achieved at slip angles around 10 degrees for many road tyres. Motorsport and high performance low profile tyres produce their maximum forces at much lower slip angles, as low as three or four degrees. Figure 5.29 shows a typical plot of lateral force  $F_y$  with slip angle  $\alpha$  for increasing tyre load with the camber angle set at zero. For the convenience of plotting results in the positive quadrant, negative slip angle is used in this plot. From the plot it can be seen that the cornering stiffness  $C_\alpha$  is the gradient of the curve measured at zero slip angle at a given tyre load. As the tyre load increases so does the cornering stiffness, although it will be seen later that at higher tyre loads the magnitude of the cornering

**FIGURE 5.29**

Plotting lateral force versus slip angle.

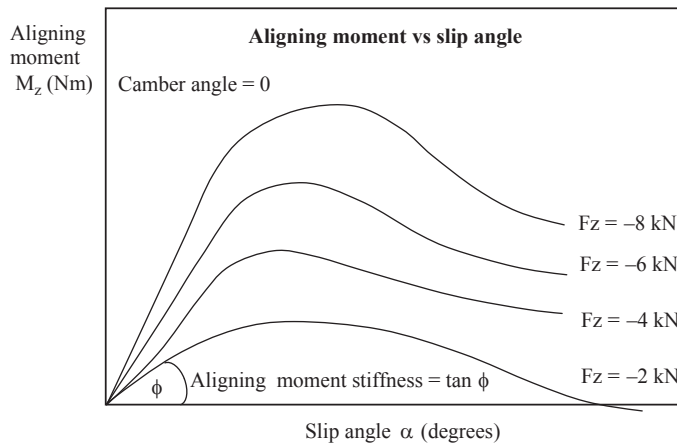
stiffness begins to level off. In Figure 5.29 the curves are shown to pass through the origin. In practice a small offset in lateral force will be apparent at zero slip angle due to the effects of conicity and plysteer discussed earlier.

Looking back to Figure 5.28 it can be seen that as the shape of the lateral stress distribution is approximately triangular, the lateral force  $F_y$  acts through the centroid of this area that is to the rear of the wheel centre line by a distance referred to as the pneumatic trail. Inspection of Figure 5.28 should indicate that as the slip angle increases the line of action of  $F_y$  moves forward reducing the pneumatic trail eventually to zero. The aligning moment  $M_z$  is the product of the lateral force and the pneumatic trail and will reduce accordingly, eventually becoming negative usually for lightly loaded tyres at high slip angles. In these situations the extent of sliding occurs to such an extent throughout the contact patch that the lateral stress distribution approaches the shape of the  $\mu p$  curve moving the centroid through which  $F_y$  acts forward of the centre. A typical plot of aligning moment with slip angle, for a given tyre load and zero camber angle, is shown in Figure 5.30. From the plot it can be seen that the aligning moment stiffness is the gradient of the curve measured at zero slip angle at a given tyre load.

The aligning moment curve is nearly linear at low slip angles and reaches a maximum value for most tyres at around half the slip angle required for peak lateral force. The gradient of the curve measured at a zero slip angle is the aligning moment stiffness. In Figure 5.30 the curves are shown to pass through the origin. In practice a small offset in aligning moment will be apparent at zero slip angle due to the effects of conicity and plysteer discussed earlier.

#### 5.4.8 The effect of camber angle

The lateral force that arises due to an inclination of the tyre from the vertical is referred to as camber thrust. The SAE definition of positive camber angle is taken

**FIGURE 5.30**

Plotting aligning moment versus slip angle.

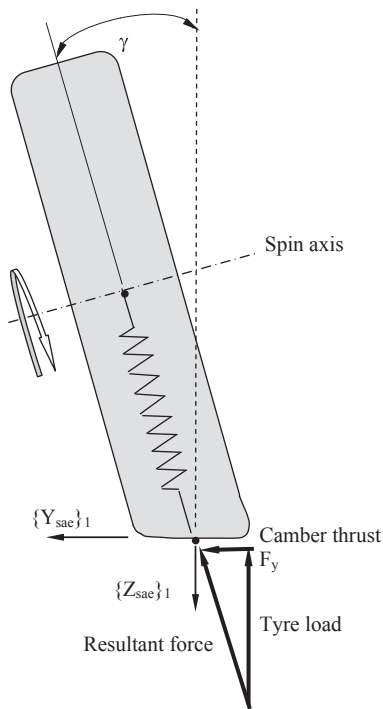
for the top of the tyre leaning outwards relative to the vehicle. The fact that this differs from one side of the vehicle to the other does not lead to consistency when developing a tyre model. For understanding it is useful to remember that the camber thrust will always act in the direction that the tyre is inclined as shown in Figure 5.31. For the SAE system shown here a positive camber angle  $\gamma$  will produce a positive camber thrust for all tyres on the vehicle modelled in that system.

If the tyre is inclined at a camber angle  $\gamma$ , then deflection of the tyre and the associated radial stiffness will produce a resultant force,  $F_R$ , acting towards the wheel centre. Resolving this into components will produce the tyre load and the camber thrust.

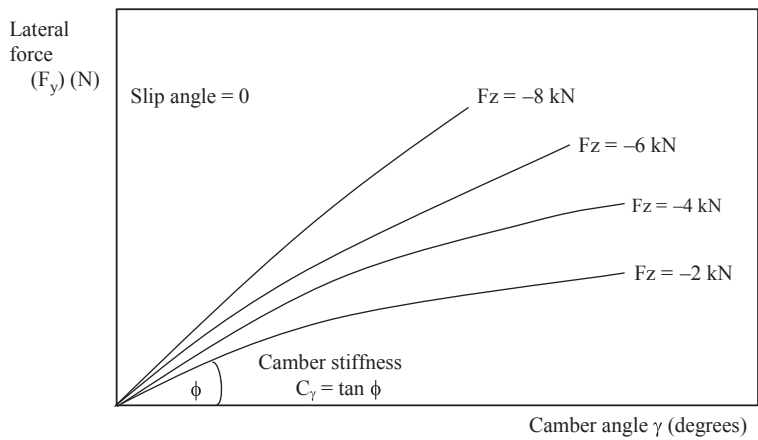
An alternative explanation provided in Milliken and Milliken (1998) compares a stationary and rolling tyre. For the stationary tyre, experimental observations of tread in the contact patch indicate a curved shape. As the tyre rolls the tread moving through the contact patch is constrained by the road to move along a straight line, the net reaction of these forces being the camber thrust.

Figure 5.32 shows a typical plot of lateral force  $F_y$  with camber angle  $\gamma$  for increasing tyre load with the slip angle set to zero. From the plot it can be seen that the camber stiffness  $C_\gamma$  is the gradient of the curve measured at zero camber angle at a given tyre load.

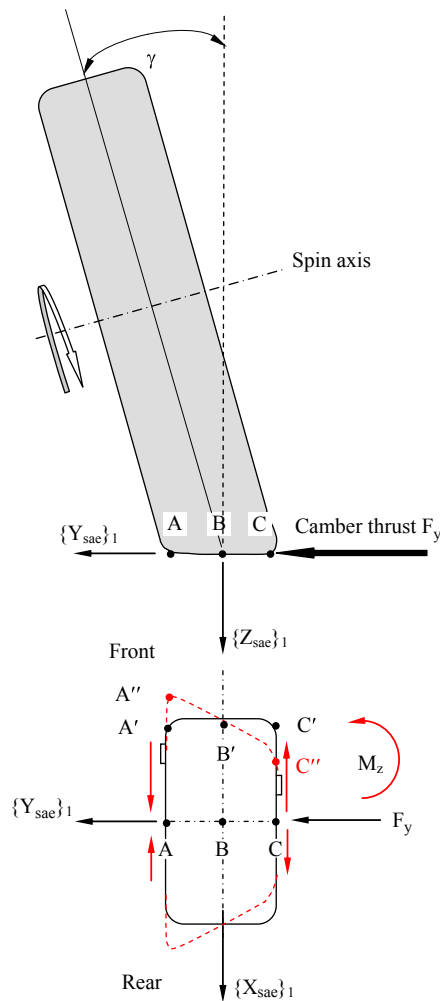
In order to understand why a cambered tyre rolling at zero slip angle produces an aligning moment, it is useful to consider the effect of the shape of the contact patch. Consider the situation shown in Figure 5.33 where the wheel and tyre are rolling at a camber angle  $\gamma$  with the slip angle equal to zero. The lower part of Figure 5.33 is a plan view on the tyre contact patch. The three points A, B and C, shown in Figure 5.33, are initially in line across the centre of the contact patch. If the tyre rolls so that point B moves to B' at the rear of the contact patch and the rubber in the centre line is not subjected to any longitudinal stress.



**FIGURE 5.31**  
Generation of lateral force due to camber angle.

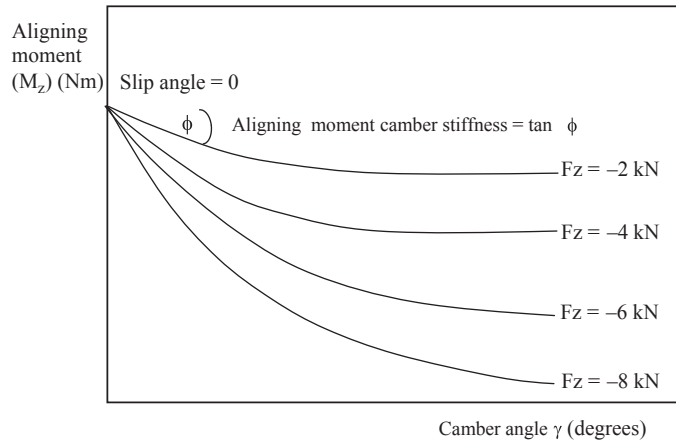


**FIGURE 5.32**  
Plotting lateral versus camber angle.

**FIGURE 5.33**

Generation of self-aligning moment due to camber angle.

Due to the camber the tyre will corner and point A on the inside of the tyre will roll at a smaller radius of bend to point C on the outside of the tyre. If the tyre rubber was not subject to any longitudinal stress these points would move to A'' and C'' respectively to preserve the total circumference on the inner and outer edges. If it is presumed that the stiffness of the tyre restricts this and the points remain in line across the rear of the contact patch (A', B' and C') then a longitudinal compressive stress acts on the inner A side and a tensile stress acts on the outer C side at the front of the contact patch. A similar effect occurs at the rear of the contact patch but the increased pressure in the front of the footprint of a rolling tyre means a net torque into the turn is developed when the effects are summed along the contact patch

**FIGURE 5.34**

Plotting aligning moment versus camber angle — note the reversed sign compared to slip angle aligning moment.

length, as shown in Figure 5.33. The geometric aspect is less obvious in motorcycle tyres but the mechanism is essentially similar.

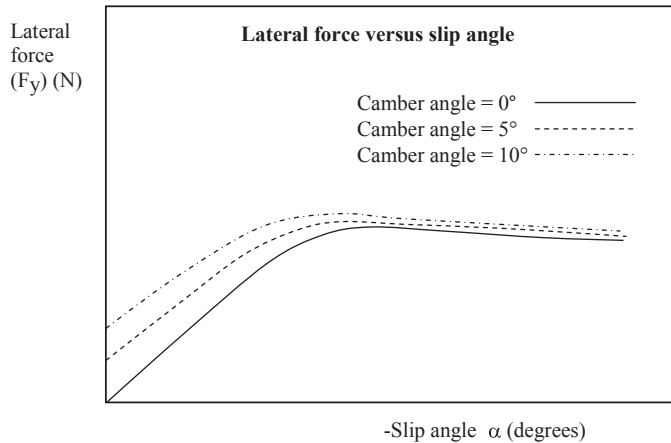
A typical plot of aligning moment with camber angle, for a given tyre load and zero slip angle, is shown in Figure 5.34. From the plot it can be seen that the aligning moment camber stiffness is the gradient of the curve measured at zero camber angle at a given tyre load.

The lateral forces due to camber angle tend to be small when compared with those resulting from slip angle for a typical car tyre. In the linear range it would not be untypical to generate as much as 20 times the amount of lateral force per degree of slip angle compared with that generated per degree of camber angle. For motorcycle tyres, with a more rounded profile, it is possible for riders to incline the motorcycle to produce camber angles between the tyre and road in excess of  $45^\circ$ , resulting in camber thrust being the most significant component of lateral force.

#### 5.4.9 Combinations of camber and slip angle

The treatment so far has considered the generation of lateral force due to slip angle and camber angle in isolation. For a road car, while slip angle dominates the generation of lateral force, some amount of camber will occur at the same time. The effect of adding camber to slip angle is shown in Figure 5.35 where for a given tyre load the lateral force against slip angle curve is plotted at 0, 5 and  $10^\circ$  of camber angle. It should be noted that the curves here are plotted with assisting camber angle where the wheels are leaning into the turn. A similar reduction in lateral force will occur where the camber angle is reversed and the wheels lean out of the turn.

At zero degrees of slip angle the introduction of camber angle introduces an offset from the origin, this being the camber thrust discussed earlier occurring at

**FIGURE 5.35**

The effect of combined camber and slip angle on lateral force.

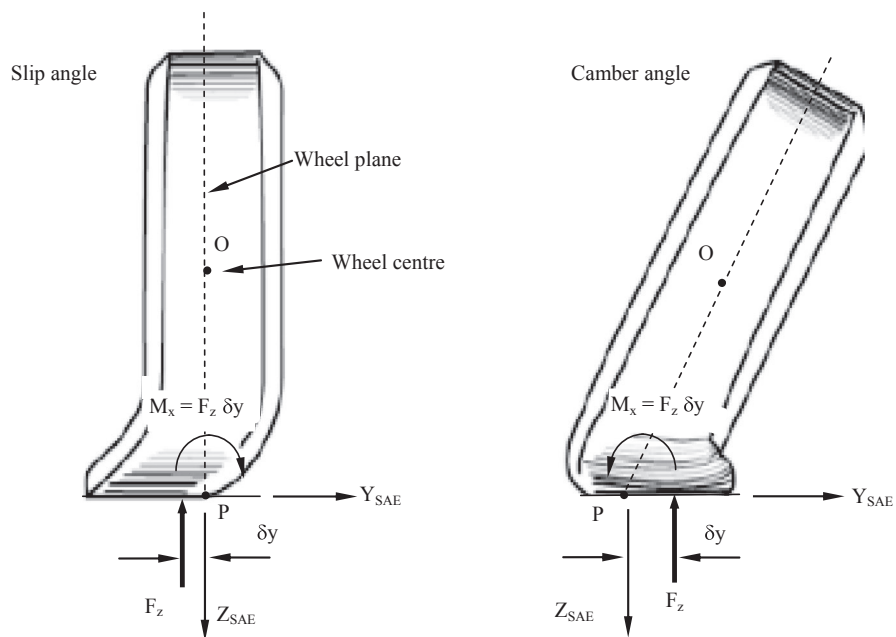
a zero slip angle. The small offsets in lateral force due to conicity and plysteer, discussed in Section 5.2.3, are ignored in Figure 5.35. In the linear range the contributions in lateral force due to slip and camber may be added together but during the transition towards sliding it can be seen that the additive effect of camber will reduce, although the peak value of lateral force is still increased. The maximum increase in peak lateral force will occur at different camber angles for different wheel loads. Thus for a given tyre on a given vehicle it is possible (Milliken and Milliken, 1998) to optimise camber angle for a given combination of slip angle and tyre load.

#### 5.4.10 Overturning moment

Two of the components of moment acting in the tyre contact patch have been discussed. The generation of rolling resistance moment was described while discussing the free rolling tyre in Section 5.4.3. The self-aligning moment arising due to slip or camber angle was discussed in Sections 5.4.7 and 5.4.8. For completeness the final component of moment acting at the tyre contact patch that requires description is the overturning moment that would arise due to deformation in the tyre as shown in Figure 5.36. The forces and moments as computed in the SAE reference frame are formulated to act at P, this being the point where the wheel plane intersects the ground plane at a point longitudinally aligned with the wheel centre.

In Figure 5.36 it can be seen that distortion of the side walls results in a lateral shift of the contact patch that may result from either slip angle or camber angle or a combination of the two. The resulting offset tyre load introduces an additional component of moment  $M_x$ . Attention to the sign convention associated with the tyre reference frame is again needed if the moment is to be included in a tyre model. In Figure 5.36, to assist understanding,  $F_z$  is represented as the tyre load acting on the tyre rather than the negative normal force computed in the  $Z_{SAE}$  direction.



**FIGURE 5.36**

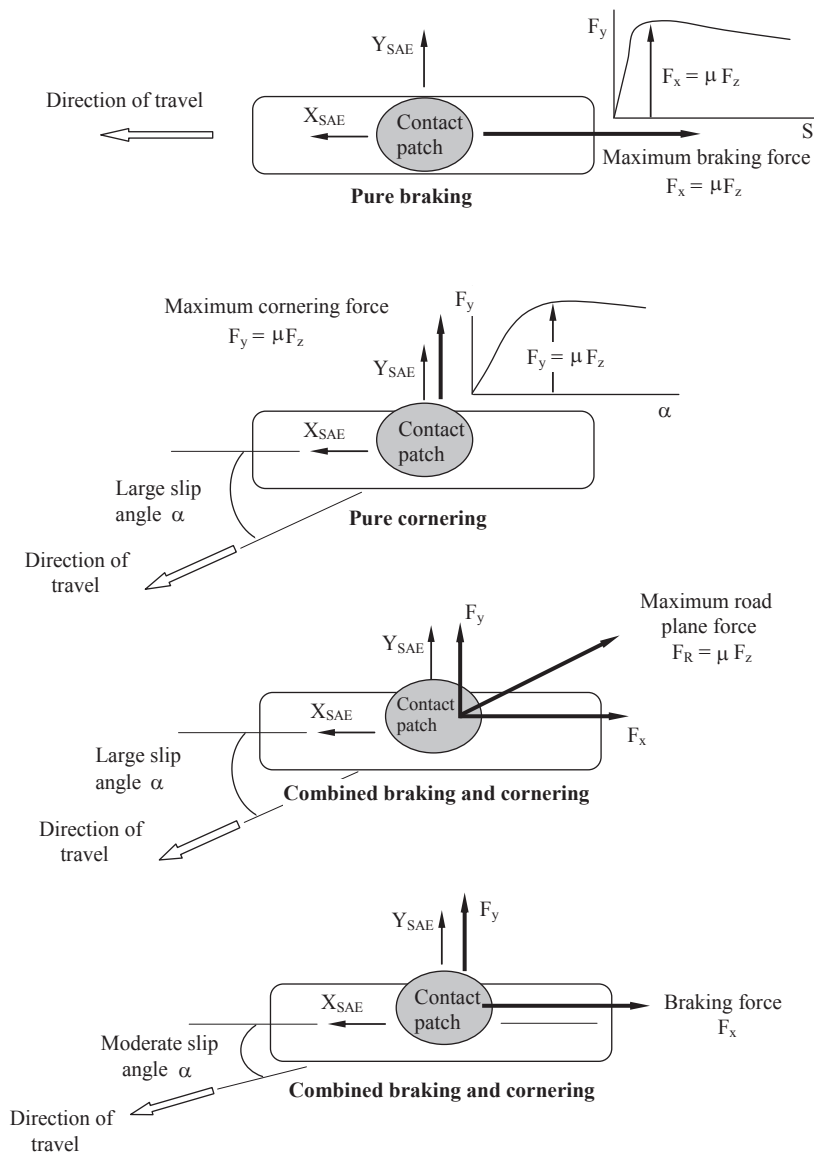
Generation of overturning moment in the tyre contact patch.

A consideration of the overturning moment is generally more important where relatively large displacements in the tyre occur, as with aircraft tyres (Smiley, 1957; Smiley and Horne, 1960). Overturning effects are also of major importance for motorcycle tyres, particularly in terms of matching the behaviour of front and rear tyres. The lateral offset,  $\delta y$ , also applies to the longitudinal forces and is responsible for the ‘stand up under light braking’ that all motorcycles display.

#### 5.4.11 Combined traction and cornering (comprehensive slip)

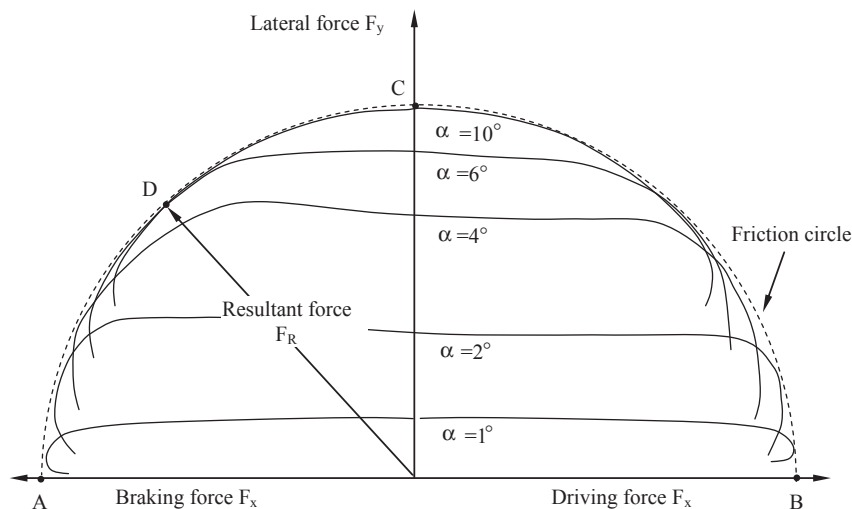
The treatment of longitudinal braking or diving forces and lateral cornering forces has so far dealt with the two components of force in isolation. The simulation of vehicle behaviour involving tyre forces acting in this manner leads to what is termed pure cornering or pure tractive (i.e. driving or braking) behaviour. In reality longitudinal and lateral forces often occur simultaneously during vehicle manoeuvres. A typical situation would be to initiate braking before entering a bend and continue braking into the corner. It is also typical, once the driver feels sufficient confidence, to begin applying throttle, and driving forces, during cornering before exiting the bend. For such situations a tyre model must be able to deal with combined tractive and cornering forces, a situation referred to as comprehensive slip.

The basic law of friction relating frictional force to normal force can be of assistance when considering combinations of longitudinal driving or braking forces with

**FIGURE 5.37**

Pure and combined braking and cornering forces.

lateral cornering forces. The treatment here concentrates on lateral forces due to slip angle with camber angle set to zero. Figure 5.37 initially shows a tyre subject to pure braking or cornering force where in each case the slip in the ground plane is such that the tyre force produced is a peak value, this being  $\mu F_z$ , the peak coefficient of friction multiplied by tyre load.

**FIGURE 5.38**

Plotting lateral force against longitudinal force (friction circle).

For pure cornering the peak force will occur at a relatively large slip angle where in Figure 5.37 some lateral distortion of the contact patch is indicated together with a small amount of pneumatic trail.

For a tyre running at a large slip angle with additional braking force the resultant ground plane force is still equal to  $\mu F_z$  but the resultant force direction opposes the direction of sliding. The longitudinal and lateral forces  $F_x$  and  $F_y$  are now components of the resultant force. Thus it can be seen that the simultaneous action of longitudinal and lateral slip reduces the amount of cornering or braking/driving force that may be obtained independently.

Figure 5.38 shows a plot of lateral force against longitudinal force for a range of slip angles at a given tyre load and with the camber angle set to zero. The x-axis represents the tyre running at zero slip angle running from a maximum braking force value equal to  $\mu F_z$  at point A to a maximum driving force value equal to  $\mu F_z$  at point B, these points being consistent with the slip ratios that would produce peak force for a straight running tyre.

In the absence of braking or driving force the maximum lateral force equal to  $\mu F_z$  that can be achieved occurs at point C. Measurements of  $F_y$  at points along the y-axis intersected by curves at the set slip angles shown would provide a plot of lateral force against slip angle at the given tyre load. As driving or braking force is added the maximum resultant force that can be achieved is defined by points lying on a curve of radius  $\mu F_z$  referred to as the 'Friction Circle' or sometimes the 'Friction Ellipse' as some tyres will have more capability in traction or cornering leading to an elliptical boundary shape.

Figure 5.38 shows, for a typical tyre, the general form of the friction circle diagram for the full range of driving and braking forces. Note that only lateral forces

due to positive slip angle are presented and a similar diagram would exist for measurements taken at negative slip angles. Point D represents an example of a position where the tyre is operating at the friction limit for combined braking and cornering, as shown in Figure 5.37 where it is clear that the amount of braking or cornering force that could be produced independently is reduced and that the magnitude of  $F_R$  is simply

$$F_R = \sqrt{F_x^2 + F_y^2} \quad (5.42)$$

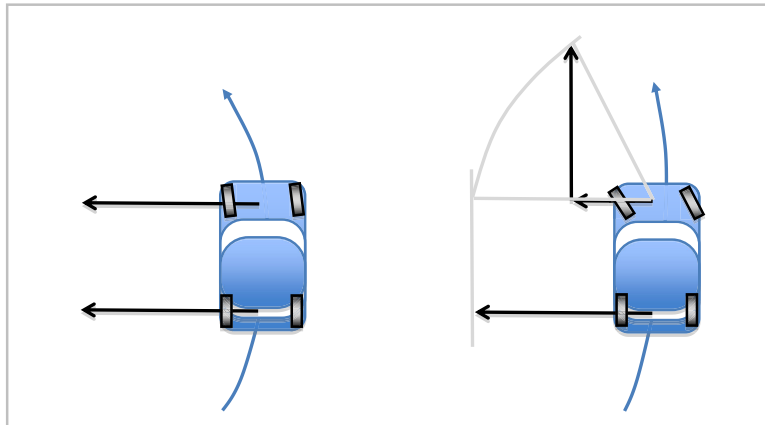
It can also be noted that the curves are not symmetric in that lateral forces initially increase slightly as braking force is applied. As discussed earlier the braking force adds circumferential tension to the tyre material entering the contact patch. This stress stiffening effect can be seen to raise the lateral force slightly, while the reversal of longitudinal force to driving leads to a reduction. As the curves approach the friction limit it can be observed that they turn inwards. For a fixed slip angle the longitudinal slip is increased moving along the curves, for either braking or driving, until the point where both lateral force and longitudinal force reduce, hence causing the curves to bend back.

By plotting aligning moment against longitudinal force, it can be seen that the opposite can occur (Phillips, 2000) and that adding braking force reduces aligning moment and adding driving force raises it. Referring back to Figure 5.37 the bottom diagram shows a tyre running at moderate slip angle producing a lateral force  $F_y$  along a line of action set back from the centre by the pneumatic trail. The contact patch is shown displaced laterally due to the cornering force so that for simultaneous braking the braking force produces a moment that would subtract from the existing aligning moment due to the product of lateral force and pneumatic trail. From the diagram at the bottom of Figure 5.37 it is also clear that the simultaneous application of a driving force would produce a moment that would add to the aligning moment due to the product of lateral force and pneumatic trail. At higher braking forces the effect may cause the aligning moment to go negative.

The friction circle or ellipse is also a way to monitor the performance of a race car driver using instrumented measurements of lateral and longitudinal accelerations, sometimes called the ‘g–g’ diagram. Comparing this diagram with known tyre data it is possible to see how well the driver performs keeping the vehicle close to the friction limits of the tyres.

A similar exercise is possible using an MBS model of a vehicle with a road model to represent the circuit. With the MBS model, extraction of longitudinal and lateral tyre force time histories is possible for a simulated lap of the circuit. This in theory allows investigation into the influence of tyre or vehicle model parameter changes and steering inputs on tyre limit behaviour.

One final point is worth mentioning on the subject of comprehensive slip. It is sometimes imagined that a spinning wheel is very helpful for cornering, particularly at the front axle. This is not the case, as can be seen in Figure 5.39. When a vehicle is

**FIGURE 5.39**

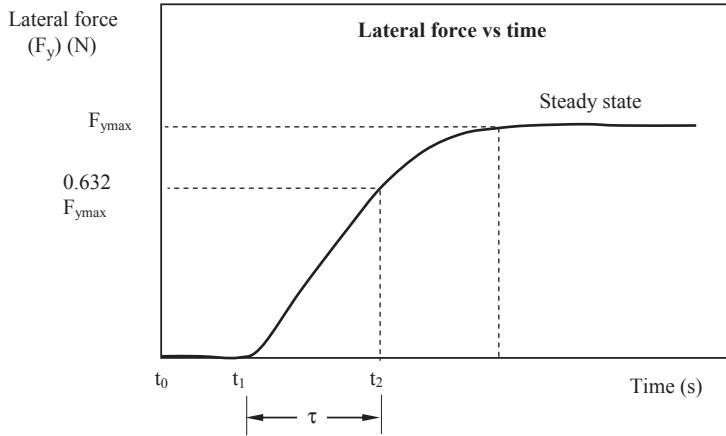
A comparison of cornering with pure lateral slip compared with pure longitudinal slip at maximum steer angle.

travelling in a curved path using only lateral forces from the tyre, the centripetal acceleration can be seen to be given by the lateral forces on the tyre, acting in a direction perpendicular to the direction of travel. The length of the arrow in the diagram can be taken as an indication of the maximum frictional forces. In contrast, when the steering is turned to the maximum angle of around  $30^\circ$  and the tyre force applied as a pure longitudinal force (with respect to the tyre reference frame) then the lateral force available at the front axle is reduced to about the sine of  $30^\circ$  — about half of what it was previously.

The large imbalance between front and rear causes the car to turn out of the turn and take a substantially wider path than previously. Further, the component along the tangent of the vehicle path causes the vehicle to accelerate, further reducing the available centripetal acceleration; for a 50% reduction in centripetal force and a 10% increase in speed, the radius of turn is increased by around 140% — which is to say it is 2.4 times its original value. The origin of the myth of ‘front wheel drive pulling into the corner’ is unknown but like many myths it has a persistence beyond reason.

#### 5.4.12 Relaxation length

For cornering it has been shown that the generation of lateral force due to slip angle is of prime importance. In practice the generation of lateral force is not instantaneous but is subject to a delay generally referred to as ‘tyre lag’. Without modelling some form of time lag a tyre model will compute the force and pass this to the MBS vehicle model so that the lateral force is applied instantly at that integration time step. It has been shown (Loeb et al., 1990) that the tyre must roll a certain distance, ‘relaxation length’, for the tyre to deflect sufficiently to generate the lateral force.

**FIGURE 5.40**

Development of lateral force following step steering input.

A possible method to measure relaxation length is to set the tyre up at a given slip angle in a test rig with the drum or belt not yet moving. On starting the machine the distance rolled before the forces and moments reach steady state can be recorded hence giving a measure of relaxation length. An improved method for vehicle dynamics (Loeb et al., 1990) is to start the test with the tyre running at low speed in the straight ahead configuration and apply a rapid step input of steer angle to the tyre producing a time history plot, similar to that shown in Figure 5.40, indicating the build-up in lateral force.

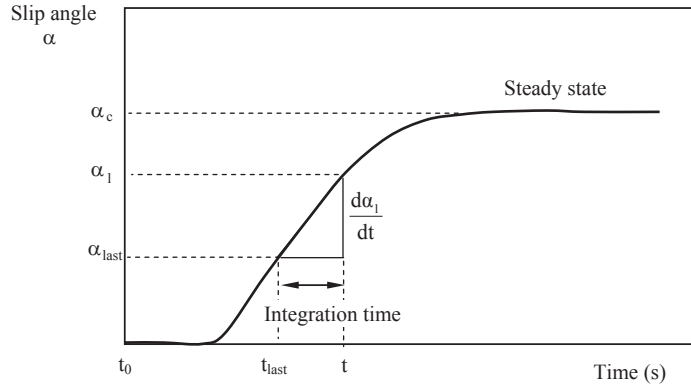
The results obtained (Loeb et al., 1990) for the lateral force response appear exponential, indicating a first-order dynamic system where the time constant  $\tau$ , equal to  $t_2 - t_1$  in Figure 5.40, is the time required to achieve 63.2% of the final steady state response.

Incorporation of a lag effect for tyre lateral force within an MBS program requires an understanding of the mathematical integration process used to solve the equations of motion as discussed in Chapter 3 of this book. A typical approach taken is to compute a theoretical value of slip angle,  $\alpha_l$ , that includes a lag effect and to input this to the appropriate tyre-model algorithm for lateral force due to slip angle. As a starting point the tyre relaxation length  $L_R$  is taken as an input parameter from which, for a forward speed  $V_x$ , the time constant  $\tau$  can be found using

$$\tau = L_R / V_x \quad (5.43)$$

Thus by this definition the relaxation length  $L_R$  is the distance through which the tyre must roll in order to develop 63.2% of the required lateral force. This leads to an initial expression

$$\frac{d\alpha_l}{dt} = \frac{\alpha_c - \alpha_l}{\tau} \quad (5.44)$$

**FIGURE 5.41**

Build up of slip angle in an MBS model to represent tyre lag.

where:

$\alpha_c$  is the computed value of slip angle (instantaneous) at the current time

$\alpha_l$  is the value of slip angle corrected to account for lag.

An estimate of the term  $\frac{d\alpha_l}{dt}$  in Eqn (5.44) can be obtained from Eqn (5.45).

Additional understanding of the terms can be obtained by reference to Figure 5.41 where for clarity the integration time step,  $t - t_{last}$ , is shown with exaggerated magnitude

$$\frac{d\alpha_l}{dt} \approx \frac{\alpha_l - \alpha_{last}}{t - t_{last}} \quad (5.45)$$

where

$\alpha_{last}$  is the value of  $\alpha_l$  computed at the last successful integration time step

$t$  is the current simulation time

$t_{last}$  is the time for the last successful integration time step

An estimate of the term  $\frac{\alpha_c - \alpha_l}{\tau}$  in Eqn (5.44) can be obtained from

$$\frac{\alpha_c - \alpha_l}{\tau} = \frac{\alpha_c - \alpha_{last}}{\tau} \quad (5.46)$$

Combining Eqns (5.45) and (5.46) allows the value of  $\alpha_l$  required to compute lateral force at the current time step to be obtained from

$$\alpha_l = \left( \frac{\alpha_c - \alpha_{last}}{\tau} \right) (t - t_{last}) + \alpha_{last} \quad (5.47)$$

Using data for the baseline vehicle used throughout this text it is possible to carry out a calculation to estimate a time delay, tyre lag, for a vehicle travelling at 100 kph. The load on the tyre  $F_z$  is taken as 4500 N and the radial stiffness of

the tyre  $k_z$  is taken as 160 N/mm. From this it is possible to calculate the static tyre deflection  $\delta_z$

$$\delta_z = \frac{F_z}{k_z} = 28.1 \text{ mm} \quad (5.48)$$

Referring back to Eqn (5.10), the effective rolling radius,  $R_e$ , can be calculated using the tyre deflection  $\delta_z$  from Eqn (5.48) and an unloaded tyre radius,  $R_u$ , of 318.5 mm from

$$R_e = R_u - \frac{\delta_z}{3} = 309.1 \text{ mm} \quad (5.49)$$

Typically a tyre would roll through between 0.5 and 1 revolution (Gillespie, 1992) in order to develop the lateral force following a change in slip angle. If we assume that the tyre must complete 0.5 revolutions then for a speed of 100 kph the tyre lag on this vehicle is 0.035 s.

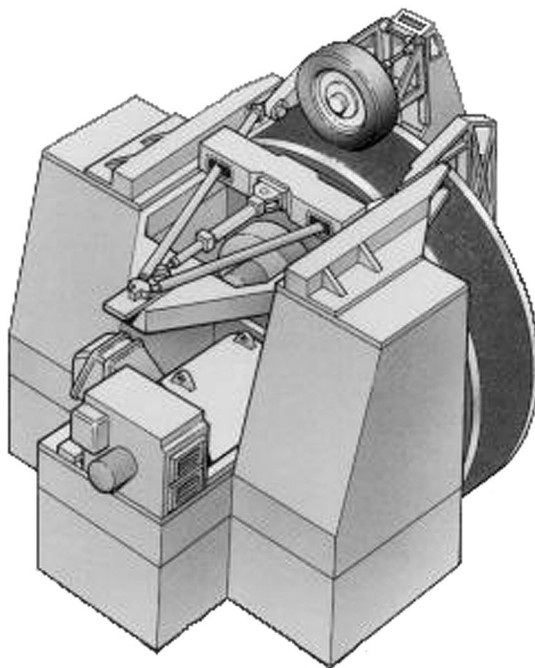
## 5.5 Experimental testing

In order to obtain the data needed for the tyre modelling required for simulation, a series of tests may be carried out using tyre test facilities, typical examples being the machines that are illustrated in Figures 5.42 and 5.43. The following is typical of tests performed (Blundell, 2000a) to obtain the tyre data that supports the baseline vehicle used throughout this text. The measurements of forces and moments were taken using the SAE coordinate system for the following configurations:

1. Varying the vertical load in the tyre 200, 400, 600, 800 kg.
2. For each increment of vertical load the camber angle is varied from  $-10$  to  $10^\circ$  with measurements taken at  $2^\circ$  intervals. During this test the slip angle is fixed at  $0^\circ$ .
3. For each increment of vertical load the slip angle is varied from  $-10$  to  $10^\circ$  with measurements taken at  $2^\circ$  intervals. During this test the camber angle is fixed at  $0^\circ$ .
4. For each increment of vertical load the slip and camber angle are fixed at zero degrees and the tyre is gradually braked from the free rolling state to a fully locked skidding tyre. Measurements were taken at increments in slip ratio of 0.1.

The test programme outlined here can be considered a starting point in the process of obtaining tyre data to support a simulation exercise. In practice obtaining all the data required to describe the full range of tyre behaviour discussed in the preceding sections will be extremely time consuming and expensive. The test programme described here does not, for example, consider effects such as varying the speed of the test machine, changes in tyre pressure or wear, changes in road texture and surface contamination by water or ice. The testing is also steady state and does not consider the transient state during transition from one orientation to another.

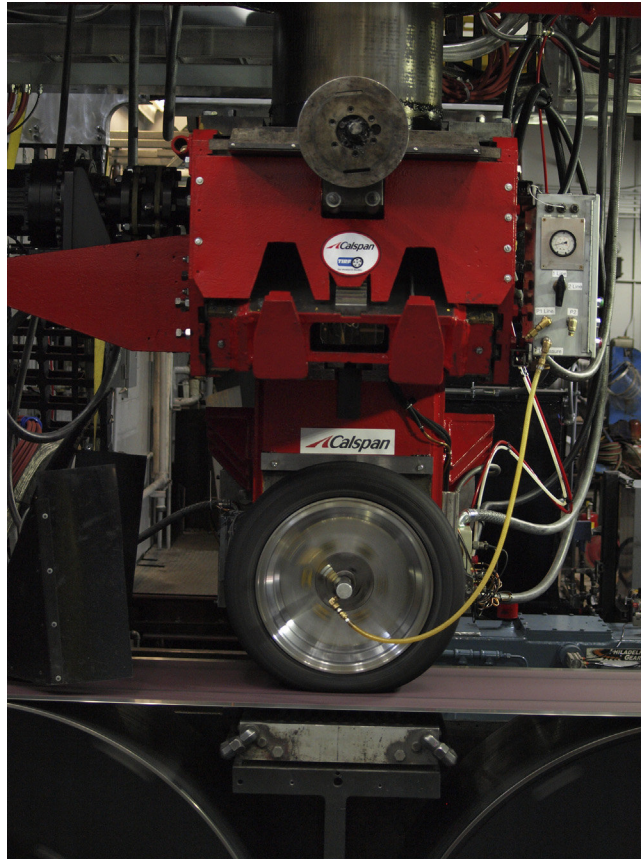


**FIGURE 5.42**

High Speed Dynamics Machine for tyre testing formerly at Dunlop Tyres Ltd.

Most importantly the tests do not consider the complete range of combinations that can occur in the tyre. The longitudinal force testing described is limited by only considering the generation of braking force. To obtain a complete map of tyre behaviour it would also, for example, be necessary to test not only for variations in slip angle at zero degrees of camber angle but to repeat the slip angle variations at selected camber angles. For comprehensive slip behaviour it would be necessary at each slip angle to brake or drive the tyre from a free rolling state to one that approaches the friction limit, hence deriving the 'friction circle' for the tyre.

Extending a tyre test programme in this way may be necessary to generate a full set of parameters for a sophisticated tyre model but will significantly add to the cost of testing. Obtaining data requires the tyre to be set up at each load, angle or slip ratio and running in steady state conditions before the required forces and moments can be measured. By way of example the basic test programme described here required measurements to be taken for the tyre in 132 configurations. Extending this, using the same pattern of increments and adding driving force, to consider combinations of slip angle with camber or slip ratio would extend the testing to 1452 configurations. In practice this could be reduced by judicious selection of test configurations but it should be noted the tests would still be for a tyre at constant pressure and constant speed on a given test surface. Examples of test results for a wider range of tyres and settings can be obtained by general reference to the tyre-specific

**FIGURE 5.43**

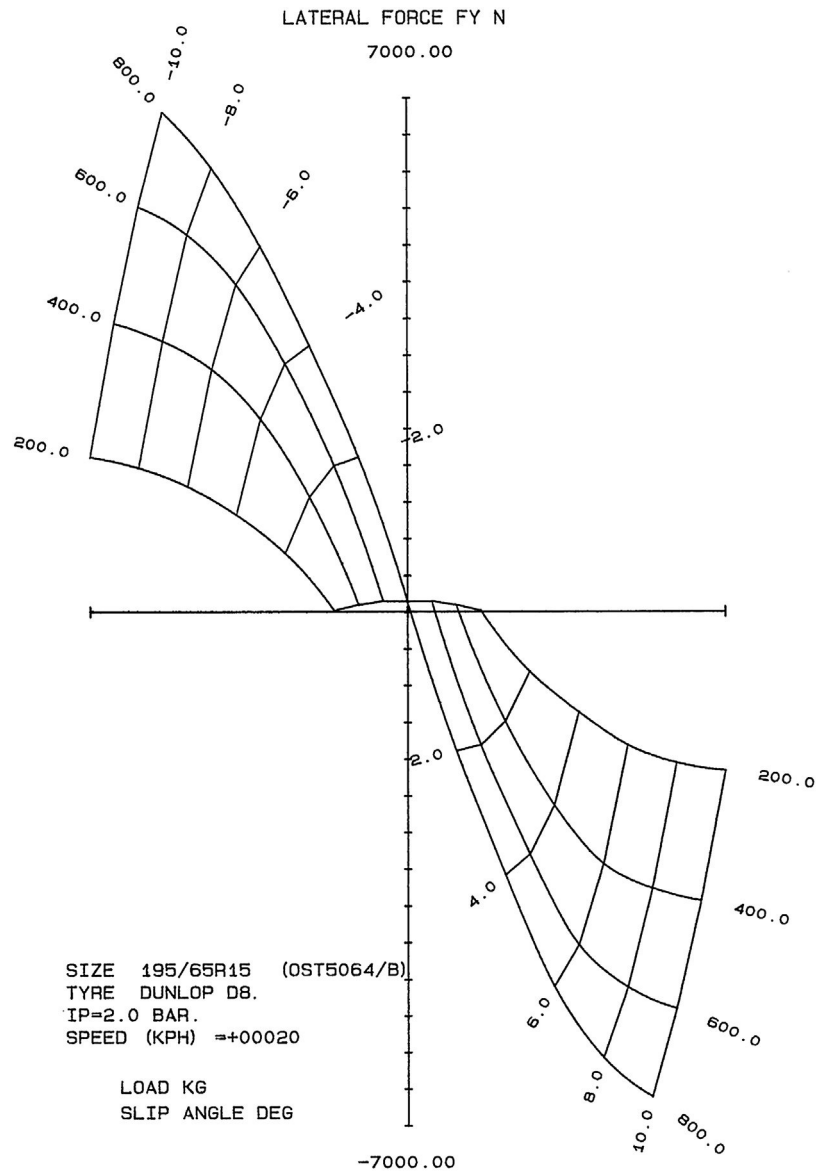
Flat Bed Tyre Test machine.

*(Courtesy of Calspan.)*

publications quoted in this chapter and in particular to the textbook by Pacejka (2012).

For the tyre tests described here the following is typical of the series of plots that would be produced in order to assess the force and moment characteristics. The results are presented in the following [Figures 5.44–5.53](#) where a carpet plot format is used for the lateral force and aligning moment results:

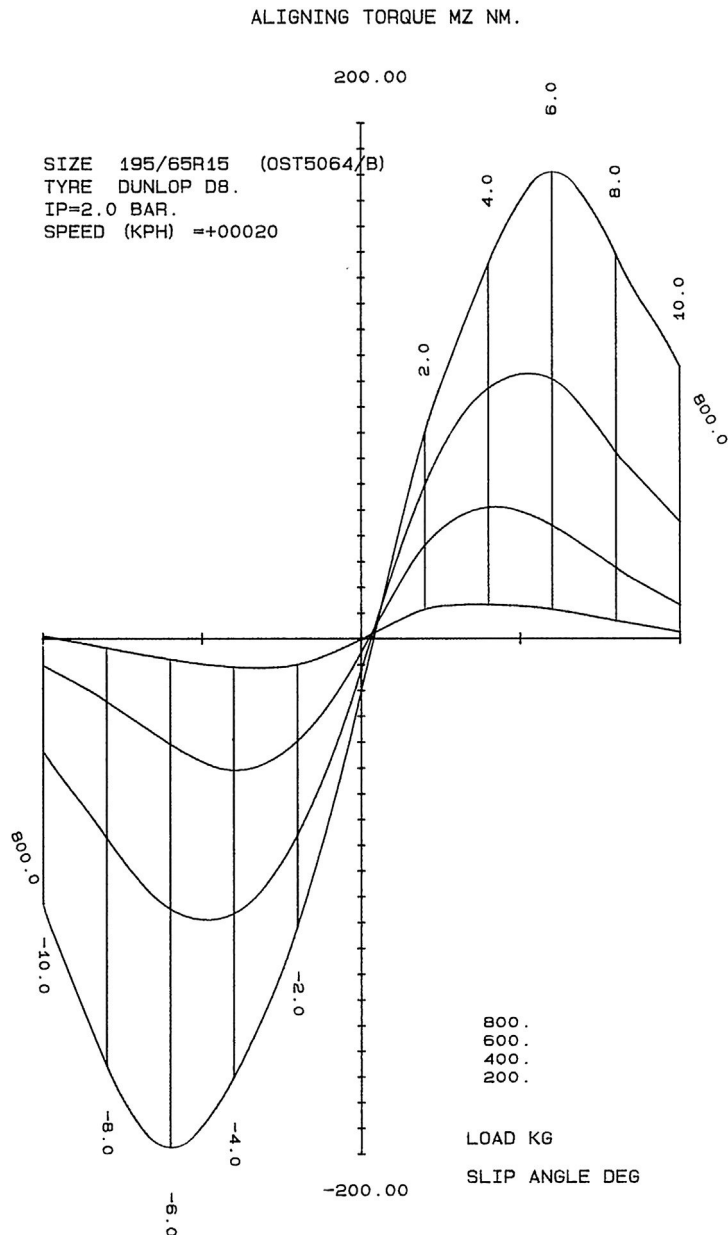
1. Lateral force  $F_y$  with slip angle  $\alpha$
2. Aligning moment  $M_z$  with slip angle  $\alpha$
3. Lateral force  $F_y$  with aligning moment  $M_z$  (Gough Plot)
4. Cornering stiffness with load
5. Aligning stiffness with load
6. Lateral force  $F_y$  with camber angle  $\gamma$


**FIGURE 5.44**

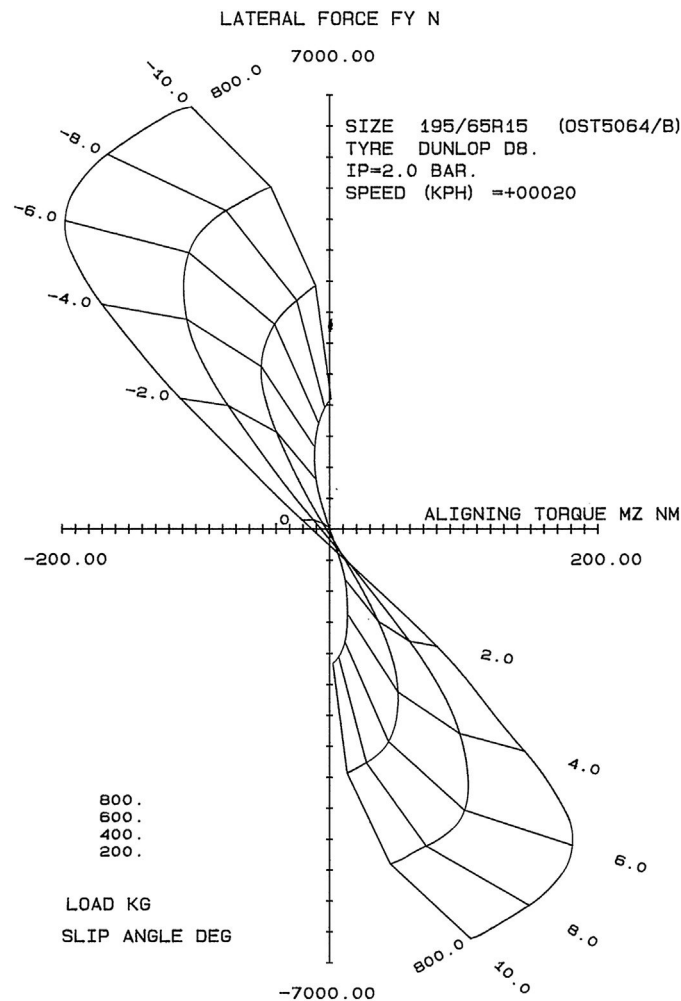
Lateral force  $F_y$  with slip angle  $\alpha$ .

(Courtesy of Dunlop Tyres Ltd.)

7. Aligning moment  $M_z$  with camber angle  $\gamma$
8. Camber stiffness with load
9. Aligning camber stiffness with load
10. Braking force with slip ratio

**FIGURE 5.45**Aligning moment  $M_z$  with slip angle  $\alpha$ .

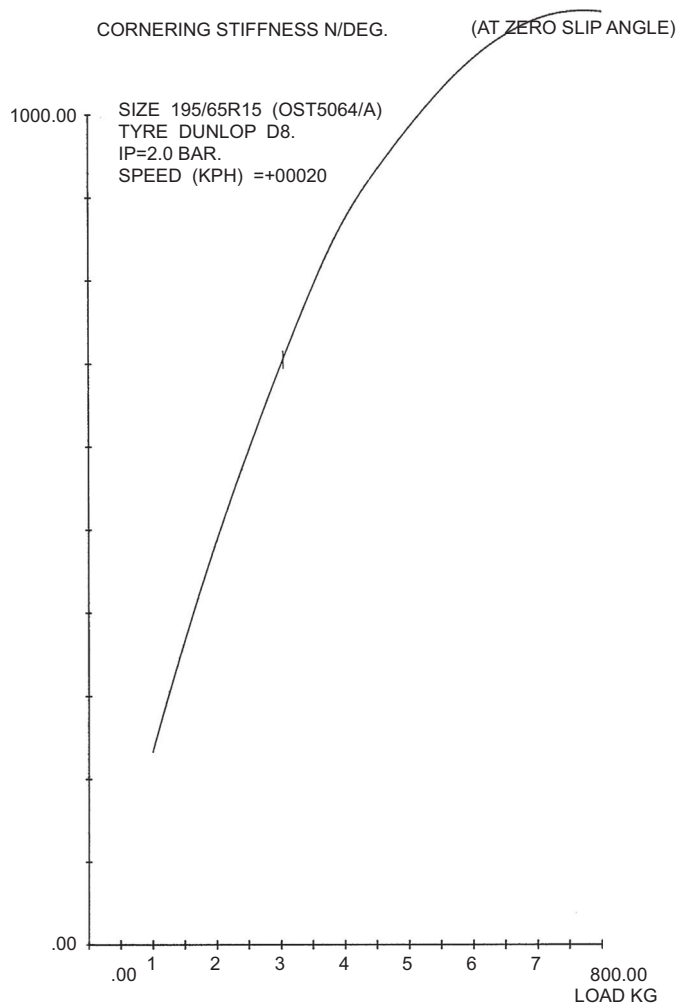
(Courtesy of Dunlop Tyres Ltd.)

**FIGURE 5.46**

Lateral force  $F_y$  with aligning moment  $M_z$  (Gough Plot).

(Courtesy of Dunlop Tyres Ltd.)

Before continuing with the treatment of tyre modelling, readers should note the findings (van Oosten et al., 1999) of the TYDEX Workgroup. In this study a comparison of tyre cornering stiffness for a tyre tested on a range of comparable tyre test machines gave differences between minimum and maximum measured values of up to 46%. Given the complexities of the tyre models that are described in the following section the starting point should be a set of measured data that can be used with confidence to form the basis of a tyre model.

**FIGURE 5.47**

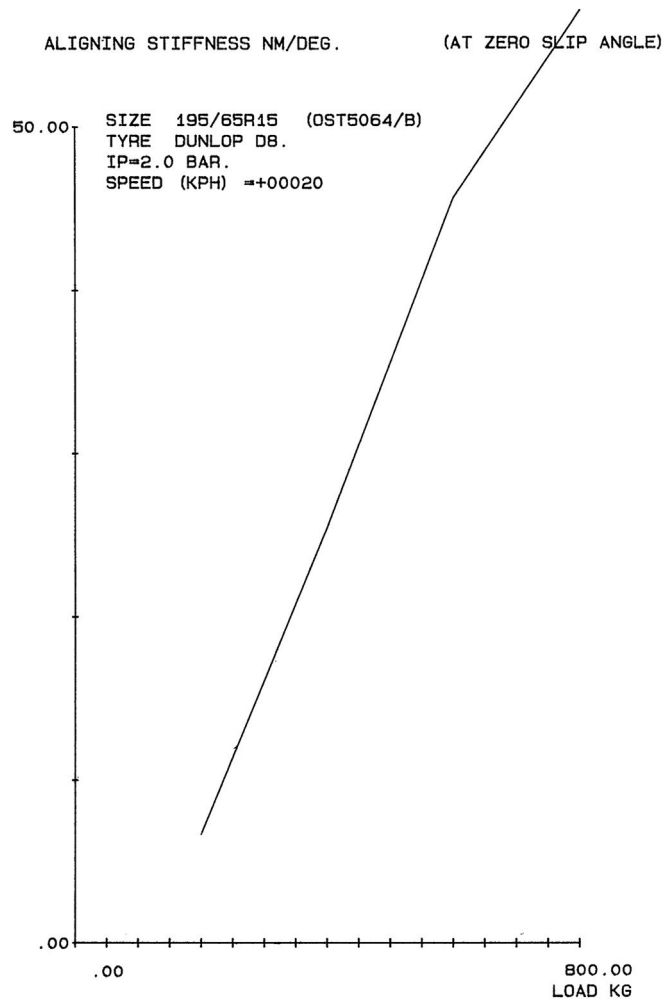
Cornering stiffness with load.

(Courtesy of Dunlop Tyres Ltd.)

## 5.6 Tyre Modelling

### 5.6.1 Overview

The modelling of the forces and moments at the tyre contact patch has been the subject of extensive research in recent years. A review of some of the most common tyre models was provided by Pacejka and Sharp (1991), where the authors state that it is necessary to compromise between the accuracy and complexity of the model. The

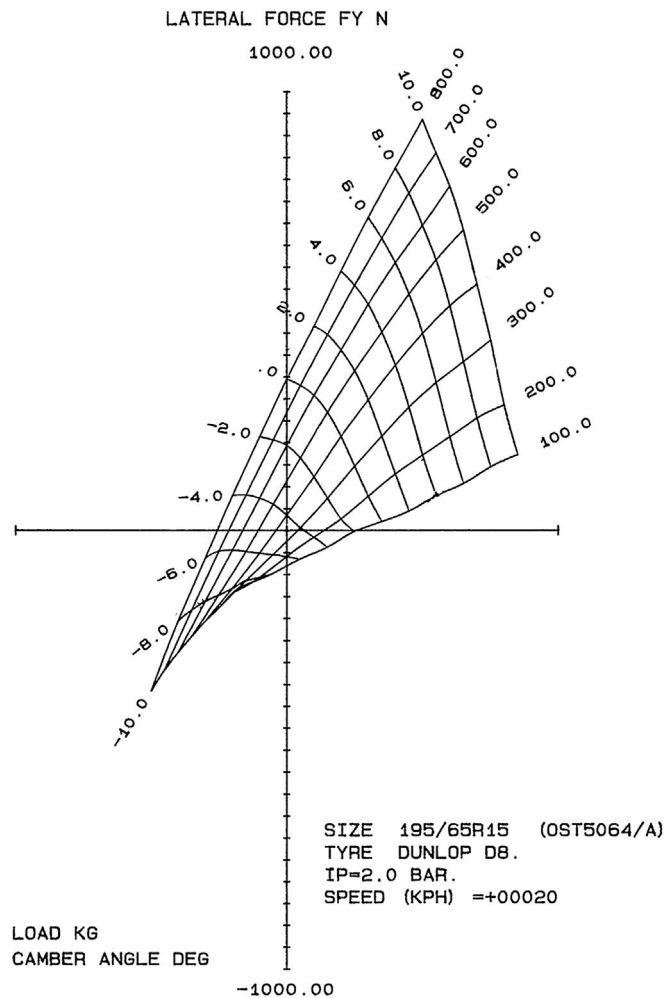
**FIGURE 5.48**

Aligning stiffness with load.

*(Courtesy of Dunlop Tyres Ltd.)*

authors also state that the need for accuracy must be considered with reference to various factors including the manufacturing tolerances in tyre production and the effect of wear on the properties of the tyre. This would appear to be a valid point not only from the consideration of computer modelling and simulation but also in terms of track testing where new tyres are used to establish levels of vehicle performance. A more realistic measurement of how a vehicle is going to perform in service may be to consider testing with different levels of wear or incorrect pressure settings.

One of the methods discussed by Pacejka and Sharp (1991) focuses on a multi-spoke model developed by Sharp where the tyre is considered to be a series of radial

**FIGURE 5.49**

Lateral force  $F_y$  with camber angle  $\gamma$ .

(Courtesy of Dunlop Tyres Ltd.)

spokes fixed in a single plane and attached to the wheel hub. The spokes can deflect in the radial direction and bend both circumferentially and laterally. Sharp provides more details on the radial-spoke model approach in (Sharp & El-Nashar, 1986), (Sharp, 1990) and (Sharp, 1993). The other method of tyre modelling reviewed is based on the 'Magic Formula' that will be discussed in more detail later in this section. Another review of tyre models is given by Pacejka (1995), where the influence of the tyre is discussed with regard to 'active' control of vehicle motion. The radial-spoke and 'Magic Formula' models are again discussed. More recently, the



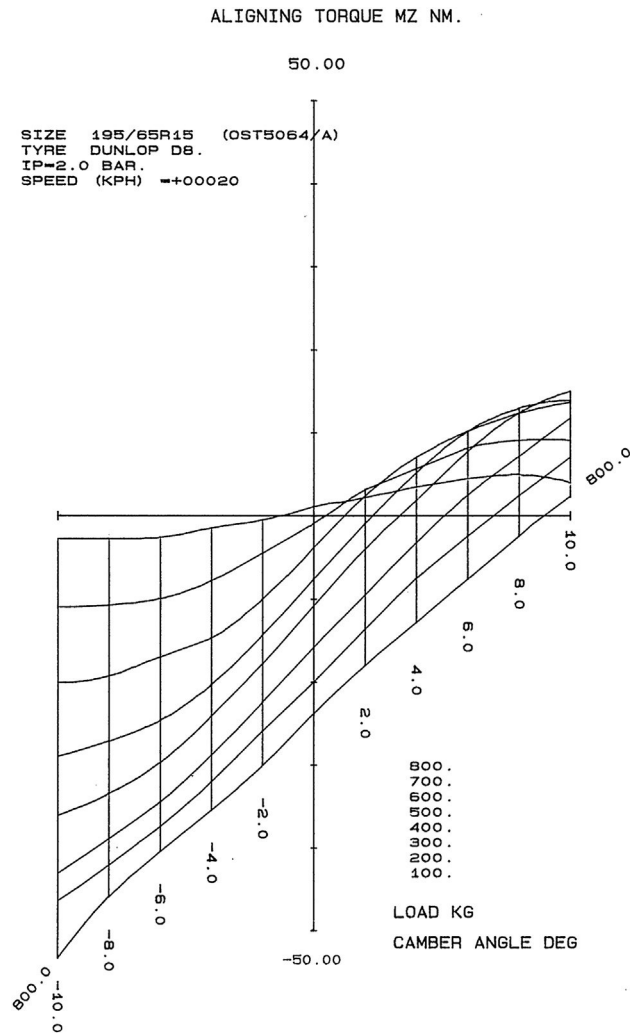


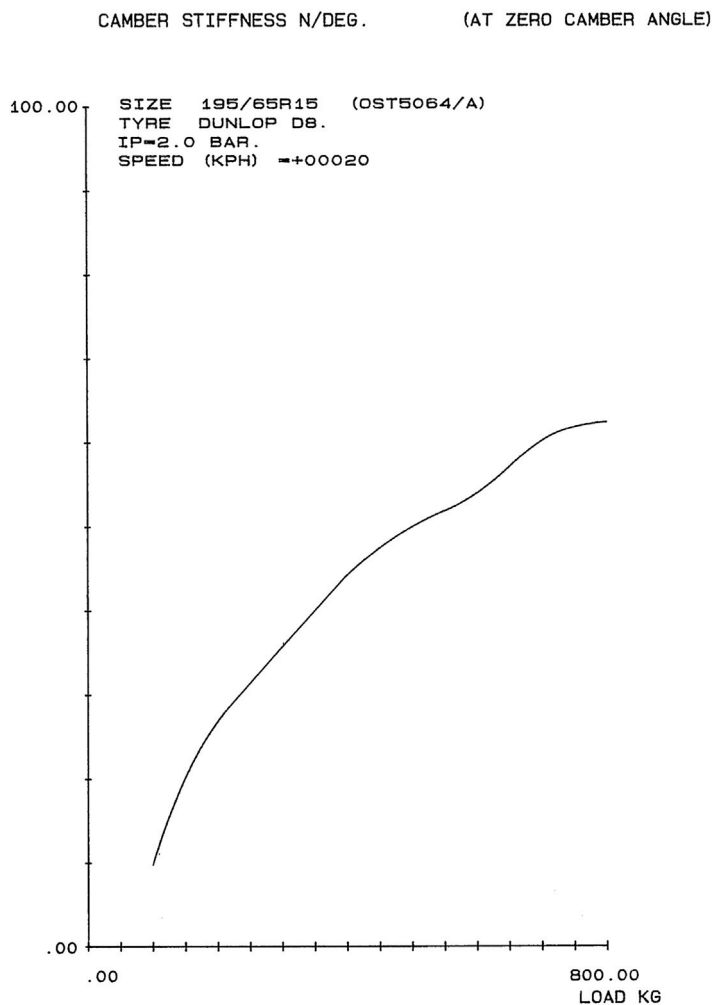
FIGURE 5.50

Aligning moment  $M_z$  with camber angle  $\gamma$ .

(Courtesy of Dunlop Tyres Ltd.)

definitive text on tyre modelling is in its third iteration (Pacejka, 2012), covering many types of model.

Before considering tyre models in more detail it should be stated that tyre models are generally developed according to the type of application the vehicle simulation will address. For ride and vibration studies the tyre model is often required to transmit the effects from a road surface where the inputs are small but of high frequency. In the simplest form the tyre may be represented as a simple compression only spring and

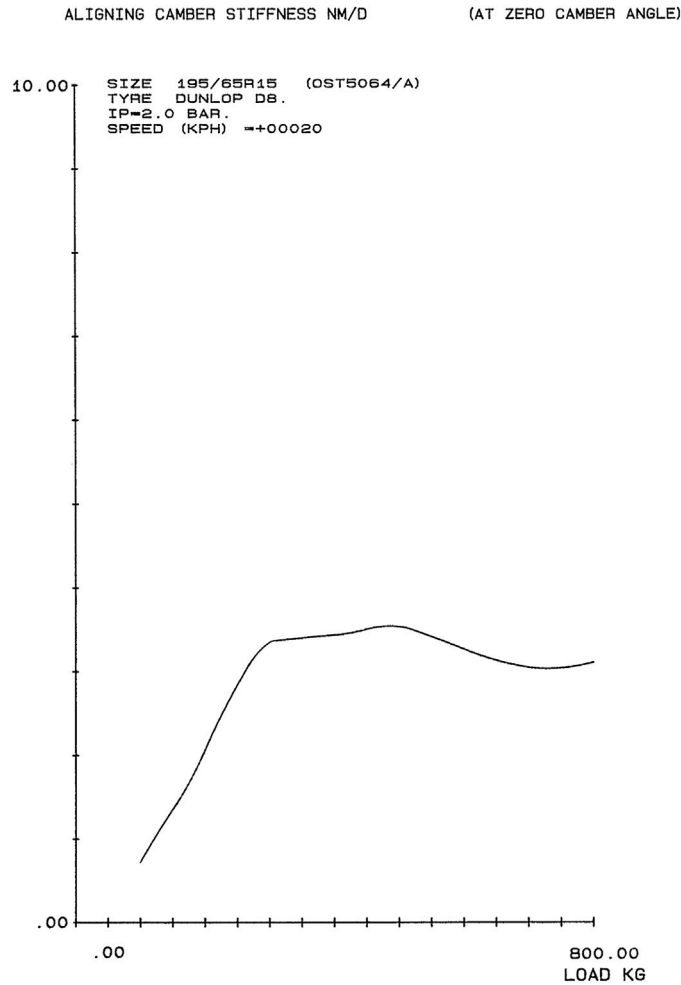
**FIGURE 5.51**

Camber stiffness with load.

*(Courtesy of Dunlop Tyres Ltd.)*

damper acting between the wheel centre and the surface of the road. The simulation may in fact recreate the physical testing using a four-poster test rig with varying vertical inputs at each wheel. A concept of the tyre model for this type of simulation is provided in [Figure 5.54](#) where for clarity only the right side of the vehicle is shown.

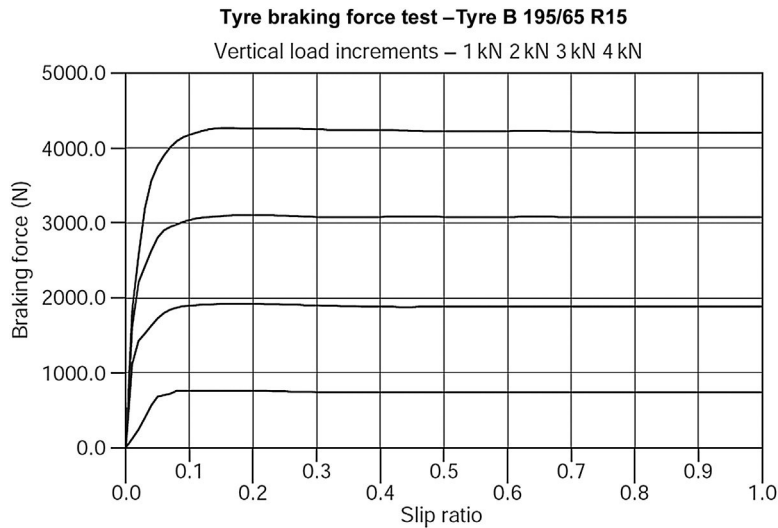
In suspension loading or durability studies the tyre model must accurately represent the contact forces generated when the tyre strikes obstacles such as potholes and road bumps. In these applications the deformation of the tyre as it contacts the obstacle is of importance and is a factor in developing the model. These sort of tyre models are often developed for agricultural or construction type vehicles

**FIGURE 5.52**

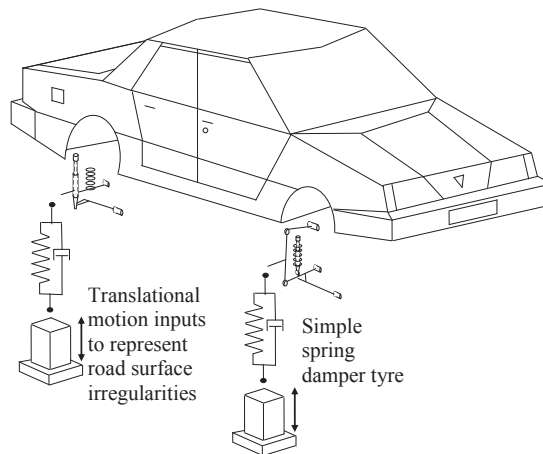
Aligning camber stiffness with load.

*(Courtesy of Dunlop Tyres Ltd.)*

used in an off road environment and dependent on the tyre to a larger extent in isolating the driver from the ground surface inputs. An example of this sort of tyre model is described by Davis (1974) where a radial spring model was developed to envelop irregular features of a rigid terrain. The tyre is considered to be a set of equally spaced radial springs that when in contact with the ground will provide a deformed profile of the tyre as it envelops the obstacle. The deformed shape is used to redefine the rigid terrain with an 'equivalent ground plane'. The concept of an equivalent ground plane model was used in the early ADAMS/Tire™ model for the durability application but has the main limitation that the model is not

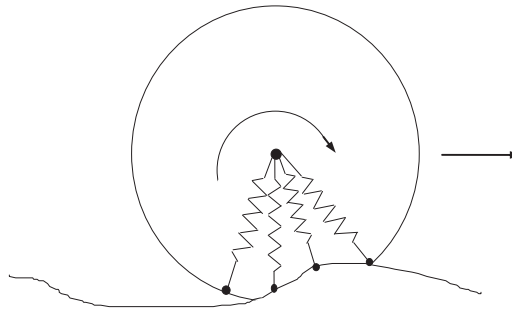
**FIGURE 5.53**

Braking force with slip ratio.

**FIGURE 5.54**

A simple tyre model for ride and vibration studies.

suitable for very small obstacles that the tyre might completely envelop. This is clarified by Davis (1974) where it is stated that the wave length of surface variations in the path of the tyre should be at least three times the length of the tyre to ground contact patch. The other and most basic limitation of this type of model is that the simulation is restricted to straight-line motion and would only consider the vertical and longitudinal forces being generated by the terrain profile. An example of a radial spring tyre model is shown in Figure 5.55.

**FIGURE 5.55**

A radial spring terrain enveloping tyre model.

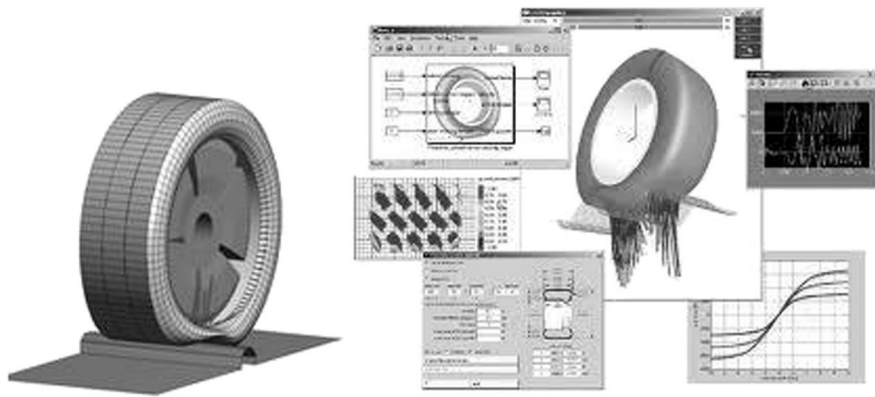
The work carried out by Kisielewicz and Ando (1992) describes how two different programs have been interfaced to carry out a vehicle simulation where the interaction between the tyre and the road surface has been calculated using an advanced nonlinear finite element analysis program.

More recently, advances in computational budget have allowed the development of a number of tyre models that begin to approach a causal description of the tyre structure and its interaction with terrain. A number of high quality terrain-interaction models are now available such as FTire and CDTire. These models are characterised by an elegant formulation that is somewhere between a coarse finite element model with spatially distributed discrete stiffness and a modal model with scalar degrees of freedom coupled to physical shapes (Figure 5.56).

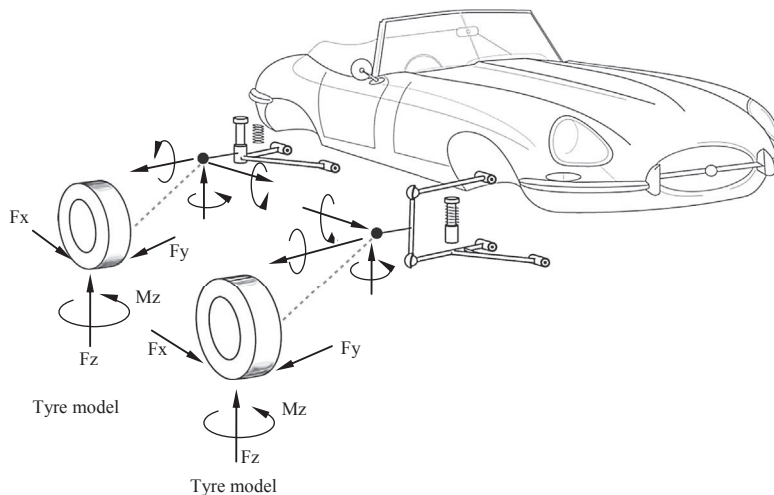
For vehicle handling studies we are generally concerned with the manoeuvring of the vehicle on a flat road surface and elaborate contact formulations such as these are not necessary and a single point of contact is used to calculate aggregate slip states in the contact patch. This is the so-called ‘point follower’ formulation. Whichever type of contact formulation is used, the function of the tyre model is to establish the forces and moments occurring at the tyre to road contact patch and resolve these to the wheel centre and hence into the vehicle as indicated in Figure 5.57.

For each tyre the tyre model will calculate the three orthogonal forces and the three orthogonal moments that result from the conditions arising at the tyre to road surface contact patch. These forces and moments are applied at each wheel centre and control the motion of the vehicle. In terms of modelling the vehicle is actually ‘floating’ along under the action of these forces at each corner. For a handling model the forces and moment at the tyre to road contact patch, which are usually calculated by the tyre model, are:

1.  $F_x$  — longitudinal tractive or braking force
2.  $F_y$  — lateral cornering force
3.  $F_z$  — vertical normal force
4.  $M_z$  — aligning moment

**FIGURE 5.56**

CD Tire (left) from the Fraunhofer Institute and FTire (right) from Cosin are both examples of semi-causal brush models that have become widely used in the last decade.

**FIGURE 5.57**

Interaction between vehicle model and tyre model.

The other two moments that occur at the patch,  $M_x$  the overturning moment and  $M_y$  the rolling resistance moment, are generally not significant for a handling tyre model for passenger cars. The calculation of forces and moments at the contact patch is the essence of a tyre model and will be discussed in more detail later.

As a simulation progresses and the equations for the vehicle and tyre are solved at each solution point in time there is a flow of information between the vehicle model and the tyre model. The tyre model must continually receive information

about the position, orientation and velocity at each wheel centre and also the topography of the road surface in order to calculate the forces and moment at the contact patch. The road surface is often flat but may well have changing frictional characteristics to represent varying surface textures or changes between dry, wet or ice conditions. Inclined or cambered road surfaces can also be modelled if needed. The information from the wheel centre such as the height, camber angle, slip angle, spin velocity and so on are the inputs to the tyre model at each point in time and will dictate the calculation of the new set of forces at the contact patch.

These newly computed tyre conditions are then fed back to the vehicle model at each wheel centre. This will produce a change in the vehicle position at the next solution point in time. The conditions at each wheel centre will change and will be relayed back to the tyre model again. A new set of tyre forces and moments will then be calculated and so the process will continue.

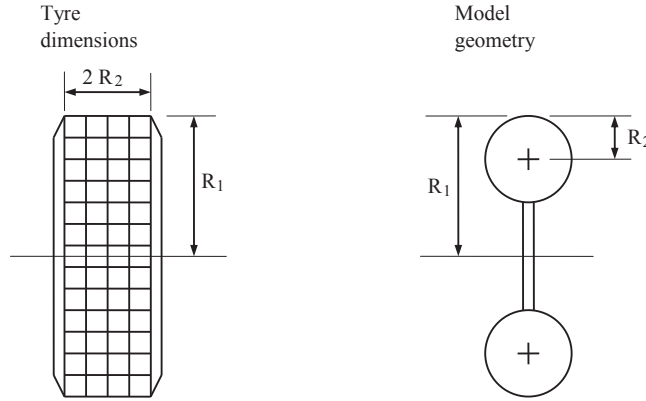
The treatment of tyre models that follows in this section is based on methods that have been developed for vehicle handling simulations. A later section will deal with tyre models for durability analysis. As stated earlier, the computation of vertical force is straightforward based on the equations in [Section 5.4.2](#). For the handling models described here the ‘model’ focuses on the calculation of longitudinal driving or braking forces and lateral forces. The formulation of rolling resistance and aligning moments is also covered. Before discussing individual tyre models it is necessary to describe the calculations carried out in the main MBS program to provide the tyre model with the necessary position, orientation and velocities of the road wheel.

## 5.6.2 Calculation of tyre geometry and velocities

A tyre model, for handling or durability analysis, requires input regarding the position and orientation of the wheel relative to the road together with velocities used to determine the slip characteristics. The implementation of these computations as a tyre model with an MBS program is best described using the full three-dimensional vector approach outlined in Chapter 2. The following description is based on the methods used in MSC.ADAMS but is applicable to any vehicle simulation model requiring tyre force and moment input. As a starting point the tyre can be modelled using the input radii  $R_1$  and  $R_2$  as shown in [Figure 5.58](#).

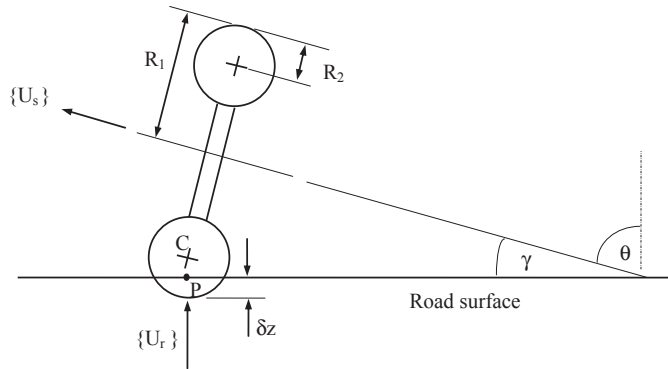
Using the tyre model geometry based on a torus it is possible to determine the geometric outputs that are used in the subsequent force and moment calculations. Consider first the view in [Figure 5.59](#) looking along the wheel plane at the tyre inclined on a flat road surface.

The vector  $\{U_s\}$  is a unit vector acting along the spin axis of the tyre. The vector  $\{U_r\}$  is a unit vector that is normal to the road surface and passes through the centre of the tyre carcass at C. The contact point P between the tyre and the surface of the road is determined as the point at which the vector  $\{U_r\}$  intersects the road surface. For the purposes of this document it is assumed the road is flat and only one point of contact occurs.



**FIGURE 5.58**

Tyre model geometry.



**FIGURE 5.59**

Inclined tyre geometry.

The camber angle  $\gamma$  between the wheel plane and the surface of the road is calculated using

$$\gamma = \pi/2 - \theta \quad (5.50)$$

where

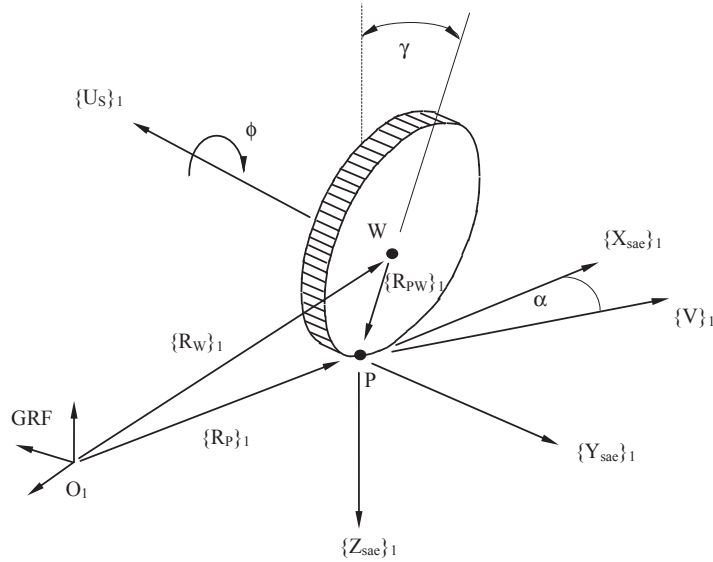
$$\theta = \cos^{-1} (\{U_r\} \bullet \{U_s\}) \quad (5.51)$$

The vertical penetration of the tyre  $\delta z$  at point P is given by

$$\delta z = R_2 - |CP| \quad (5.52)$$

In order to calculate the tyre forces and moment it is also necessary to determine the velocities occurring in the tyre. In [Figure 5.60](#) the SAE coordinate system is located at the contact point P. This is established by the three unit vectors





**FIGURE 5.60**

Tyre geometry and kinematics.

$\{X_{sae}\}_1$ ,  $\{Y_{sae}\}_1$  and  $\{Z_{sae}\}_1$ . Note that referring back to Chapter 2 the subscript 1 indicates that the components of a vector are resolved parallel to reference frame 1, which in this case is the Ground Reference Frame (GRF).

Using the triangle law of vector addition it is possible to locate the contact point P relative to the fixed GRF  $O_1$

$$\{R_P\}_1 = \{R_W\}_1 + \{R_{PW}\}_1 \quad (5.53)$$

At this stage it should be said that the vector  $\{R_{PW}\}_1$  represents the loaded radius and not the effective rolling radius of the tyre. Should this be significant for the work in hand, such as the modelling and simulation of ABS (Ozdalyan and Blundell, 1998) then further modification of the tyre model may be necessary.

If the angular velocity vector of the wheel is denoted by  $\{\omega\}_1$  then the velocity  $\{V_P\}_1$  of point P is given by

$$\{V_P\}_1 = \{V_W\}_1 + \{V_{PW}\}_1 \quad (5.54)$$

where

$$\{V_{PW}\}_1 = \{\omega\}_1 \times \{R_{PW}\}_1 \quad (5.55)$$

It is now possible to determine the components of  $\{V_P\}_1$  that act parallel to the SAE coordinate system superimposed at P. The longitudinal slip velocity  $V_{XC}$  of point P is given by

$$V_{XC} = \{V_P\}_1 \cdot \{X_{sae}\}_1 \quad (5.56)$$

The lateral slip velocity  $V_Y$  of point P is given by

$$V_Y = \{V_P\}_1 \bullet \{Y_{sae}\}_1 \quad (5.57)$$

The vertical velocity  $V_Z$  at point P, which will be used to calculate the damping force in the tyre, is given by

$$V_Z = \{V_P\}_1 \bullet \{Z_{sae}\}_1 \quad (5.58)$$

Considering the angular velocity vector of the wheel  $\{\omega\}_1$  in more detail we can represent the vector as follows. The wheel develops a slip angle  $\alpha$  that is measured about  $\{Z_{sae}\}_1$ , a camber angle  $\gamma$  that is measured about  $\{X_{sae}\}_1$  and a spin angle  $\phi$  that is measured about  $\{U_S\}_1$ . The total angular velocity vector of the wheel is the summation of all three motions and is given by

$$\{\omega\}_1 = \dot{\alpha} \{Z_{sae}\}_1 + \dot{\gamma} \{X_{sae}\}_1 + \dot{\phi} \{U_S\}_1 \quad (5.59)$$

It is possible to consider an angular velocity vector  $\{\omega_S\}_1$  that only considers the spinning motion of the wheel and does not contain the contributions due to  $\alpha$  and  $\gamma$ . This vector for angular velocity that only considers spin is given by

$$\{\omega_S\}_1 = \dot{\phi} \{U_S\}_1 \quad (5.60)$$

Using this it is possible to determine  $V_C$  the ‘circumferential velocity’ component of point P relative to the centre of the wheel W and measured parallel to  $\{X_{sae}\}_1$

$$V_C = (\{\omega_S\}_1 \times \{R_{PW}\}_1) \bullet \{X_{sae}\}_1 \quad (5.61)$$

At this stage it is worth considering the usual two-dimensional representation of longitudinal slip for straight-line braking. Referring back to [Section 5.4.4](#) a definition of slip ratio,  $S$ , during braking was given by

$$S = \frac{V - \omega_B R_e}{V} \quad (5.62)$$

Based on the velocities, which have been determined for the three-dimensional case it is now possible to calculate a longitudinal slip ratio,  $S$ , during braking which is given by

$$S = \frac{V_{XC}}{V_X} \quad (5.63)$$

For this formulation of slip ratio  $V_{XC}$  can be considered to be the contact patch velocity relative to the road surface. This is equivalent to  $V - \omega_B R_e$  in the two-dimensional model, albeit using the loaded radius in the vector based formulation. The circumferential velocity  $V_C$  of P measured relative to the wheel centre can be subtracted from  $V_{XC}$  to give  $V_X$  the actual longitudinal velocity of P ignoring the

rotation effect. This can be thought of as the velocity of an imaginary point in the ground that follows the contact patch and is also equivalent to  $V$  in the two-dimensional model. During traction, the longitudinal slip ratio is formulated using

$$S = \frac{V_{xc}}{|V_c|} \quad (5.64)$$

The lateral slip of the contact patch relative to the road is defined by the slip angle  $\alpha$ , where

$$\alpha = \arctan \{V_Y/V_X\} \quad (5.65)$$

During braking a lateral slip ratio  $S_\alpha$  is computed as

$$S_\alpha = |\tan \alpha| = |V_Y/V_X| \quad (5.66)$$

During braking  $S_\alpha$  will have a value of zero when  $V_Y$  is zero and can have a maximum value of 1.0, which equates to a slip angle  $\alpha$  of  $45^\circ$ . Slip angles in excess of this are not usual for vehicle handling but may occur in other applications where a tyre model is used, for example to simulate aircraft taxiing on a runway.

During traction the formulation becomes:

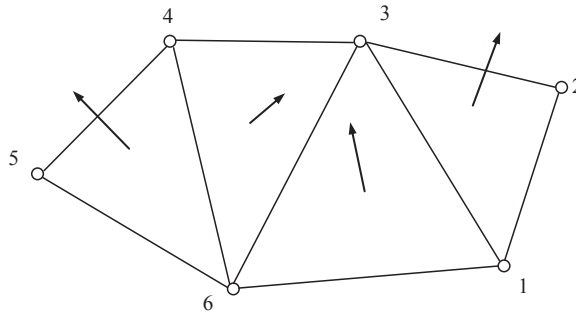
$$S_\alpha = (1 - S) |\tan \alpha| = |V_Y/V_C| \quad (5.67)$$

### 5.6.3 Road surface/terrain definition

It is easy to forget when discussing tyres that the resultant forces and moments are a product, as with any friction-based model, of the interaction between the tyre and the road and not a property of the tyre in isolation. Early tyre models treated the road or terrain as simply an infinite flat surface requiring only a vertical datum for definition. This was extended to two-dimensional models that allowed a terrain to be defined as a series of sections. This allowed for durability type analysis, with forward motion only, to encounter potholes, bumps and other road obstacles. A typical current approach is to represent the geometry and frictional characteristics of the road surface or terrain using a finite element approach as shown in [Figure 5.61](#).

The road surface is defined as a system of triangular patches. As with finite elements the outward normal of the element is defined by numbering the nodes for each element using a sequence that is positive when considering a rotation about the outward normal. For each element it is possible to define frictional constants that are factored with the friction parameters associated with a tyre property file. This would allow simulations when the vehicle encounters changing road conditions as with driving from dry to wet conditions or simulation of ‘mu-split’ conditions where one side of the vehicle is braking in the dry and the other on ice.

The ability to model a road surface as a continuous three-dimensional surface is required not only for durability work but is also needed for handling simulations on

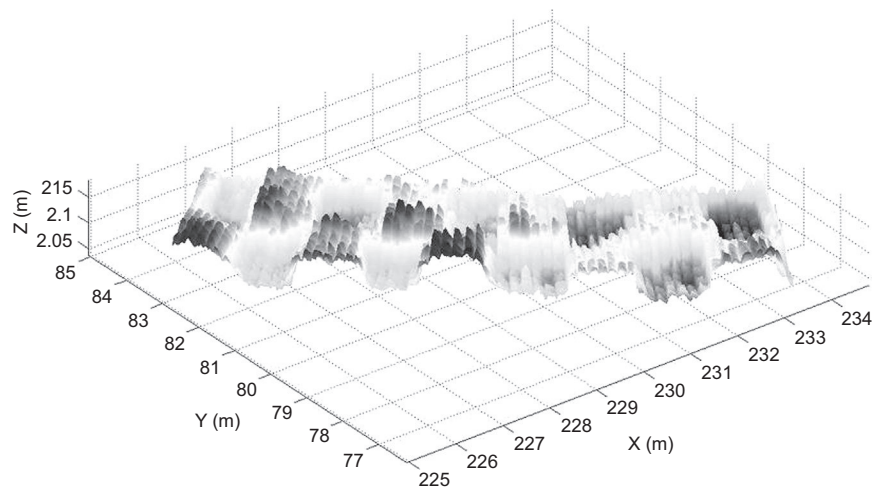
**FIGURE 5.61**

Definition of road surface using finite element approach.

non-flat road surfaces. As with any finite element model the accuracy of the road surface definition will be a function of the mesh refinement. Current practice amongst some users in industry is to model the road using the more sophisticated mesh generation tools provided with finite element-style software and then run a small translation programme to reformat the data file. The widespread adoption of the so-called ‘TeimOrbit®’ format has facilitated the separation of road generation, sometimes entirely, from the task of multibody modelling. The format appears widely used, being cited by MSC Software, Altair, Cosin and Intec Dynamics help documentation for roads. Other items are also stored in TeimOrbit files. XML files are also widely used as an easily readable and versatile storage format for finite element-style road grid descriptions. In principle any file format that is topologically similar to a finite element mesh will work for the gridded representation of road surface data, with ‘nodes’ (points in space) and ‘elements’ (a list of typically three node numbers that form a triangular facet) – so in principle a stereo lithography (STL) file could be used. However, TeimOrbit and XML appear to dominate applications at the time of writing, for artificial roads.

For durability and detailed ride work, the ready availability of scanned road surface data, (typically on grids around  $1 \times 1$  cm) has led to dissatisfaction with the previous methods, which generate unreasonably large data files. This in turn has led to a standard initially defined and used by the Daimler Group known as ‘Curved Rectangular Grid’. It gets around the ‘small difference of large numbers’ problem of storing detailed road information over a long physical distance by separating macro information (essentially the centreline of the road, including elevation) and micro information (the ‘shape of the stones’) in two separate data groups. Any given point is synthesised by combining the two pieces of information. A public version was launched in February 2009 in an ‘open’ form and has been released as a stable protocol with open source Matlab tools available to support it as of July 2013. Sample data from the so-called ‘OpenCRG’ format is shown in [Figure 5.62](#).

It can be seen that there are a number of options available for defining and using road data files; history suggests it is likely that only a small number will persist but

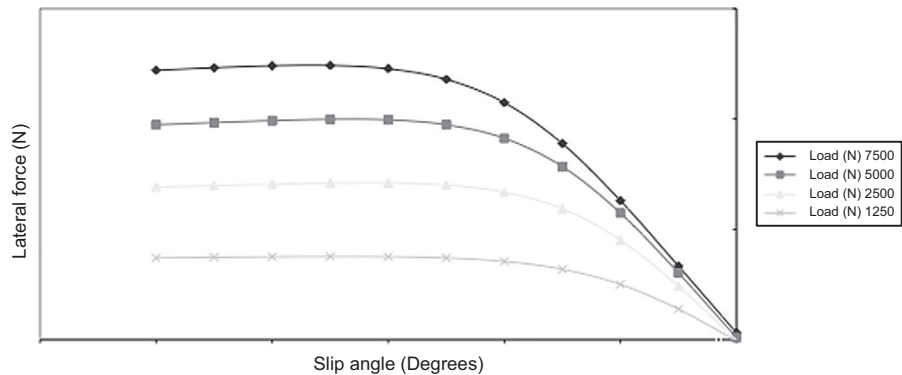


**FIGURE 5.62**  
A sample OpenCRG file rendered using Matlab. Note that the vertical scale is somewhat exaggerated compared to the two horizontal scales.

unfortunately gives few clues as to which ones; for the time being the analyst needs to be adequately familiar with all of them and intimately familiar with the ones supported by the package at hand.

**5.6.4 Interpolation methods**

Early tyre models for handling used the results of laboratory rig testing directly to generate ‘look-up’ tables of data that were used directly by the tyre model to interpolate the lateral force and aligning moment at the contact patch. Figure 5.63 illustrates a sample of some results that might typically be obtained from a tyre rig test



**FIGURE 5.63**  
Interpolation of measured tyre test data as carried out by Microsoft Excel.

where for variations in vertical load  $F_z$  the lateral forces  $F_y$  are plotted as a function of slip angle at a given camber angle.

For this set of data the independent variables that are set during the test are the camber angle, the vertical force and the slip angle. The measured dependent variable is the lateral force. Using this measured data the tyre model, really a method rather than a model, uses a curve fit to obtain a value for the lateral force for the value of  $F_z$  and slip angle determined by the wheel centre position and orientation. If the instantaneous camber angle lies between two sets of measured data at different camber angles set during the test then the tyre model can use linear, quadratic or some other interpolation scheme between the two camber angles. If the instantaneous camber angle is for example  $2.4^\circ$  and measured data is available at  $2^\circ$  and  $3^\circ$ , then the curve fitting as a function of  $F_z$  and slip angle is carried out at the two bounding camber angles and the linear interpolation is carried out between these two points. The approach described here for lateral force is applied in exactly the same manner when determining by interpolation, a value for the aligning moment. There are some disadvantages in using an interpolation tyre model:

1. The process of interpolating large quantities of data (easily 4000 plus items at a single tyre pressure) at every integration step in time may not be an efficient simulation approach and is often considered to result in increases in computer solution times for the analysis of any given manoeuvre.
2. This sort of model does not lend itself to any design modification or optimisation involving the tyre. The tyre must already exist and have been tested. In order to investigate the influence of tyre design changes on vehicle handling and stability then the tyre model must be reduced to parameters that can be related to the tyre force and moment characteristics. This has led to the development of tyre models represented by formulae that will now be discussed.

### 5.6.5 The 'Magic Formula' tyre model

For smooth road handling work, the tyre model that is now most well established and has generally gained favour is based on the work by Pacejka and as mentioned earlier is referred to as the 'Magic Formula'. The Magic Formula is not a predictive tyre model but is used to empirically represent and interpolate previously measured tyre force and moment curves. The early version (Bakker et al., 1986, 1989) is sometimes referred to as the 'Monte Carlo version' due to the conference location at which this model was presented in the 1989 paper. The tyre models discussed here are based on the formulations described in Bakker et al. (1989) and a later version Pacejka and Bakker, 1993), which is sometimes referred to as Version 3 of the Magic Formula. Mention of Version 4 appears to have disappeared from the literature.

The Magic Formula model is undergoing continual development, which is reflected in a further publication (Pacejka and Besselink, 1997) where the model is not restricted to small values of slip and the wheel may also run backwards. The

authors also discuss a relatively simple model for longitudinal and lateral transient responses restricted to relatively low time and path frequencies. The tyre model in this paper also acquired a new name and was referred to as the 'Delft Tyre 97' version, which seems to have reverted over time to MF-Tyre 5.0; 'Delft-Tyre' has become an umbrella term to include not only the base MF-Tyre model but a modified version of it suitable for intermediate frequency events, known as SWIFT (short wavelength intermediate frequency tyre).

MF-Tyre 5.0 switched to a 'normalised' formulation that ostensibly gives improved numerical conditioning, and version 6.1 includes direct support for different tyre pressures, which had been hitherto absent. TNO Automotive have become de facto custodians of the Magic Formula approach and have libraries for it and other tyre models that can be used directly with a number of popular software packages, including Matlab, Simpack, ADAMS, DADS and many others (TNO, 2012). While not apparently claiming a monopoly on the term, it seems that 'MF-Tyre' has become associated specifically with the TNO implementation. At the time of writing, the latest version of the Magic Formula is 6.1.2. Between version 3 and this version, a separate motorcycle version has been introduced ('Pac-MC' or 'MF-MC') and then merged with the basic formulation since version 6.0.

Other authors have developed systems closely based around the Magic Formula approach. The BNPS model (Schuring et al., 1993) is a particular version of the Magic Formula that automates the development of the coefficients working from measured test data. The model name BNPS is in honour of Messrs Bakker, Nyborg and Pacejka who originated the Magic Formula and the S indicates the particular implementation developed by Smithers Scientific Services Inc. MSC Software have a proprietary take on the Magic Formula that they label 'Pac2002' (Kuiper and Van Oosten, 2007). It includes stationary operation by switching to a different model at speeds below walking speed in order to avoid the divide-by-zero problem at rest. Since Magic Formula version 6.0, so-called 'turn-slip' behaviour has been modelled to allow parking loads to be captured.

The general acceptance of the Magic Formula is reinforced by the work carried out at Michelin and described in Bayle et al. (1993). In this paper the authors describe how the Magic Formula has been tested at Michelin and 'industrialised' as a self-contained package for the pure lateral force model. The authors also considered modifications to the Magic Formula to deal with the complicated situation of combined slip. To the authors' knowledge, Pirelli have also had (and may still retain) their own flavour of the Magic Formula model. The US military organisation TARDEC have published Matlab routines for handling the PAC2002 formulation model (Goryca, 2010).

It may thus be seen that talk of 'The' Magic Formula is therefore no longer appropriate and care must be taken to establish which one is in use. The general difficulty of obtaining tyre test data means that tyre data tends to get 'squirreled away' in personal archives and transported from role to role and between organisations; the backwards compatible formulation of most multibody software in use means that original 1987 data sets can still be used in, for example, Simpack, and samples of

1989 ('Pac89') data sets are included in ADAMS/Car at the time of writing. Interchanging coefficient sets between versions may produce that most insidious of errors, the 'plausible-but-wrong' result — although it equally may produce pure garbage. A clear distinction between the different versions available is therefore necessary in the mind of the analyst.

In the original Magic Formula paper, the authors in Bakker et al. (1986) discuss the use of formulae to represent the force and moment curves using established techniques based on polynomials or a Fourier series. The main disadvantage with this approach is that the coefficients used have no engineering significance in terms of the tyre properties and as with interpolation methods the model would not lend itself to design activities. This is also reflected in Sitchen (1983) where the author describes a representation based on polynomials where the curves are divided into five regions but this still has the problem of using coefficients that do not typify the tyre force and moment characteristics.

The generic Magic Formula has been developed using mathematical functions that relate:

1. The lateral force  $F_y$  as a function of slip angle  $\alpha$ .
2. The aligning moment  $M_z$  as a function of slip angle  $\alpha$ .
3. The longitudinal force  $F_x$  as a function of longitudinal slip  $\kappa$ .

When these curves are obtained from steady state tyre testing and plotted, the general shape of the curves is similar to that indicated in Figure 5.64. It is important to note that the data used to generate the tyre model is obtained from steady state testing. The lateral force  $F_y$  and the aligning moment  $M_z$  are measured during pure cornering, i.e. cornering without braking, and the longitudinal braking force during pure braking, i.e. braking without cornering.

The basis of this model is that tyre force and moment curves obtained under pure slip conditions and shown in Figure 5.64 look like sine functions that have been modified by introducing an arctangent function to 'stretch' the slip values on the x-axis.

The general form of the model as presented in Bakker et al. (1986) is:

$$Y(X) = D \sin [ C \arctan \{ Bx - E ( Bx - \arctan ( Bx ) ) \} ] \quad (5.68)$$

where

$$Y(X) = y(x) + S_v \quad (5.69)$$

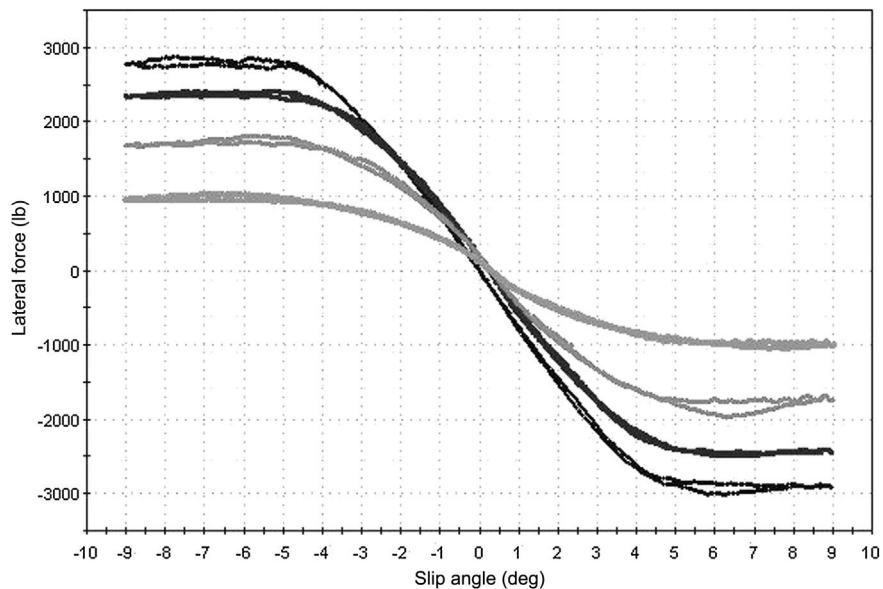
$$x = X + S_h \quad (5.70)$$

$S_h$  = horizontal shift

$S_v$  = vertical shift

In this case  $Y$  is either the side force  $F_y$ , the aligning moment  $M_z$  or the longitudinal force  $F_x$  and  $X$  is either the slip angle  $\alpha$  or the longitudinal slip, for which



**FIGURE 5.64**

Typical form of tyre force curves from testing. Note the presence of noise on the data despite high quality data acquisition.

(Courtesy of Calspan.)

Pacejka uses  $\kappa$ . The physical significance of the coefficients in the formula becomes more meaningful when considering Figure 5.65.

For lateral force or aligning moment the offsets  $S_v$  and  $S_h$  arise due to adding camber or physical features in the tyre such as conicity and plysteer. For the longitudinal braking force this is due to rolling resistance.

Working from the offset XY axis system the main coefficients are:

D — is the peak value.

C — is a shape factor that controls the ‘stretching’ in the x direction. The value is determined by whether the curve represents lateral force, aligning moment or longitudinal braking force. The very first published implementation of the Magic Formula (Bakker et al., 1986) set C as a constant:

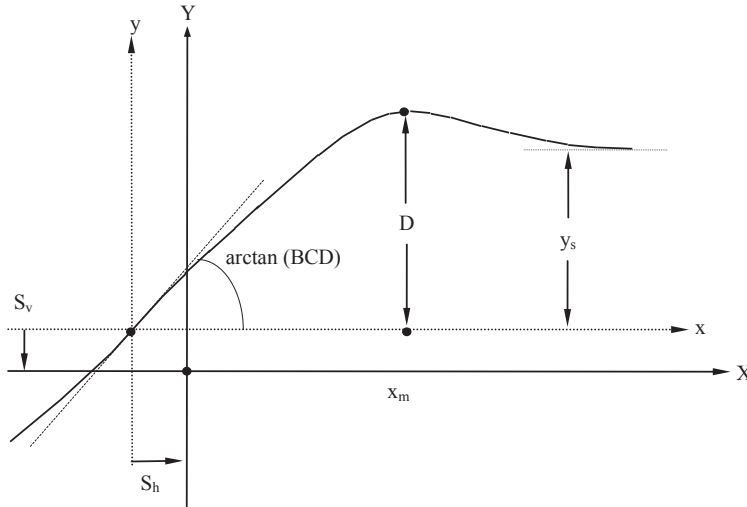
1.30 — lateral force curve.

1.65 — longitudinal braking force curve.

2.40 — aligning moment curve.

As experience emerged with using the formula, C became another one of the constants to be varied during the empirical fit from Version 2 (‘Pacejka 89’) on.

B — is referred to as a ‘stiffness’ factor. From Figure 5.65 it can be seen that BCD is the slope at the origin, i.e. the cornering stiffness when plotting lateral force. Obtaining values for D and C leads to a value for B.

**FIGURE 5.65**

Coefficients used in the 'Magic Formula' tyre.

$E$  – is a 'curvature' factor that effects the transition in the curve and the position  $x_m$  at which the peak value if present occurs.  $E$  is calculated using:

$$E = \frac{B_{X_m} - \tan(\pi / 2C)}{B_{X_m} - \arctan(B_{X_m})} \quad (5.71)$$

$y_s$  – is the asymptotic value at large slip values and is found using:

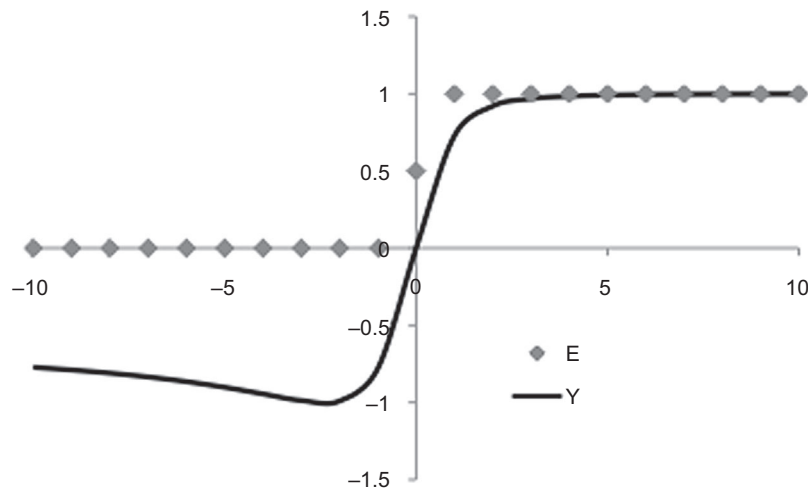
$$y_s = D \sin(\pi C / 2) \quad (5.72)$$

The curvature factor  $E$  can be made dependent on the sign of the slip value plotted on the  $x$ -axis.

$$E = E_0 + \Delta E \operatorname{sgn}(x) \quad (5.73)$$

This will allow for the lack of symmetry between the right and left side of the diagram when comparing driving and braking forces or to introduce the effects of camber angle  $\gamma$ . This effect is illustrated in Pacejka and Bakker (1993) by the generation of an asymmetric curve using coefficients  $C = 1.6$ ,  $E_0 = 0.5$  and  $\Delta E = 0.5$ . This is recreated here using the curve shape illustrated in Figure 5.66. Note that the plots have been made non-dimensional by plotting  $y/D$  on the  $y$ -axis and  $BCx$  on the  $x$ -axis.

Version 3 of the Magic Formula is discussed in this text rather than the later normalised versions because the algebra can be more readily assimilated. However, the principle of the later normalised versions remains identical but allows the use of

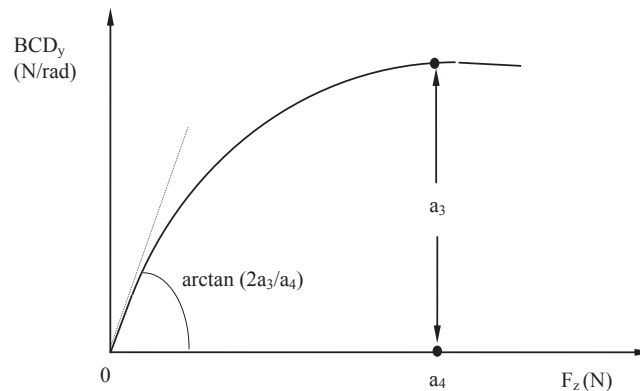
**FIGURE 5.66**

Generation of an asymmetric curve using different values of  $E$ .

convenient scaling factors to manipulate measured data on different surfaces, of which more later.

Version 3 utilises a set of coefficients  $a_0, a_1, a_2, \dots$  as shown in [Tables 5.3](#) and [5.4](#). In [Figure 5.67](#) it can be seen that at zero camber the cornering stiffness  $BCD_y$  reaches a maximum value defined by the coefficient  $a_3$  at a given value of vertical load  $F_z$  that equates to the coefficient  $a_4$ . This relationship is illustrated in [Figure 5.67](#) where the slope at zero vertical load is taken as  $2a_3/a_4$ .

The load  $a_4$ , sometimes referred to as the ‘reference load’, is not to be confused with the Rated Load for the tyre. ETRTO, the European Tyre and Rim Technical Organisation, have identified a load beyond which tyres should not be used. This

**FIGURE 5.67**

Cornering stiffness as a function of vertical load at zero camber angle.

is a mean vehicle corner load and reflects some kind of boundary between acceptable durability and unacceptable durability. Rated Load is codified on the tyre sidewall by a so-called 'Load Index', a two- or three-digit code that can be converted into an actual vehicle corner load with the help of the relevant standard. Rated Load generally has some kind of legislative significance, and varies with tyre pressure as well as details of the tyre construction and so on. This is the reason why many vehicles are specified with different tyre pressures for the laden condition. Unfortunately, no clue about the value of the coefficient  $a_4$  is available on the sidewall of the tyre.

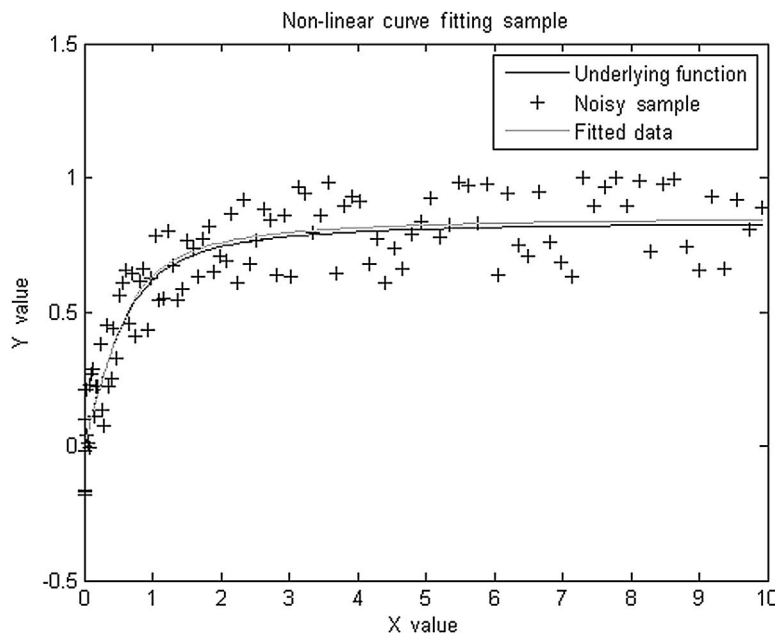
The model has been extended to deal with the combined slip situation where braking and cornering occur simultaneously. A detailed account of the combined slip model is given in Pacejka and Bakker (1993) and for more recent implementations in Pacejka (2012). The definitive equation guide from TNO (2010) contains some small but distinct differences from the implementation in the textbook, but has less illuminating commentary. The equations for pure slip only and as developed for the Monte Carlo model (Bakker et al., 1989) are summarised in Table 5.3 and similarly for Version 3 (Pacejka and Bakker, 1993) in Table 5.4. As can be seen a large number of parameters are involved and great care is needed to avoid confusion between each version, something that has become steadily more difficult over time as previously noted.

Apart from implementing the model into a MBS analysis program for vehicle simulation some method is needed to obtain the coefficients from raw test data.

The authors in van Oosten and Bakker (1993) describe their work using measured data and software developed at the TNO Road-Vehicles Research Institute to apply a regression method and obtain the coefficients. The authors in Schuring et al. (1993) have also automated the process for the BNPS version of the model.

The fitting of data for Magic Formula models is often regarded with a certain amount of mystique but there is no particular reason why this should be so, as should already be apparent. Many students are familiar with the 'trendline' function in Microsoft Excel that uses a least-squares method to find the line that is 'least bad' at representing the data. Microsoft Excel also has a solver that will attempt to minimise or maximise the value of a cell by adjusting one or more parameters using the function called 'solver' — note that this is currently unavailable in non-Windows versions of Excel. To use this method, a measured data set at one load from a tyre test machine is subtracted from the output from a calculation of the formula using some seed value coefficients. The resulting value represents an error in the fit with the seed values of coefficients. The solver can be set to minimise the error by varying the parameters. Alternatively, other fitting methods can be used by using a code such as Matlab or Octave. The definitive 'Numerical Recipes' (Press et al, 2011) recommends the Nelder Mead method and provides a 'downhill simplex' implementation of it; both Matlab and Octave have ready-coded versions available.

Some interesting wrinkles lie within such an approach and an important one is illustrated here. A sample curve is generated by setting B, C, D and E to unity. A fictitious data set is invented by adding noise to the data set and then a generic curve-fitting process is let loose on it. A lot of noise can be added and the fitting

**FIGURE 5.68**

A generic (Nelder Mead downhill simplex) fit on a fictitious data set with large amounts of noise.

routine generally converges well on the original curve when inspected visually (Figure 5.68).

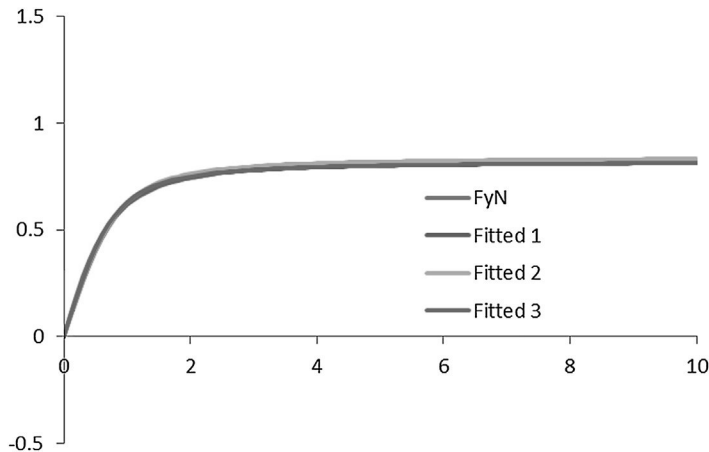
If the exercise is repeated with different sets of noise — generated using a typical computer-style ‘random’ function — then the repeated attempts at fitting all converge well and are difficult to distinguish from one another, or the original function, when plotted together, as shown in Figure 5.69.

However, examination of the fitted coefficients themselves gives a different view, as shown in Table 5.1.

1. The fitted parameters vary from run to run, particularly E.
2. The resulting fitted values are nothing like the original parameters, having differences up to 30% in the case of B, C and D and over 160% in the case of E.

In Sharp (1992) a suggested approach is to use an appreciation of the properties of the Magic Formula to fix C based on the values suggested in Pacejka and Bakker (1993) for lateral force, longitudinal force and aligning moment. For each set of load data it is then possible to obtain the peak value D and the position at which this occurs  $x_m$ . Using the slope at the origin and the values for C and D it is now possible to determine the stiffness factor B and hence obtain a value for E.

Using exactly this method can result in a disappointing fit but a small variation on it can give good results. Figure 5.70 shows a comparison between the Sharp method

**FIGURE 5.69**

Results of three different curve fitting exercises with different sets of noise generated to the same parameters.

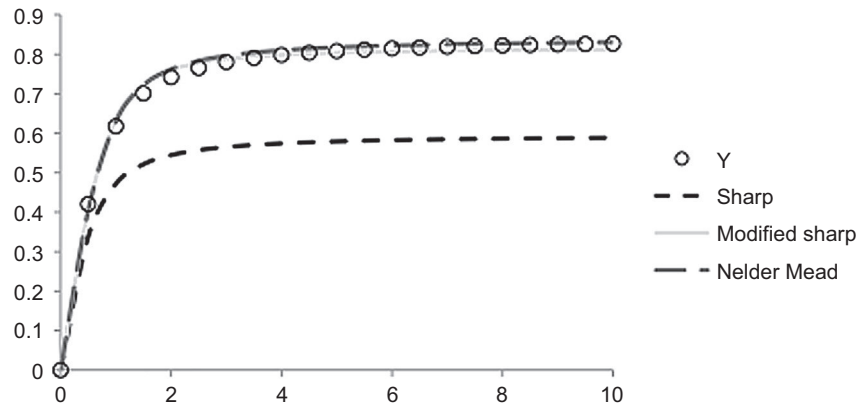
**Table 5.1** Results of Three Different Curve Fitting Exercises with Different Sets of Noise Generated to the Same Parameters

	Actual	Fitted 1	Fitted 2	Fitted 3
B	1	1.2588	1.2113	1.2889
C	1	0.8226	0.8461	0.8480
D	1	0.8540	0.8656	0.8444
E	1	-0.07021	-0.6698	-0.6361

implemented directly and a variation on it where B and D are calculated according to the Sharp protocol while C and E are found by minimising the error between the fitted and measured data using a least squares method.

The comparisons between coefficients are shown in Table 5.2 and are different again. Both C and E need modifying, although this is not unreasonable given the Sharp approach of picking C based on some prior feel for the data; the reference set does not particularly resemble any known tyre characteristic.

It is apparent using the modified Sharp method on this data set that there is a significant interplay between C and E, and also that the overall quality of the fit is not unduly improved by exploring this interplay for levels of C higher than about 2 (it is possible to get an acceptable fit at least up to  $C = 5$ ). In general it is the authors' observation that for sample, artificial data sets such as the modified Sharp method appear quite unpromising but on real tyre data sets they give rather acceptable results. In the examples given in Figure 5.71, below, only E needed finding via the least squares method.



**FIGURE 5.70**  
A comparison of the Sharp, modified Sharp and Nelder Mead fits on a reference data set.

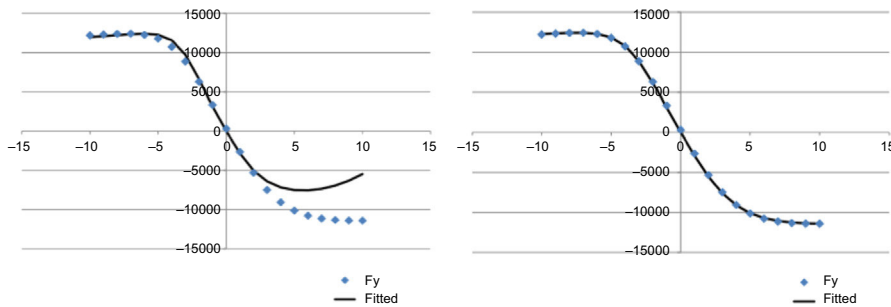
Table 5.2 A Comparison of Fits Using Other Methods			
	Actual	Fitted (Sharp)	Fitted (Modified Sharp)
B	1	0.7825	0.7825
C	1	1.3	3
D	1	0.8270	0.3391
E	1	0.8130	−30.49

Having obtained these terms at each load, the various coefficients are determined using curve fitting techniques to express B, C, D and E as functions of load. One issue with this method is that the value of C arbitrarily chosen may not subsequently emerge as the best. Applying the method described to a set of four measurements at zero camber angle gives the results for coefficients B, D and E against load as shown in Figure 5.72.

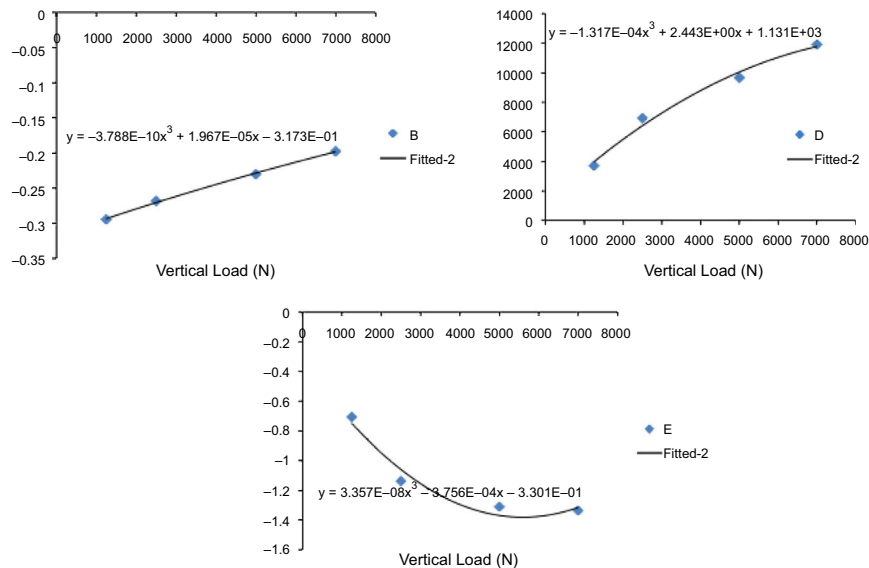
It can be seen in Figure 5.72 above that the quadratic fit for the parameter E, for example, is not terribly good compared to the measured data. Also apparent in the same figure is that extrapolating significantly beyond the fitted region may be even more unwise than is normal for extrapolation, which is already quite unwise.

A generic issue that occurs when deriving the coefficients for this model is whether those that have a physical significance should be fixed to match the tyre or set to values that give the best curve fit. Given the empirical nature of the model and the stated aim of reproducing measured data, it seems clear to the authors that fit should be favoured over everything else.

Considering the equations given in Tables 5.3 and 5.4, it can be seen that the dependency on vertical load is linear for the Version 3 model with the exception of cornering stiffness, as already discussed. This linear dependency on load

**FIGURE 5.71**

An improved fit (right) compared to the Sharp method (left) can be made by finding  $E$  using a curve fitting method and relinquishing Eqn 5.71.

**FIGURE 5.72**

Fitted values for  $B$ ,  $D$  and  $E$  plotted against vertical load in  $N$ .

continues into version 6.1. Thus it may be seen that some degradation of fit may be expected at very low loads where  $E$  may be poorly served by a linear relationship.

Despite the linear dependency for  $D$  and  $E$  on load apparently being at odds with the measured data as shown in Figure 5.72, comparisons of output from the Magic Formula with measured test data (Bakker et al., 1986, 1989) indicate good correlation. A study in Makita and Torii (1992) comparing the results of this model with those obtained from vehicle testing under pure slip conditions also indicates the high degree of accuracy that can be obtained using this tyre model.



**Table 5.3** Pure Slip Equations for the 'Magic Formula' Tyre Model**General Formula**

$$\begin{aligned}
y(x) &= D \sin[\text{Carctan}\{Bx - E(Bx - \arctan(Bx))\}] \\
Y(X) &= y(x) + S_v \\
x &= X + S_h \\
B &= \text{stiffness factor} \\
C &= \text{shape factor} \\
D &= \text{peak factor} \\
S_h &= \text{horizontal shift} \\
S_v &= \text{vertical shift} \\
B &= dy/dx|_{x=0}/CD \\
C &= (2/\pi) \arcsin(y_g/D) \\
D &= y_{\max} \\
E &= (Bx_m - \tan(\pi/2C))/(Bx_m - \arctan(Bx_m))
\end{aligned}$$

**Lateral Force**

$$\begin{aligned}
X_y &= \alpha \\
Y_y &= F_y \\
D_y &= \mu_y F_z \\
\mu_y &= a_1 F_z + a_2 \\
BCD_y &= a_3 \sin(2 \arctan(F_z/a_4)) (1 - a_5 |\gamma|) \\
C_y &= a_0 \\
E_y &= a_6 F_z + a_7 \\
B_y &= BCD_y / C_y D_y \\
S_{hy} &= a_8 \gamma + a_9 F_z + a_{10} \\
S_{vy} &= a_{11} F_z \gamma + a_{12} F_z + a_{13}
\end{aligned}$$

**Longitudinal Force**

$$\begin{aligned}
X_x &= \kappa \\
Y_x &= F_x \\
D_x &= \mu_x F_z \\
\mu_x &= b_1 F_z + b_2 \\
BCD_x &= (b_3 F_z^2 + b_4 F_z) \exp(-b_5 F_z) \\
C_x &= b_0 \\
E_x &= b_6 F_z^2 + b_7 F_z + b_8 \\
B_x &= BCD_x / C_x D_x \\
S_{rx} &= b_9 F_z + b_{10} \\
S_{vx} &= 0
\end{aligned}$$

**Aligning Moment**

$$\begin{aligned}
X_z &= \alpha \\
Y_z &= M_z \\
D_z &= c_1 F_z^2 + c_2 F_z \\
BCD_z &= (c_3 F_z^2 + c_4 F_z)(1 - c_6 |\gamma|) \exp(-c_5 F_z) \\
C_z &= c_0 \\
E_z &= (c_7 F_z^2 + c_8 F_z + c_9) (1 - c_{10} |\gamma|) \\
B_z &= BCD_z / C_z D_z \\
S_{hz} &= c_{11} \gamma + c_{12} F_z + c_{13} \\
S_{vz} &= (c_{14} F_z^2 + c_{15} F_z) \gamma + c_{16} F_z + c_{17}
\end{aligned}$$

Monte Carlo Version, also known as 'Pac89' and identifiable by the presence of the coefficient  $a_0$  as distinct from the original version presented in 1986 that had fixed values for  $C$ .

**Table 5.4** Pure Slip Equations for the 'Magic Formula' Tyre Model**General Formula**

$$y(x) = D \sin[\text{Carctan}\{Bx - E(Bx - \arctan(Bx))\}]$$

$$Y(X) = y(x) + S_v$$

$$x = X + S_h$$

B = stiffness factor

C = shape factor

D = peak factor

$S_h$  = horizontal shift

$S_v$  = vertical shift

$$B = dy/dx|_{x=0}/CD$$

$$C = (2/\pi) \arcsin(y_s/D)$$

$$D = y_{\max}$$

$$E = (Bx_m - \tan(\pi/2C))/(Bx_m - \arctan(Bx_m))$$

**Lateral Force**

$$X_y = \alpha$$

$$Y_y = F_y$$

$$D_y = \mu_y F_z$$

$$\mu_y = (a_1 F_z + a_2) (1 - a_{15} \gamma^2)$$

$$BCD_y = a_3 \sin(2 \arctan(F_z/a_4)) (1 - a_5 |\gamma|)$$

$$C_y = a_0$$

$$E_y = (a_6 F_z + a_7) (1 - (a_{16} \gamma + a_{17}) \text{sgn}(\alpha + S_{hy}))$$

$$B_y = BCD_y / C_y D_y$$

$$S_{hy} = a_8 F_z + a_9 + a_{10} \gamma$$

$$S_{vy} = a_{11} F_z + a_{12} + (a_{13} F_z^2 + a_{14} F_z) \gamma$$

**Longitudinal Force**

$$X_x = \kappa$$

$$Y_x = F_x$$

$$D_x = \mu_x F_z$$

$$\mu_x = b_1 F_z + b_2$$

$$BCD_x = (b_3 F_z^2 + b_4 F_z) \exp(-b_5 F_z)$$

$$C_x = b_0$$

$$E_x = (b_6 F_z^2 + b_7 F_z + b_8) (1 - b_{13} \text{sgn}(\kappa + S_{hx}))$$

$$B_x = BCD_x / C_x D_x$$

$$S_{hx} = b_9 F_z + b_{10}$$

$$S_{vy} = b_{11} F_z + b_{12}$$

$$\text{Brake force only } (b_{11} = b_{12} = b_{13} = 0)$$

**Aligning Moment**

$$X_z = \alpha$$

$$Y_z = M_z$$

$$D_z = (c_1 F_z^2 + c_2 F_z) (1 - c_{18} \gamma^2)$$

$$BCD_z = (c_3 F_z^2 + c_4 F_z) (1 - c_6 |\gamma|) \exp(-c_5 F_z)$$

$$C_z = c_0$$

$$E_z = (c_7 F_z^2 + c_8 F_z + c_9) (1 - (c_{19} \gamma + c_{20})^*$$

$$*\text{sgn}(\alpha + S_{hz})) / (1 - c_{10} |\gamma|)$$

$$B_z = BCD_z / C_z D_z$$

$$S_{hz} = c_{11} F_z + c_{12} + c_{13} \gamma$$

$$S_{vz} = c_{14} F_z + c_{15} + (c_{16} F_z^2 + c_{17} F_z) \gamma$$

Version 3, also known as the 'Pac94' version, identifiable by the addition of coefficients  $a_{15}$ – $a_{17}$  to improve camber sensitivity.

More detailed texts on tyre modelling (Pacejka, 2012) have additional information about pushing and pulling the Magic Formula around to deliver longitudinal forces and aligning moments. The essential elegance of the Magic Formula is that the same basic expression can be used to calculate all the tyre responses of interest. Tyres operate in conditions other than pure slip and the Magic Formula implementation also allows for the calculation of weighting functions to overlay pure lateral slip functions with mitigation due to longitudinal slip and vice versa. In-depth discussion is beyond the scope of this text but they use another form of the Magic Formula to produce a shape that can reproduce a wide variety and character of weighting functions.

The latest version of the Magic Formula (6.1 at the time of writing) has over 200 parameters and many nested loops of one Magic Formula inside another with complex weighting functions. Even for experienced practitioners this can be more or less impossible to assimilate and one of the longstanding criticisms of the Magic Formula model is that it is very difficult to construct a hypothetical tyre with '+10% cornering stiffness' to see the influence on the vehicle. The more recent embodiments of the Magic Formula, as well as switching to the normalised formulation shown in Table 5.5, have added simple scaling factors such as  $\lambda_{KY}\alpha$  (referred to as LKY inside the property files) to allow just such manipulation. All fitting of the model to measured data is done with these scaling factors set at unity. This allows the intriguing possibility of taking high quality flat-track data, with elaborate fits for all the comprehensive slip character of the tyre and so on, and scaling it according to measured data on a vehicle or tyre test trailer (See Figure 5.73) using high quality load wheel instrumentation to reflect the same tyre on a variety of different surfaces. While still in its infancy, this process shows good promise even with the very noisy data sets that might be acquired on a rolling vehicle, as can be seen in Figure 5.73.

A fundamental limitation of most point followers including the Magic Formula concept is that they are limited to steady state calculations of the tyre forces. Additional layers of calculation can be added to the scheme to permit the representation of transient delays in force build-up. The most commonly described method is the so-called 'relaxation length'. When considering a tyre to which a step change in, say, slip angle is applied, a finite length must be travelled over the ground before the tyre carcass assumes its steady state shape. This length is broadly unchanged with forward speed, and so characterising it as a length and not a time delay is logical. However, it is frequently referred to as a single length as if it were an invariant property for the tyre; there is good evidence that it varies with both slip angle and vertical load. Typical values for relaxation length with passenger car tyres are of the order of 0.3 m at 2000 N, rising to 0.8 m at 7000 N. Practically, they are often measured with a 'plank' style test machine at quite low rolling velocities.

Many texts, Pacejka (2012) and Zegelaar (1998) for example, quote the idea that lateral relaxation length is a function of cornering stiffness divided by static lateral stiffness and that a similar treatment can be applied to longitudinal forces and slip states. In principle a first order differential equation — like charging a

<b>Table 5.5</b> Pure Lateral Slip Equations for the Magic Formula Model in Later Versions
<b>Load Dependency</b>
Scaled nominal load: $F'_{z0} = \lambda F_{z0} F_{z0}$
Vertical load increment: $df_z = \frac{F_z - F'_{z0}}{F'_{z0}}$
<b>Inflation Pressure Dependency</b>
Pressure Increment: $dp_i = \frac{P - P_0}{P_0}$
<b>Pure Slip: MF-Tyre</b>
$F_{yp} = D_y \sin[C_y \arctan \{ B_y \alpha_y - E_y (B_y \alpha_y - \arctan(B_y \alpha_y)) \}] + S_{vy}$ $\alpha_y = \alpha_F + S_{Hy}$ $C_y = P_{Cy1} \lambda_{Cy}$ $D_y = \mu_y F_z \zeta_2$ $\mu_y = (p_{Dy1} + p_{Dy2} df_z) (1 + p_{py3} dpi + p_{py4} dpi^2) (1 - p_{Dy3} \gamma^2) \lambda_{\mu y}$ $E_y = (p_{Ey1} + p_{Ey2} df_z) (1 + p_{Ey5} \gamma^2 - (p_{Ey3} + p_{Ey4} \gamma) \operatorname{sgn}(\alpha_y)) \lambda_{Ey}$ $K_{y\alpha} = p_{Ky1} \cdot F'_{z0} (1 + p_{py1} dpi) \cdot \sin \left[ p_{Ky4} \arctan \left\{ \frac{F_z}{(p_{Ky2} + p_{Ky5} \gamma^2) F'_{z0} \cdot (1 + p_{py2} dpi)} \right\} \right] (1 - p_{Ky3}  \gamma ) \lambda_{Ky\alpha} \cdot \zeta_3$ $K_{y\alpha\alpha} = p_{Ky1} \cdot F'_{z0} (1 + p_{py1} dpi) \cdot \sin \left[ p_{Ky4} \arctan \left\{ \frac{F_z}{p_{Ky2} F'_{z0} \cdot (1 + p_{py2} dpi)} \right\} \right] \lambda_{Ky\alpha}$ $\text{MF-Tyre 5.2 } K_{yy0} = (p_{Hy3} K_{y\alpha\alpha} + F_z (p_{Vy3} + p_{Vy4} df_z)) \lambda_{Ky\gamma}$ $\text{MF-Tyre 6.0 } K_{yy0} = (p_{Ky6} + p_{Ky7} df_z) F_z \lambda_{Ky\gamma}$ $\text{MF-Tyre 6.1 } K_{yy\gamma} = (p_{Ky6} + p_{Ky7} df_z) F_z \lambda_{Ky\gamma} (1 + p_{py\gamma} dpi)$ $B_y = \frac{K_{y\alpha}}{C_y D_y}$ $S_{Hy0} = (p_{Hy1} + p_{Hy2} df_z) \lambda_{Hy}$

Continued

**Table 5.5** Pure Lateral Slip Equations for the Magic Formula Model in Later Versions—cont'd

$$\text{MF-Tyre 5.2} \quad S_{Hy\gamma} = p_{Hy3} \cdot \gamma \cdot \lambda_{Ky\gamma}$$

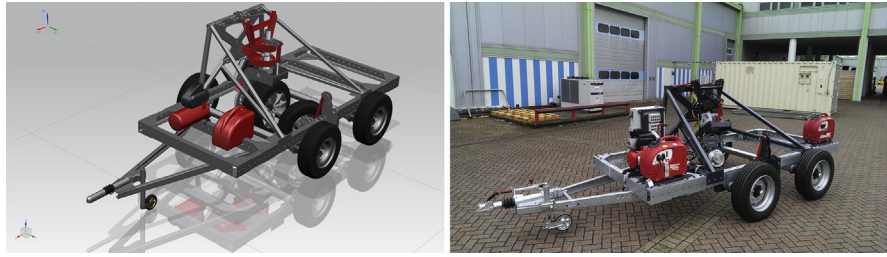
$$\text{MF-Tyre 6.0, 6.1} \quad S_{Hy\gamma} = \frac{K_{y\gamma\gamma} \gamma - S_{Vy\gamma}}{K_{y\alpha}} \zeta_0 + \zeta_4 - 1$$

$$S_{Hy} = S_{Hy0} + S_{Hy\gamma}$$

$$S_{Vy0} = F_z (p_{Vy1} + p_{Vy2} df_z) \lambda_{V\gamma} \cdot \lambda_{\mu\gamma} \cdot \zeta_2$$

$$S_{Vy\gamma} = F_z (p_{Vy3} + p_{Vy4} df_z) \gamma \cdot \lambda_{Ky\gamma} \cdot \lambda_{\mu\gamma} \cdot \zeta_2$$

$$S_{Vy} = S_{Vy0} + S_{Vy\gamma}$$

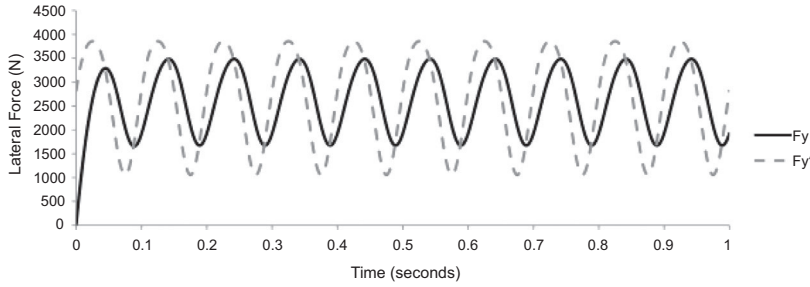
**FIGURE 5.73**

Loughborough Tyre Test Trailer — CAD model and actual trailer.

(Courtesy of G. Mavros, Loughborough University.)

capacitor — can be used to add a delay to the output of the steady state calculations. It can be applied as either an output filter, limiting rates of change of developed forces, or an input filter, limiting the rate of change of slip state before passing it to the steady state calculations. The disadvantage of applying it as an input filter is that it produces no response to a change in vertical load at a constant slip angle, something that real tyres are known to do (Pacejka and Takahashi, 1992).

An issue with naively delaying the output force with a simple delay function is illustrated in Figure 5.74. A linear cornering stiffness model is calculated on the basis outlined previously, based on vertical force  $F_z$  and two coefficients,  $a_3$  and  $a_4$ , according to the equation given in Table 5.3. A contact patch longitudinal and lateral velocity are presumed constant to give a slip angle  $\text{atan}(V_x/V_y)$  and time varying  $F_z$  is applied to give a steady state prediction for lateral force,  $F_y$ . This prediction is then subject to a simple first order lag equation with  $\tau$  set at 0.02 s, representing a relaxation length of  $\tau V_x$ . With  $V_x$  set to 20 m/s, this gives a relaxation length of around 0.4 m. The resulting lagged  $F_y$  is shown in Figure 5.74.

**FIGURE 5.74**

A naive first order lag applied to tyre force prediction.

It can be seen that while predicting the delayed build up of force in a plausible way, the lag also suggests that the cornering stiffness of tyre dynamically exceeds its steady state value during periods of decreasing load. A moment's consideration suggests that were the tyre operating in its saturated region that the model predicts a transient increase in friction coefficient, which appears unlikely. Measurements of real tyres do not reproduce this behaviour, see Zegelaar (1998).

There are multiple solutions to this but none of them are trivial. One is to detect the impending 'crossover' condition and reduce the value of  $\tau$  in response to it, as shown in Figure 5.75.

The expressions used to produce Figure 5.75 are:

For  $F'_y - F_y > 100\text{N}$

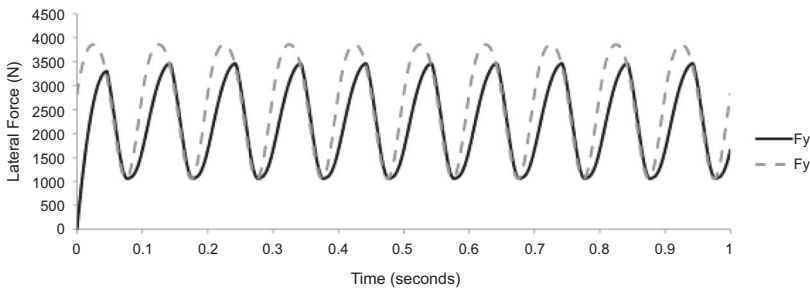
$$\tau = 0.02 \quad (5.74)$$

For  $10\text{N} < F'_y - F_y > 100\text{N}$

$$\tau = 0.0002(F'_y - F_y) \quad (5.75)$$

For  $F'_y - F_y < 10\text{N}$

$$\tau = 0.002 \quad (5.76)$$

**FIGURE 5.75**

An asymmetric lag more capable of predicting transient response to a change in load.

$$\dot{F}_y = \left( \frac{F'_y - F_y}{\tau} \right) \quad (5.77)$$

It can be seen that with some effort a continuously differentiable form of this type of model can be formulated and implemented in most general-purpose software. With this type of relaxation, sometimes labelled nonlinear relaxation or given some other title in the software, the point follower model can be persuaded to give useful results up to around wheel hop frequency. This is useful for conceptual studies of rough road handling although detailed modelling of rough road load variation need to take into account the flexible belt dynamics of the tyre, which pushes the modelling bandwidth quite high. An intermediate step is the so-called rigid belt or rigid ring model, which can be used for conceptual studies with high bandwidth systems such as ABS as long as road features are not so complex that multiple points of contact occur.

### 5.6.6 The Fiala tyre model

The Fiala tyre model (Fiala, 1954) is probably most well known to MSC ADAMS users as it is provided as a standard feature of the program. Although limited in capability this model has the advantage that it only requires 10 input parameters and that these are directly related to the physical properties of the tyre. The input parameters are shown in Table 5.6

The parameters  $R_1$ ,  $R_2$ ,  $k_z$ ,  $\zeta$ , are all used to formulate the vertical load in the tyre and are required for all tyre models that are used here, including the Pacejka and Interpolation models. As the Fiala model ignores the influence of camber angle, the coefficient, which defines lateral stiffness due to camber angle  $C_\gamma$ , is not

**Table 5.6** Fiala Tyre Model Input Parameters

$R_1$	– The unloaded tyre radius (units - length)
$R_2$	– The tyre carcass radius (units - length)
$k_z$	– The tyre radial stiffness (units - force/length)
$C_s$	– the longitudinal tyre stiffness. This is the slope at the origin of the braking force $F_x$ when plotted against slip ratio (units - force)
$C_\alpha$	– lateral tyre stiffness due to slip angle. This is the cornering stiffness or the slope at the origin of the lateral force $F_y$ when plotted against slip angle $\alpha$ (units - force/radians)
$C_\gamma$	– lateral tyre stiffness due to camber angle. This is the cornering stiffness or the slope at the origin of the lateral force $F_y$ when plotted against camber angle $\gamma$ (units - force/radians)
$C_r$	– the rolling resistant moment coefficient which when multiplied by the vertical force $F_z$ produces the rolling resistance moment $M_y$ (units - length)
$\zeta$	– the radial damping ratio. The ratio of the tyre damping to critical damping. A value of zero indicates no damping and a value of one indicates critical damping (dimensionless)
$\mu_0$	– the tyre to road coefficient of ‘static’ friction. This is the y intercept on the friction coefficient versus slip graph, effectively the peak coefficient of friction
$\mu_1$	– the tyre to road coefficient of ‘sliding’ friction occurring at 100% slip with pure sliding

used. This means that the generation of longitudinal forces, lateral forces and aligning moments with the Fiala model is controlled using just five parameters ( $C_s$ ,  $C\alpha$ ,  $C_r$ ,  $\mu_0$  and  $\mu_1$ ).

Tyre states are passed into the model. These are:

- longitudinal slip state (slip ratio) as a decimal fraction with  $-1$  representing a locked wheel,  $0$  representing a free-rolling wheel and greater than zero representing driving
- lateral slip angle in radians
- presentation (inclination) angle in radians
- vertical deflection of the footprint imposed by the road in mm
- vertical footprint velocity
- a flag denoting spin direction

The effective friction coefficient  $\mu$  is determined as a function of the comprehensive slip ratio  $S_{L\alpha}$  as shown in Figure 5.61. The comprehensive slip ratio  $S_{L\alpha}$  is taken to be the resultant of a longitudinal slip coefficient  $S_x$  and a lateral slip coefficient  $S\alpha$ .

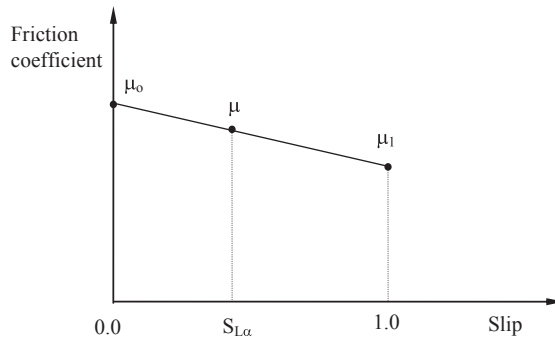
$$S_{L\alpha} = \sqrt{S^2 + S\alpha^2} \quad (5.78)$$

The instantaneous value of the tyre to road friction coefficient  $\mu$  can then be obtained by linear interpolation (Figure 5.76):

$$\mu = \mu_0 - S_{L\alpha} (\mu_0 - \mu_1) \quad (5.79)$$

Despite the advantage of a simple parameter set the main limitations of the model include:

1. The model cannot in any sense represent combined cornering and braking or cornering and driving.
2. Lateral force and aligning moment resulting from camber angle are not modelled.



**FIGURE 5.76**

Linear tyre to road friction model.



3. The variation in cornering stiffness at zero slip angle with tyre load is not considered.
4. The offsets in lateral force or aligning moment at zero slip angle due to conicity and plysteer are not represented.

While points 1, 2 and 4 are acceptable simplifications, the sensitivity of tyres to vertical load plays a primary role in shaping whole vehicle behaviour, as explored in Chapter 6. Because it cannot represent this behaviour, the Fiala model has no practical use in ground vehicle modelling — it is, in fact, that most damaging of calculations, the plausible-but-wrong type.

### 5.6.7 The Harty tyre model

A desire to have a simple, computationally light tyre model that captured directionally the behaviour of real tyres led to the development of a TIRSUB routine for use with MSC ADAMS in early 1996. The model has progressed over time and has been migrated into a more modern TYR501 subroutine, still in FORTRAN. The source code is reproduced in Appendix B. The starting point for the model was the sample routine for coding the Fiala model supplied with the software and so useful aspects of the Fiala model were re-used.

As with the Fiala model, tyre states are presumed passed into the model:

- longitudinal slip state,  $S_x$  (slip ratio) as a decimal fraction with  $-1$  representing a locked wheel,  $0$  representing a free-rolling wheel and greater than zero representing driving
- lateral slip angle in radians,  $\alpha$
- presentation (inclination) angle in radians,  $\gamma$
- vertical deflection of the footprint imposed by the road in mm  $DZ$
- vertical footprint velocity,  $VZ$
- a flag denoting spin direction

Vertical load,  $F_z$ , is calculated using a linear stiffness and damping ratio and is not described explicitly as it is somewhat trivial. The existing Fiala friction coefficient treatment is retained. It is observed that, over the load range of interest in road vehicle handling, both the slip angle at which peak side force is produced and the slip ratio at which peak longitudinal force is produced do not actually vary much. Thus they are declared as constant. While clearly an approximation, it remains useful.

It is noted that measured tyre data for both lateral and longitudinal force in pure slip displays a characteristic convex form that can be generated using a single additional parameter, a curvature factor,  $A$ :

$$|S_x| \leq S_{cx} : F_x = \mu F_z \left( 1 - e^{\left( A \frac{S_x}{S_{cx}} \right)} \right) \frac{|S_x|}{S_x} \quad (5.80)$$

$$|S_x| > S_{cx} : F_x = \mu F_z \frac{|S_x|}{S_x} \quad (5.81)$$

For the lateral slip angle case, some ability to scale lateral and longitudinal forces independently is desired and so an additional scaling factor,  $B$ , is introduced. Load sensitivity is also present in the lateral formulation (but not the longitudinal formulation; this is arguably a simplification too far) by introducing a reference load  $R_z$  and a scaling-factor-to-vertical force sensitivity term, labelled  $dB\_dF_z$  in the software code:

$$|\alpha| \leq \alpha_{CY} : F_{ya} = \left( B + (|F_z| - R_z) \frac{dB}{dF_z} \right) \mu F_z \left( 1 - e^{\left( \frac{\lambda_y - \alpha}{\alpha_{CY}} \right)} \right) \frac{|\alpha|}{\alpha} \quad (5.82)$$

$$|\alpha| > \alpha_{CY} : F_{ya} = \left( B + (|F_z| - R_z) \frac{dB}{dF_z} \right) \mu F_z \frac{|\alpha|}{\alpha} \quad (5.83)$$

With inclination angle/presentation angle, the typical taut string idea that force is generated in the plane of the tyre is used with a simple camber coefficient,  $C_\gamma$ . However, a naïve formulation using only the tangent of the angle tends to produce unreasonably high forces as the tyre leans over more, and also does not respect the friction limit. Therefore a formulation is made that sets a fractional threshold,  $T$ , beyond which a clipping function is used:

$$|\gamma| \leq \frac{\mu T}{C_\gamma} : F_{y\gamma} = -F_z \tan(C_\gamma \gamma) \quad (5.84)$$

$$|\gamma| > \frac{\mu T}{C_\gamma} : F_{y\gamma} = \frac{|\gamma|}{\gamma} F_z \left[ D (1 - e^{-E\gamma}) - T \mu C_\gamma \right] \quad (5.85)$$

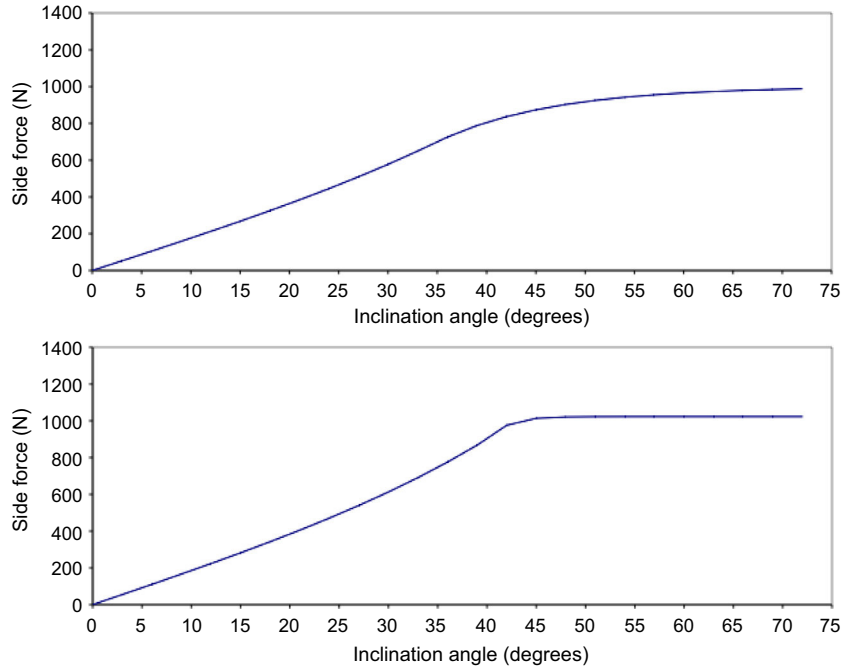
$$D = \frac{\left( \frac{1}{(1 - T C_\gamma)} \right)}{\left( \cos \left( \arctan \left( \frac{T_\mu}{C_\gamma} \right) \right) \right)^2} \quad (5.86)$$

$$E = (T - 1) \mu C_\gamma \quad (5.87)$$

This apparently complex form is entirely empirical and produces a smooth blend to the friction limit even at extended lean angles, as may be seen in [Figure 5.77](#), where the upper figure was produced with a threshold of 0.7 and the lower figure with a threshold of 0.9.

Lateral forces due to slip and inclination angles are added algebraically to generate a total side force.

This side force and the longitudinal forces together are combined into a comprehensive slip force that is compared to the available friction in a straightforward Pythagorean way. The multiplier  $B$ , described earlier, is re-used to make the ‘circle of



**FIGURE 5.77**

Harty tyre model showing camber clipping with different clip thresholds,  $T = 0.7$  (upper),  $T = 0.9$  (lower).

friction' into something elliptical; it can be seen that the ellipse can go either way depending on whether  $B$  is greater than or less than unity.

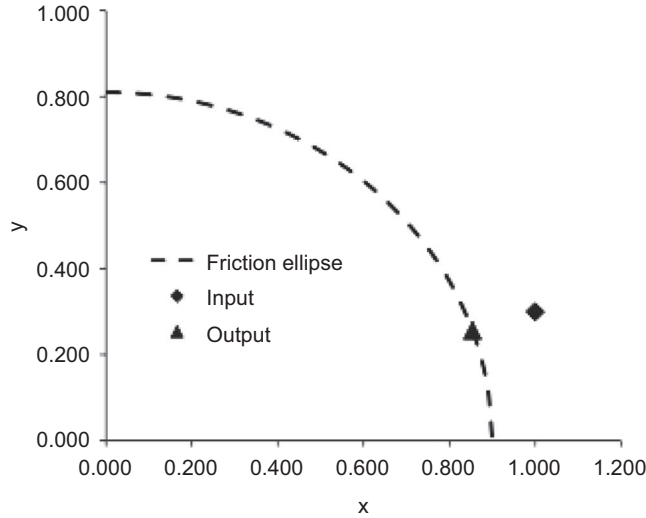
$$F_x^2 + (F_{y\alpha} + F_{y\gamma})^2 > 1 : \quad (5.88)$$

$$F_{x1} = \sqrt{\frac{\mu F_z^2}{(F_{y\alpha} + F_{y\gamma})^2 + B^2}} \frac{F_x}{|F_x|} \quad (5.89)$$

$$F_{y1} = \sqrt{\left(1 - \frac{F_{x1}}{\mu F_z}\right)^2 (\mu F_z B)^2} \frac{F_y}{|F_y|} \quad (5.90)$$

Although appearing algebraically complex, this is a very simple formulation that simply truncates the combined force vector as it crosses the friction ellipse. The concept is illustrated in normalised form in [Figure 5.78](#).

This formulation behaves well at low levels of slip but becomes confused when the slip levels become high in the tyre, such as the case of a locked wheel or with

**FIGURE 5.78**

A normalised representation of the comprehensive slip formulation within the Harty tyre model at low levels of slip.

gross wheelspin. Under these circumstances a different approach is taken, which is to calculate the velocity vector of the contact patch with respect to the ground and to apply a simple Coulomb-style frictional force in the opposite direction. Note that there are none of the numerical difficulties associated with the Coulomb formulation around zero speed because this model is only applied when the contact-patch-to-ground speed is substantial:

$$F_{x2} = \sqrt{\frac{\mu F_z^2}{1 + \frac{\tan^2 \alpha}{S_x^2}}} \frac{F_x}{|F_x|} \quad (5.91)$$

$$F_{y2} = \sqrt{\left(1 - \frac{F_{x2}}{\mu F_z}\right)^2 (\mu F_z)^2} \frac{F_y}{|F_y|} \quad (5.92)$$

Note that when using this formulation, slip angle  $\alpha$  is expressed in radians and longitudinal slip,  $S_x$  is a fraction and not percentage. These two slip quantities are simply blended between a longitudinal slip ratio of 50% and 100% in either direction, which is to say only in the gross wheelspin or near lock condition. The resulting force vectors appear directionally correct. However, it must be said that this final elaboration in the comprehensive slip calculation is really only necessary for simulations where this deep slip condition is of interest, such as perhaps in a video game or perhaps in some A:B comparison animation of, say, behaviour with locked wheels versus behaviour without; in general this deep slip model is unnecessary.

Once the aggregate contact forces have been rescaled for comprehensive slip, computations for the various moments can be made using them.

Overturning moment or capsize moment is calculated with a very simple function involving the width of the tyre and an overturning moment coefficient,  $C_{MX}$ ; a value of unity for  $C_{MX}$  produces no overturning moment and mimics a blade wheel. Values of greater than unity saturate the aligning moment when an imagined contact point has moved laterally by half the width of the tyre. For car tyres, this happens at around  $5^\circ$  of inclination angle but for motorcycle tyres it is often at or beyond  $45^\circ$  of inclination that the tyre contact centre of pressure is closest to the edge of the tyre. Simple trigonometry is used to move the notional contact point laterally and the vertical load is multiplied by this lever arm:

$$|\gamma| \frac{W}{2} (C_{MX} - 1) > \frac{W}{2} : M_x = \frac{W}{2} \frac{|\gamma|}{\gamma} F_z \quad (5.93)$$

$$|\gamma| \frac{W}{2} (C_{MX} - 1) \leq \frac{W}{2} : M_x = \gamma \frac{W}{2} (C_{MX} - 1) F_z \quad (5.94)$$

Aligning moment is also comprised of two components, one from slip and one from camber in a similar manner to side force.

A notional contact patch length is calculated geometrically from the deflection passed in to the routine and a notional pneumatic trail used with the slip-angle-induced side force to generate an aligning moment. Note that the side force is that produced by the comprehensive slip truncation  $F_{y1}$  or  $F_{y2}$  depending on the operating regime. It is observed studying measured tyre data sets that the pneumatic trail decreases in a broadly linear way with increasing slip angle, and so the calculation of pneumatic trail is kept deliberately simple apart from the addition of an empirical, MF-style scaling factor:

$$F_{ya}' = F_{ya} \frac{F_y}{F_{yJ}}, J = 1, 2 \quad (5.95)$$

$$L = 2 \left( r_u^2 - (r_u - \Delta z)^2 \right) \quad (5.96)$$

$$M_{za} = -F_{ya}' L \left( 1 - \frac{\alpha}{\alpha_{CY}} \right) \left( T_{PFZ2} F_z^2 + T_{PFZ} F_z + T_{PC} \right) \quad (5.97)$$

It will be noted that progressing beyond the critical slip angle produces increasingly reversed values of aligning torque, which is unlikely to be true in practice. In most vehicle dynamics simulations (and real driving conditions), slip angles of this magnitude constitute an error condition; in any case reliable tyre measurements in this region are difficult because of the high thermal loads on the tyre — again this formulation is a clear choice of ‘simple and wrong’ over ‘complicated and wrong’.

For camber-induced moments, it is noted that they are in the opposite sense to steer torque moments. A ‘pneumatic lead’ – conceptually opposite of pneumatic trail – is introduced to calculate camber-induced moments, again scaled with an MF-style factor:

$$F_{yJ}' = F_{yJ} \frac{F_y}{F_{yJ}}, J = 1, 2 \quad (5.98)$$

$$M_{zJ} = F_{yJ}' (L_{PFZ2} F_z^2 + L_{PFZ} F_z + L_{PC}) \quad (5.99)$$

Again it is noted that the camber forces are scaled according to the comprehensive slip truncation. Naive but functional calibrations of the model may be obtained by setting all scaling values to zero save for  $T_{PC}$  and  $L_{PC}$ , which can be set to unity and 25 mm, respectively.

Final aligning and capsize moment contributions come from applied longitudinal forces with the lateral shift of the contact patch centre of pressure under nonzero inclination angles. This is particularly important for motorcycles as it generates an uncommanded steer moment that will steer into the turn and thus reduce roll rate. Its importance is relatively minor for passenger cars although it can improve prediction of overall steering return torque.

$$O_p = 2\gamma \frac{W}{2} (C_{MX} - 1) \quad (5.100)$$

$$\text{For } O_p \leq \frac{W}{2}$$

$$M_{x\gamma\kappa} = F_z O_p \quad (5.101)$$

$$M_{z\gamma\kappa} = -F_x O_p \quad (5.102)$$

$$\text{For } O_p > \frac{W}{2}$$

$$M_{x\gamma\kappa} = F_z \frac{W}{2} \frac{O_p}{|O_p|} \quad (5.103)$$

$$M_{z\gamma\kappa} = -F_x \frac{W}{2} \frac{O_p}{|O_p|} \quad (5.104)$$

The rolling resistance moment  $M_y$  is identical to the Fiala formulation and is given by:

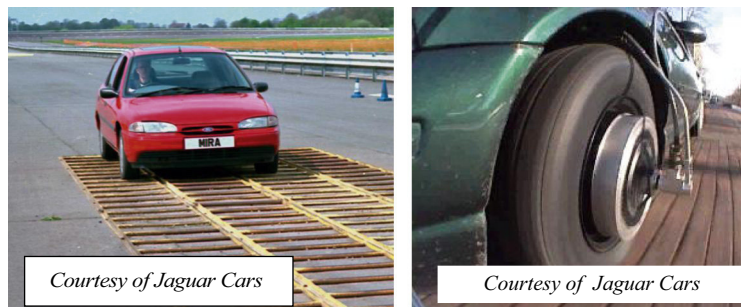
$$M_y = -C_r F_z \quad (\text{forward motion}) \quad (5.105)$$

$$M_y = C_r F_z \quad (\text{backward motion}) \quad (5.106)$$

### 5.6.8 Tyre models for durability analysis

The modelling of suspension systems for durability analysis where component loads are needed for follow on stress and fatigue analysis was discussed in Chapter 4. As discussed, the models are developed to simulate tests carried out on the proving ground to represent impacts with road obstacles of the type shown on the left side of Figure 5.79. If an actual vehicle has been taken on the proving ground and instrumented to take readings at the wheel centre, as shown on the right side of Figure 5.79, then the recorded data can be used as input to a suspension or vehicle model and the need for a durability tyre model is negated.

As vehicle design processes move towards more use of virtual prototypes, the need to carry out full dynamic simulations of proving ground procedures requires more sophisticated tyre models to interact with the terrain. In the extreme simulations recreating the testing of off road vehicles involving conditions of the type shown in Figure 5.80 requires a tyre model that can deal with a wide range of terrain. The tyre models developed for such work are based on a physical representation of



**FIGURE 5.79**

Proving ground measurements at the wheel centre for durability analysis.



**FIGURE 5.80**

Off road testing conditions involving a wide range of terrain.

the tyre carcass geometry and material rather than a mathematical model of the measured force and moment behaviour used for vehicle handling tyre models.

A tyre model that has been developed at TNO on the foundation of the Magic Formula and subsequent Delft-Tyre models has been shown to handle a wide range of road inputs allowing handling simulations to be combined with highly nonlinear road inputs. The SWIFT model is described by the authors van Oosten and Jansen (1999) as a model intended for the development of active chassis control systems and optimising vehicle ride properties with capabilities including:

1. The use of the Magic Formula for slip force calculations
2. A sophisticated contact for short wavelength slip variations
3. An effective method to model road obstacles (durability)
4. A rigid ring model to accommodate tyre belt vibrations to 80 Hz
5. Tyre characteristics that can vary with speed and load

The model has been validated through the extensive tyre test capabilities at TNO and has been shown (van Oosten and Jansen, 1999) to be accurate for durability applications, such as rolling over cleats and enveloping steps in the road surface, when comparing simulations with experimental measurements.

A comprehensive description of this complex tyre model is not possible here. Rather the reader is referred to the companion text in this series (Pacejka, 2012) where a complete chapter is dedicated to describing the formulations within the SWIFT tyre model.

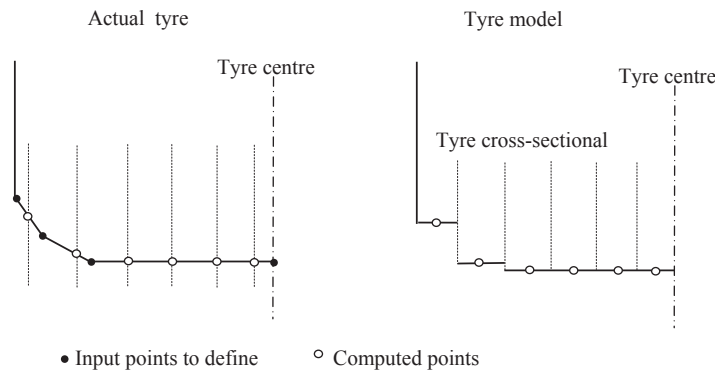
The MSC ADAMS durability tyre model (Vesimaki, 1997) was originally developed to deal with off road applications. An example of this would be the simulation of very large vehicles used by the timber industry in the forests of that author's home country, Finland. The tyre model developed for such an application would be required to deal with a vehicle cornering on a steep uneven slope where the tyres are going to encounter obstacles such as tree stumps. The requirements for such a tyre model are summarised by the author as:

1. to enable handling simulation on an uneven 3D road surface
2. to allow a road/terrain definition based on geometry
3. to accommodate varying friction over the terrain
4. to account for the cross-sectional tyre dimension and geometry

Such a model requires a physical representation of the tyre profile in order to model the boundaries of the tyre carcass as they envelop obstacles. The tyre model input consists of points that model one half of the tyre profile, as shown on the left side of Figure 5.81. The tyre model uses the input geometry to compute interpolated internal points, each of which defines the radius and lateral position of a disc representing a slice of the tyre cross-section.

As with any model based on a physical discretisation, the model refinement or number of cross-sectional elements must be such that the width of the 'slices' is sufficiently small to deal with obstacles that are narrow compared with the overall



**FIGURE 5.81**

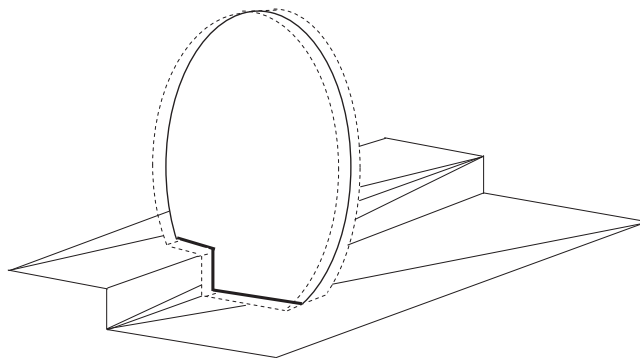
Discretisation of tyre profile for durability analysis.

width of the tyre. The road model is based on the finite element representation described in [Section 5.6.3](#).

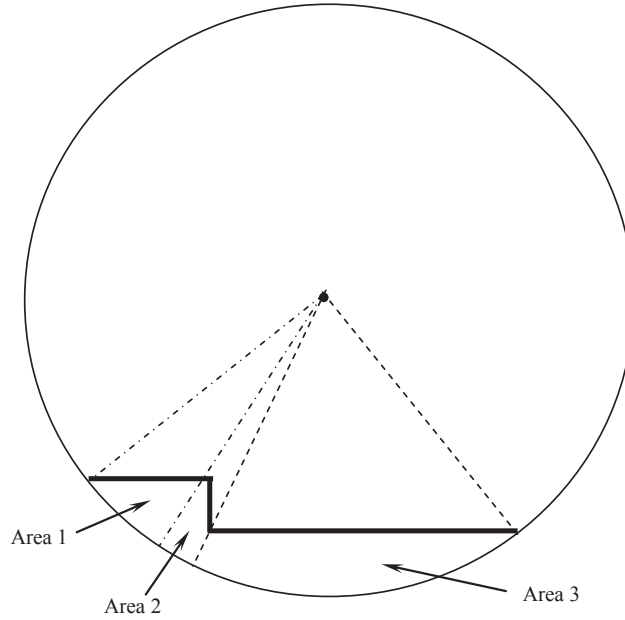
The algorithm developed carries out initial iterations to identify road elements that are subject to potential contact, at the current integration time step, before evaluating the position of each tyre element slice with each of the candidate road elements. An example of this is shown schematically in [Figure 5.82](#) where one tyre cross-sectional element is seen to intersect a step defined by a number of triangular road elements.

For each of the discrete elements used to model the tyre cross-section, the interaction with the road surface elements produces a line projection of the intersection on the tyre element.

From this it is possible to compute the area and hence volume related to the penetration of tyre cross-sectional element by the road, for example by summing the three components shown in [Figure 5.83](#). For a tyre with  $n$  cross-sectional elements,

**FIGURE 5.82**

Intersection of durability tyre model element with road surface element.

**FIGURE 5.83**

Penetration of tyre elemental slice by road surface elements.

where each element has  $m$  components of penetrated area, the effective penetrated volume,  $V_{\text{eff}}$ , for the complete tyre is given by

$$V_{\text{eff}} = \sum_{i=1}^n \sum_{j=1}^m A_j w_i \quad (5.107)$$

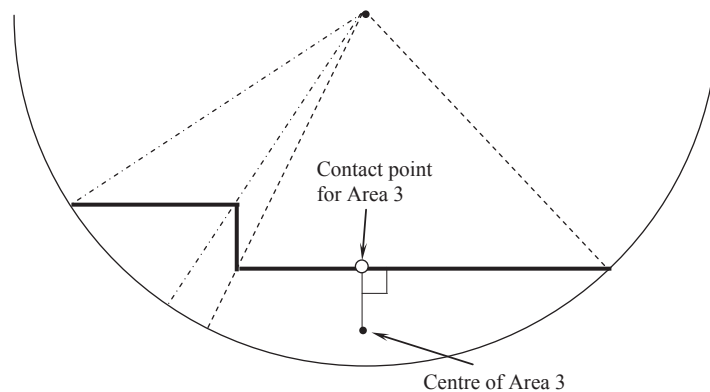
where

$A_j$  is the penetrated area of the  $j$ th component of area within the  $i$ th cross-sectional tyre element

$w_i$  is the width of the  $i$ th cross-sectional element of the tyre

The location of the effective contact point, within the deformed volume of tyre, is found by taking a weighted average of the contact point locations associated with each component of area within each cross-sectional element of the tyre. In [Figure 5.84](#) it can be seen that for the component of area shown, the associated contact point is located where the road surface is intersected by a line normal to the road surface and passing through the centre of area. On this basis the  $x$  coordinate,  $X_{\text{ecp}}$ , for the effective contact point of the tyre would be found from

$$X_{\text{ecp}} = \sum_{i=1}^n \sum_{j=1}^m \frac{A_j w_i}{V_{\text{eff}}} X_j \quad (5.108)$$

**FIGURE 5.84**

Determination of effective contact point.

where

$X_j$  is the x coordinate of the contact point for the jth component of area within ith the cross-sectional tyre element

The y and z coordinates of the effective contact point are found using the same approach used to determine  $X_{ecp}$  in Eqn (5.108). Having located an overall effective contact point for the tyre at the given moment of interaction with the terrain it is necessary to determine an effective normal vector to the road surface acting through the contact point. Once again the author (Vesimaki, 1997) uses a weighted average approach. The road normal can therefore be defined by, using again the x component  $X_{ern}$  as an example

$$X_{em} = \sum_{i=1}^n \sum_{j=1}^m \frac{A_j w_i}{V_{eff}} Xn_j \quad (5.109)$$

where

$Xn_j$  is the x component of the road normal for the jth component of area within ith the cross-sectional tyre element

Having found a volume, contact point and road normal vector for the deformed tyre at the given integration point in time it is necessary to compute a normal force acting on the tyre from the road. This involves an intermediate step where the effective volume of penetration,  $V_{eff}$ , is related to effective radial penetration of the tyre. This involves interpolation of a look-up table held within the tyre model that relates, for the defined tyre profile, the tyre penetration to penetrated volume when the tyre is compressed onto a flat surface.

The final computation required by the tyre model is to determine the effective coefficient of friction,  $\mu_{\text{eff}}$ , due to contact with the terrain

$$\mu_{\text{eff}} = \sum_{i=1}^n \sum_{j=1}^m \frac{A_j w_i}{V_{\text{eff}}} \mu_j \quad (5.110)$$

where

$\mu_j$  is the coefficient of friction associated with the  $j$ th component of area within  $i$ th the cross-sectional tyre element

For off road simulation a useful aspect of this approach is that the coefficient of friction can be factored to vary for each road element as described in [Section 5.6.3](#).

As modelling requirements pass beyond handling loads and deeper into durability modelling where the enveloping behaviour of the tyre becomes important, another category of tyre models has emerged. These models take a ‘semi-literal’ approach to modelling the structure of the tyre.

Another tyre model, FTire, specifically developed for ride and durability simulations, has been developed by COSIN Software in Germany (Gipser, 1999) and made available through an interface in MSC ADAMS.

The model comprises a rigid rim surrounded by elements with elastic interconnections that form a surrounding flexible belt or ring and has been developed to deal with frequencies up to 120 Hz and to encompass obstacles in the longitudinal direction of rolling with wavelengths half the length of the tyre contact patch. In the transverse direction the model can handle inclination of the road surface and also obstacles that vary across the tyre lateral footprint, hence the model is referred to as ‘2 ½ - dimensional’ nonlinear vibration model. The model can also accommodate the effects of stiffening and radial growth associated with high angular spin velocities. The model input parameters comprise tyre geometry and measured physical characteristics, with the optional input of natural frequencies and damping factors associated with the lower vibration modes of an unloaded tyre on a rigid rim. The belt or flexible ring is modelled as 50 to 100 lumped mass elements elastically interconnected and mounted to the rigid rim.

The elements have interconnecting stiffness to account for relative bending, extension, radial and tangential motion in the circumferential and lateral directions. The radial connection between elements on the belt and the rigid rim is a combined spring damper that allows the model to account for centrifugal dynamic stiffening at high angular spin velocities.

Each of the interconnected belt elements has up to 20 mass-less tread blocks each having nonlinear stiffness and damping in the radial, tangential and lateral directions, hence allowing the tread blocks to transmit normal forces from the road directly to the belt. Frictional forces in both the circumferential and lateral directions can be transmitted through the shear forces acting on the mass-less tread elements.

Tread blocks can be given individual heights, allowing the digitization of tread patterns without difficulty.

In this manner the pressure distribution in the contact patch can be discerned to quite high resolution. In conjunction with a good ground plane friction model, the resulting pressure and velocity distributions allow aggregate resultant forces and moments acting on the rigid rim to be found by integrating the forces acting throughout the elastic foundation of the belt.

The ‘secret sauce’ in the formulation of these types of model is the use of orthogonal shape functions to describe some aspects of the tyre carcass deformation. To explain the significance of this, it is helpful to imagine a cantilever beam subject to a tip load. One possible way of representing this is the classical expression, derived through the process of writing down an expression for shear force, and integrating it successively through bending moment, curvature, slope and finally deflection to calculate the response of the tip to a point load. For a cantilever that does not obviously deform when viewed with the naked eye, careful measurements of deflection will show this representation is usefully accurate. For a cantilever that deforms appreciably and obviously, this approximation becomes increasingly inaccurate due to geometric nonlinearity caused by the changing shape. It has long been noted in Finite Element analysis that the approximation can be improved by breaking the cantilever into multiple elements, each having an elastic stiffness. In general, it is noted that more elements increase the compliance in a convergent way until something resembling the ‘real’ answer is reached. This is excellent, but leads to a large increase in the number of computations required.

An alternative approach is to use a detailed (classical or finite element) model to calculate a deformed shape for the tip loading scenario and a so-called ‘modal stiffness’ — which takes into account the stiffness of each individual element and multiplies it by the square of the deformation associated with that mode of deformation *for each individual element*. This is visually represented with the shape function shown in Figure 5.85, where the total shape can be thought of as scaled summations

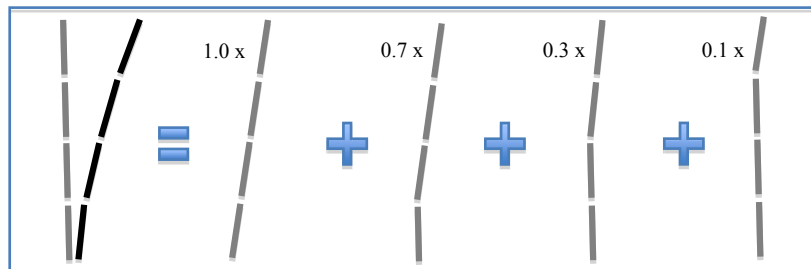


FIGURE 5.85

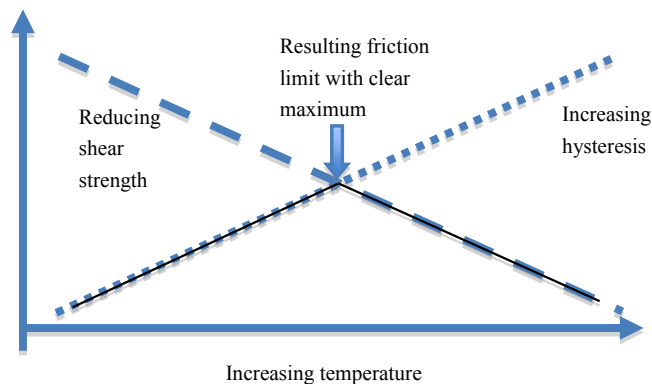
A visual representation of a fictional shape function for a cantilever modelled with rotational degrees of freedom — the shape function being [ 1.0 0.7 0.3 0.1 ].

of the individual deformations. Having calculated the shape functions externally, they are incorporated into the model that now has all the fidelity of the four element model but uses only a single tip deflection, a single so-called modal stiffness and the shape function. If four shape functions are used then the entire deformation is rendered with equal complexity in either the literal co-ordinate space or the modal co-ordinate space, but it is typically true that only a small subset of all possible shape functions are used, thus leading to a tremendous computational advantage while retaining the high fidelity compliance and deformation description of a significantly more complex model. The skill in formulating this model is all in the selection of shape functions to represent as economically as possible all the physical deformed shapes that are likely to be encountered.

Another important aspect of tyre behaviour, particularly in competition vehicles, is the variation of tyre behaviour with temperature. While the carcass stiffness is broadly provided by the metallic elements of the structure and thus is largely insensitive to the temperature variations seen in typical usage ( $-40$  to  $+100$  °C), the frictional behaviour of the contact patch is tremendously influenced over this range. All rubber-like compounds display a somewhat complex behaviour with changes in temperature and a detailed discussion is beyond the scope of this text. However, a simplified understanding can be gained by imagining there is a temperature below which the compound will behave in a much less elastic manner. When subject to intense cold, most objects become extremely brittle but the ‘glass transition’ displayed in rubber is not a change to an entirely brittle state and is something of a misnomer. The transition temperature is surprisingly close to the lower limit of the operating range, being typically between  $-50$  and  $-100$  °C for passenger car tyres. Below this limit there is an abrupt change over about a  $5$  °C temperature window in both stiffness and damping of the compound, with the modulus of elasticity increasing from a typical level around  $10$  MPa up to around  $1000$  MPa – approximately the same as polypropylene, from which many lunch-boxes are made.

While the so-called glass transition is never encountered in service, it is empirically observable that compounds with a lower glass transition temperature are ‘more rubbery’ at a typical operating temperature – which is to say they display more hysteresis, a key mechanism for producing frictional forces as described in [Section 5.3.1](#). It is also generically true that a given compound loses modulus and gains hysteresis as temperature increases.

As temperature increases, the absolute shear strength of the compound reduces – the rubber progresses through a phase somewhat like adhesive putty (which is in fact a rubber compound a long way from its glass transition point at room temperature) before it melts. Thus for use within a tyre there is a point where the increasing hysteresis is of no value because although the surface is in contact with the ground, the shear strength is so low that the contact patch is simply left behind. This means there is an optimum temperature for a given compound to operate at in terms of frictional force, as illustrated in [Figure 5.86](#).

**FIGURE 5.86**

An illustration of temperature effects on tyre friction.

Since the tread rubber also experiences a ‘cooking’ (which is to say an irreversible chemical change) effect with temperature, this optimum temperature is unfortunately not constant and varies in use. This fact severely complicates the testing of tyres and also their use in motorsport.

Another rather trivial influence of temperature is to modify the inflation pressure. This can be readily described using Boyle’s law, remembering to use absolute temperatures (Kelvin) and not Centigrade. Inflation pressure modifies the pressure distribution in the contact patch and the structural stiffness of the tyre.

It can be seen that for reasons of rubber property changes and for tyre pressure changes that the thermal modelling of tyres is also of interest.

There are two primary thermal inputs to a running tyre. The first is the hysteresis in the whole tyre assembly, which manifests itself globally as a rolling resistance. The rolling resistance torque, typically a few Nm, can be multiplied by the rotational speed to give an input power in Watts. This heat is broadly all applied in the carcass structure by the internal properties of the rubber. A tyre that has been run without excitement for some time at speed is pleasantly warm — between 40 and 50 °C — to a human when the inflation pressure is correct.

The second heat input mechanism is local sliding at the contact patch. A piece of contact patch generating a frictional force and having a certain sliding velocity is generating heat power that is the product of the in-plane force and the sliding velocity. Note that the sliding velocity is not generally the same as the velocity of the vehicle; this is only the case when the wheel is locked. This activity generates a rather large amount of power; for a single wheel carrying 400 kg sliding at 40 m/s on a surface with a friction coefficient of unity, the thermal power at the tyre/road interface is 160 kW, or over 50 English electric kettles. This heat is shared in some proportion between the tyre and the road surface itself.

In motorsport use such gross sliding is uncommon, since it rapidly consumes the tyres, but nevertheless the routine operation of motorsport tyres at 90–100 °C is due

almost entirely to their increased use of sliding behaviour in the contact patch. (Some small amount of the temperature increase also comes from direct radiative transfer from brake system components, which operate at average temperatures far higher than road cars.)

Thermal models of tyres track the heat input to the system, its distribution within the tyre and its dissipation via a convective mechanism. The net heat is divided by the thermal mass to give a change in temperature that is integrated over time to give tyre temperature. While in principle point-follower style models allow for bulk modelling of tyre temperature, they typically do not have such calculations included.

Current practice such as the Michelin TAME (Thermal And MEchanical, pronounced to rhyme with *game* and not to rhyme with *gammy*) tyre model (Hague, 2010) tends to split the tyre into two or three chunks thermally — the contact patch tread block, the rest of the carcass and perhaps the air volume, with some kind of empirical heat transfer model between them. The contact patch block may or may not contain a detailed representation of the instantaneous power input on a location-by-location basis; the remainder of the carcass is usually a single thermal mass. Heat transfer coefficients are empirically determined by experiment.

It can be seen that heat transfer is substantially affected by vehicle speed and such effects are generally captured within the model. Models such as FTire can have a thermal layer within them, that can be used with an empirically determined ablation rate to further model tyre wear. Also possible, if data is available to support it, is to predict the change in performance by looking up different frictional properties of the tread compound at different temperatures.

There appear to the authors to be two schools of thought with respect to tyre temperatures. On the one hand, practitioners who regularly use a pyrometer — a sort of thermocouple on a stick — maintain that only the core temperature of the tyre is of interest, and anyone who measures anything else is in error. On the other hand, practitioners who regularly use tyre surface measurements (typically some kind of infrared, contactless, running technology) maintain that after-the-fact measurements of core temperature reveal nothing about which corners put in the most energy and whether or not suspension geometry — most notably camber — is in need of adjustment. As ever with highly polarised discussions, there are aspects of value in both measurements and there are indeed aspects of value in interrogating models for both aspects of the tyre thermal behaviour.

---

## 5.7 Implementation with MBS

General purpose MBS software intended for use in vehicle dynamics will often have specialised modules intended for tyre modelling. Implementation of the Magic Formula tyre model and the Interpolation Method can be achieved using commercial modules, or by writing bespoke subroutines and linking these to provide a customised library function, presuming the software allows such linking (as many do). Current versions of programs such as ADAMS/Car or the Simpack Vehicle Wizard



make the incorporation of a tyre model appear seamless and can give the user a false sense of comfort with what is in fact an entirely empirical and quite complex piece of modelling. Example tyre model subroutines developed by the authors are provided in Appendix B, some of which form the basis of a tyre modelling, checking and plotting facility (Blundell, 2000a).

An example interface between MSC ADAMS and a tyre model is through a user-written TYR501<sup>1</sup> subroutine. The subroutine receives tyre states from a road/tyre interface module (usually bundled with the overall tyre modelling module but possible to implement as run time variables totally outside any additional modules) and then defines a set of three forces and three torques acting at the tyre-road surface contact patch and formulated in the TYDEX coordinate system. The equations used to formulate these forces and moments have been programmed into the subroutines to represent the various tyre models. Note that in the example provided in Appendix B the source code was migrated from an earlier TIRSUB that used the SAE co-ordinate frame; just before returning the values to the main solver, the final act is to transform the SAE forces into the TYDEX convention; this is done entirely for reasons of coding convenience and to maintain the availability of legacy support tools — spreadsheets and so on — external to the MBS software. The transformation of the forces and moments from the tyre contact patch to the wheel centre is performed internally by the program.

The TYR501 subroutine is called from within the model data set by a group of statements for each tyre on the vehicle. Appendix B contains a typical group of solver statements used to call the TYR501 model from within MSC ADAMS. It receives the tyre states listed below in the TYDEX ISO co-ordinate frame:

1. Longitudinal Slip Ratio
2. Lateral slip angle (in radians)
3. Camber angle (in radians)
4. Normal deflection of tyre into road surface
5. Normal velocity of penetration of tyre into road surface
6. Longitudinal sliding velocity of contact patch
7. Distance from wheel centre to contact point (loaded radius)
8. Angular velocity about the spin axis of the tyre
9. Longitudinal velocity of tyre tread base
10. Lateral velocity of tyre tread base

### 5.7.1 Virtual tyre rig model

Because of the complexity of the tyre model and the generic difficulty in assimilating tyre model behaviour as part of a full vehicle model, it is regarded as poor practice not to interrogate tyre models in isolation before they are used in a full

---

<sup>1</sup>The first edition of the book used a TIRSUB routine but this is no longer supported by the software vendors; TIRSUB examples are provided to illustrate the calculations only and are not usable as is.

vehicle environment. A wide range of failures in data transmission is possible, including but not limited to the use of decaNewtons, kilograms, Newtons or kilo-Newtons for forces in the measured data set, the use of radians or degrees in the measured data set, the use of ISO, SAE, modified ISO or any of the six imaginable co-ordinate systems. There is no substitute for an interrogation of the tyre model and a comparison with common sense — if it has a declared friction coefficient around unity, then frictional forces and vertical load should broadly correspond; a road-use tyre is unlikely to develop peak lateral forces much outside the slip angle range of 5 to 20°; and so on.

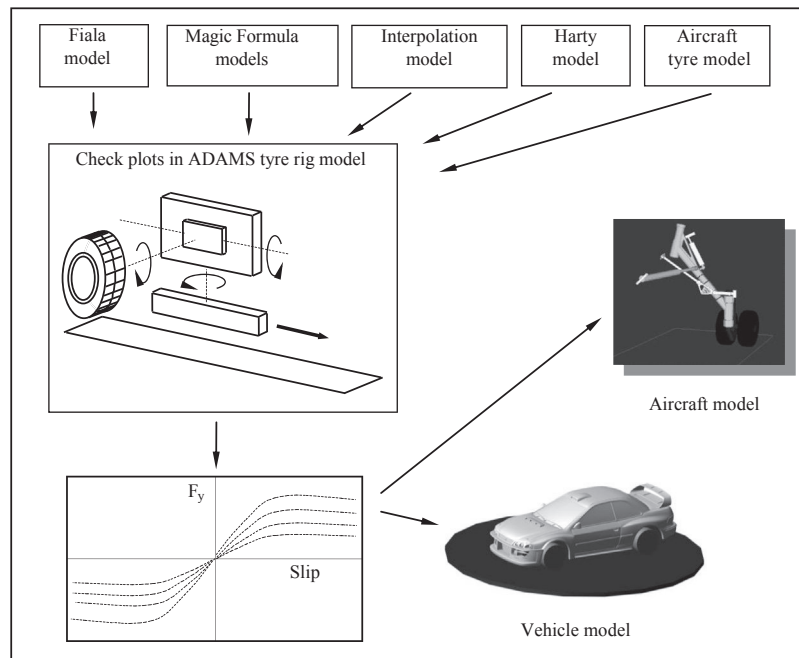
A functional model of a generic flat bed tyre test machine has been developed in MSC.ADAMS and forms part of the exemplar process described here. It is clearly desirable to use the tyre data parameters or coefficients to generate the sort of plots produced from a tyre test programme and to inspect these plots before using the data files with an actual full vehicle model. Matlab routines for directly plotting Magic Formula tyre model forces are available, as are built in test rig routines for both ADAMS/Car and the Simpack Vehicle Wizard.

The tyre rig model is also useful where test data has been used to extract mathematical model parameters. The plots obtained from the mathematical model can be compared with test data to ensure the mathematical parameters are usefully accurate, represent the actual tyre and are read by modelling software in the same manner as the parameter-fitting software (these are often quite separate software tools). The process that this involves is shown conceptually in [Figure 5.87](#). Note that this system has also been used to manage the data and model used to develop a low parameter tyre model for use in aircraft ground dynamics (Wood et al., 2012).

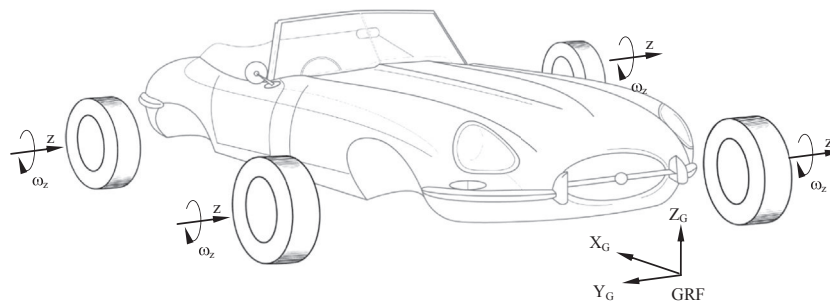
Also possible is the use of a dedicated external piece of software for handling, such as Optimum Tire from Optimum G, that can interrogate and render a tire data file with ease as well as handling a large amount of the fitting process described in [Section 5.6.5](#).

Attention must be given to the orientation as well as the location of tyre attachment frames within the model, particularly when suspension adjustments such as static toe and static camber are intended to be made parametrically. Not all tyre models run symmetrically in both forward and backward directions and so it is often good practice to ensure that all tyres are rotating the same way, as shown in [Figure 5.88](#). One possible approach with full vehicle modelling is to set up a global coordinate system or GRF where the x-axis points back along the vehicle, the y-axis points to the right of the vehicle and the z-axis is up. The local z-axis of each tyre part is orientated to point towards the left side of the vehicle so that the wheel spin vector is positive when the vehicle moves forward during normal motion. Note that this is the coordinate system as set up at the wheel centre and should not be confused with the SAE coordinate system that is used at the tyre contact patch in order to describe the forces and moments occurring there. There are many traps for the inattentive; it is clear that the tyre test rig should emulate the vehicle model attachments to be useful.

The model of the tyre test machine presented here contains a tyre part that rolls forward on a flat uniform road surface in the same way that the tyre interacts with a

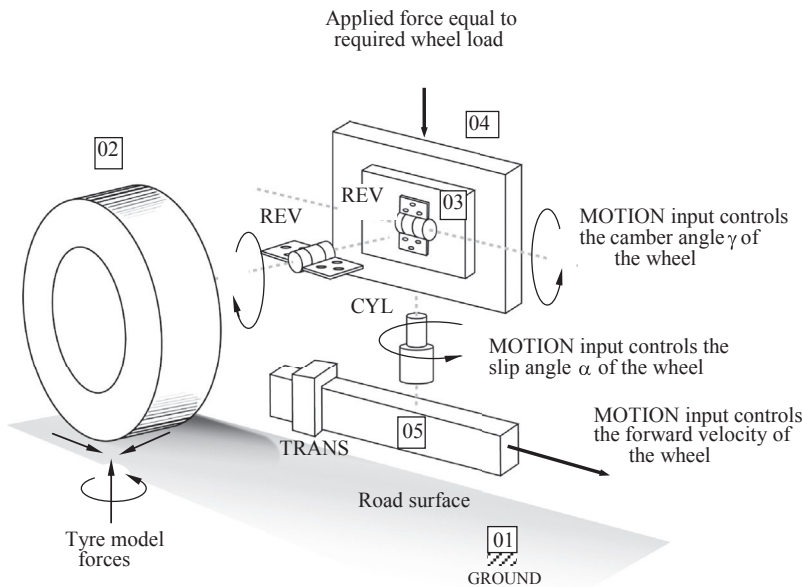
**FIGURE 5.87**

Overview of the tyre modelling system.

**FIGURE 5.88**

Orientation of tyre coordinate systems on the full vehicle model.

moving belt in the actual machine. In this model the road is considered fixed as opposed to the machine where the belt represents a moving road surface and the tyre is stationary; modelling a moving belt is surprisingly awkward in a MBS environment. Considering the system schematic of the model shown in Figure 5.89, the tyre part 02 is connected to a carrier part 03 by a revolute joint aligned with the spin axis of the wheel. The carrier part 03 is connected to another carrier part 04 by a

**FIGURE 5.89**

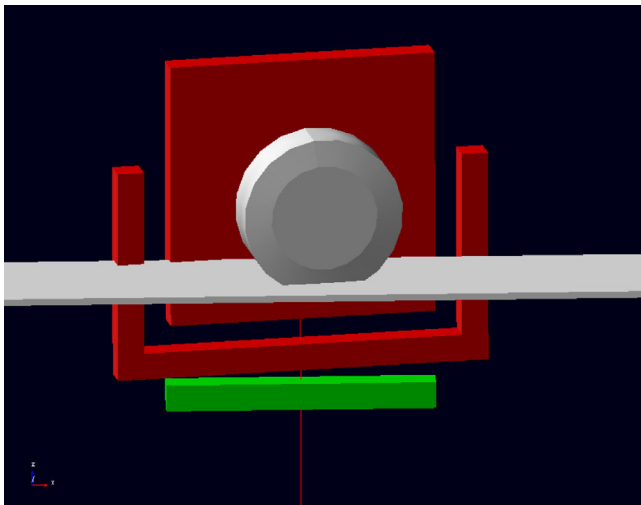
Mechanism sketch for a flat bed tyre test machine model.

revolute joint that is aligned with the direction of travel of the vehicle. A motion input applied at this joint is used to set the required camber angle during the simulation of the test process. The carrier part 04 is connected to a sliding carrier part 05 by a cylindrical joint that is aligned in a vertical direction. A rotational motion is applied at this joint that will set the slip angle of the tyre during the tyre test simulation. The cylindrical joint allows the carrier part 04 to slide up or down relative to part 05, which is important as a vertical force is applied downwards on the carrier part 04 at this joint and effectively forces the tyre down on to the surface of the road. The model has been set up to ignore gravitational forces so that this load can be varied and set equal to the required wheel vertical load that would be set during the tyre test process. The sliding carrier part 05 is connected to the ground part 01 by a translational joint aligned with the direction of travel of the wheel. A motion input applied at this joint will control the forward velocity of the tyre during the test.

The joint controlling camber angle can be located at the tyre contact patch rather than at the wheel centre. This will avoid introducing lateral velocity and hence slip angle for the change in camber angle during a dynamic simulation.

The model of the tyre test machine has two rigid body degrees of freedom as demonstrated by the calculation of the DOF balance in Table 5.7. One DOF is associated with the spin motion of the tyre, that is dependent on the longitudinal forces generated and the slip ratio. The other DOF is the height of the wheel centre above the road, that is controlled by the applied force representing the wheel load.

Table 5.7 Degree of Freedom (DOF) Balance Equation for the Tyre Rig Model			
Model Component	DOF	Number	Total DOF
Parts	6	4	24
Revolutes	−5	2	−10
Translational	−5	1	−5
Cylindrical	−4	1	−4
Motions	−1	3	−3
			$\Sigma_{\text{DOF}} = 2$



**FIGURE 5.90**  
Computer graphics for the tyre rig model.

The tyre test rig model has been used to read the tyre model data files used in a study (Blundell, 2000a) to plot tyre force and moment graphs. The graphics of the tyre rig model are shown in [Figure 5.90](#).

### 5.8 Examples of tyre model data

The results obtained from a series of tyre tests (Blundell, 2000a) have been used to set up the data needed for the various modelling approaches described here. In summary the following procedure was followed:

1. For the Interpolation method the measured numerical values were reformatted directly into the SPLINE statements within an MSC.ADAMS data file as shown in [Table 5.8](#). For each spline shown in [Table 5.8](#) the X values correspond to

<b>Table 5.8</b> Spline Data for Interpolation Model
<b>Lateral Force (n) with Slip Angle (degree) and Load (kg)</b>
SPLINE/100 ,X = -10,-8,-6,-4,-2,0,2,4,6,8,10 ,Y = 200,2148,2050,1806,1427,867,16,-912,-1508,-1881,-2067,-2151 ,Y = 400,3967,3760,3409,2727,1620,75,-1587,-2776,-3482,-3759,-3918 ,Y = 600,5447,5099,4436,3385,1962,94,-1893,-3397,-4557,-5049,-5269 ,Y = 800,6738,5969,4859,3533,2030,66,-1971,-3662,-5122,-6041,-6500
<b>Aligning Moment (nm) with Slip Angle (degree) and Load (kg)</b>
SPLINE/200 ,X = -10,-8,-6,-4,-2,0,2,4,6,8,10 ,Y = 200,4.6,-0.1,-6,-11.1,-10.9,-1.3,10.6,11.2,7.9,3.2,-0.3 ,Y = 400,-4.8,-19.6,-39,-52.1,-41.9,-6.7,35.8,49.1,38.6,23.4,10.1 ,Y = 600,-36.5,-73.1,-102.6,-107.9,-78.7,-14.2,60.6,96.2,93.4,65.8,40.7 ,Y = 800,-105.1,-181.1,-206.1,-172.4,-116.0,-23.6,79.9,143.3,172.2,141.5, 98.5
<b>Lateral Force (N) with Camber Angle (degree) and Load (kg)</b>
SPLINE/300 ,X = -10,-8,-6,-4,-2,0,2,4,6,8,10 ,Y = 100,-123.3,-96.3,-64.6,-39.3,-3,19,46,80.6,108.3,146,173.3 ,Y = 200,-142.6,-106.6,-57.3,-14.6,28,78,127,169.6,212.3,255,285.6 ,Y = 300,-173.6,-106.6,-44,20.6,87.6,159,223.6,291.3,344.3,393.3,443.6 ,Y = 400,-194,-115.6,-31.3,53,141.6,237,319.6,396.3,468.6,526.3,579 ,Y = 500,-219.6,-121.6,-17.3,91,199,304,403.3,487,572.6,651.3,717 ,Y = 600,-247.6,-128.3,-9.3,109.3,234,351,453.3,557.3,651.6,734.6,829.6 ,Y = 700,-278,-138.6,-3.6,126.3,254,381,499.3,616,723,827,922.6 ,Y = 800,-318.6,-165,-21,128,261.3,404.0,524.3,656,780,895,1012
<b>Aligning Moment (NM) with Camber Angle (degree) and Load (kg)</b>
SPLINE/400 ,X = -10,-8,-6,-4,-2,0,2,4,6,8,10 ,Y = 100,-5,-5,-4.3,-2.2,-0.9,1.2,2.6,4.2,5.8,7,6.4 ,Y = 200,-14.6,-13.7,-12,-9.2,-4.9,-0.9,3.6,6.7,9.6,11,11.7 ,Y = 300,-24.1,-22.6,-19.6,-16.7,-11.1,-4.2,2.8,8.1,11.9,15.2,17 ,Y = 400,-34.2,-31.8,-28.5,-22.9,-15.8,-8.2,-0.3,6.5,12.2,15.6,17.7 ,Y = 500,-41.5,-38,-32.7,-26.5,-18.8,-10.8,-2.5,3.9,10.7,16.5,19.6 ,Y = 600,-48.7,-43.6,-38,-31.6,-23.9,-15.9,-8.1,-0.4,6.4,12.1,16.8 ,Y = 700,-52.5,-47.5,-40.9,-34.4,-26.6,-19.5,-11.9,-4.7,1.3,7.2,12.6 ,Y = 800,-56.9,-51.3,-44.2,-37.9,-30.7,-23.9,-16.7,-10.1,-4,2.4,8.3

either the slip or camber angle and are measured in degrees. The first value in each Y array corresponds to the vertical load measured in kg. The following values in the Y arrays are the measured lateral forces (N) or the aligning moments (Nm) that correspond with the matching slip or camber angles in the X arrays. All the required conversions to the vehicle model units are carried out in the FORTRAN subroutine for the interpolation tyre model listed in Appendix B.

2. The parameters for the Fiala model were obtained by simple measurements from the plots produced during tyre testing. The Fiala model requires a single value of cornering stiffness to be defined although in reality cornering stiffness varies with tyre load. For the purposes of comparing the tyre models the parameters for the Fiala tyre model shown in Table 5.9 have been derived from the test data at the average of the front and rear wheel loads of the vehicle considered in this study. Fiala parameters obtained at front and rear wheel loads are given in Tables 5.10 and 5.11. Using the data for each of these models the tyre rig model described in the previous section was run for vertical loads of 200, 400, 600 and 800 kg. In each case the slip angle was varied between plus and minus 10°.
3. The coefficients for the Magic Formula model were provided by Dunlop Tyres using in-house software to fit the values. The Magic formula tyre model (Version 3) parameters are shown in Table 5.12. It should be noted that the parameters due to camber effects were not available from this set of tests.

Table 5.9 Fiala Tyre Model Parameters (Average Wheel Load)	
$R_1 = 318.5 \text{ mm}$	$R_2 = 97.5 \text{ mm}$
$k_z = 150 \text{ N/mm}$	$C_s = 110000 \text{ N}$
$C\alpha = 51560 \text{ N/rad}$	$C\gamma = 2580 \text{ N/rad}$
$C_r = 0.0 \text{ mm}$	$\xi = 0.05$
$M_0 = 1.05$	$M_1 = 1.05$

Table 5.10 Fiala Tyre Model Parameters (Front Wheel Load)	
$R_1 = 318.5 \text{ mm}$	$R_2 = 97.5 \text{ mm}$
$k_z = 150 \text{ N/mm}$	$C_s = 110000 \text{ N}$
$C\alpha = 54430 \text{ N/rad}$	$C\gamma = 2750 \text{ N/rad}$
$C_r = 0.0 \text{ mm}$	$\xi = 0.05$
$M_0 = 1.05$	$M_1 = 1.05$

Table 5.11 Fiala Tyre Model Parameters (Rear Wheel Load)	
$R_1 = 318.5 \text{ mm}$	$R_2 = 97.5 \text{ mm}$
$k_z = 150 \text{ N/mm}$	$C_s = 110000 \text{ N}$
$C\alpha = 46980 \text{ N/rad}$	$C\gamma = 2350 \text{ N/rad}$
$C_r = 0.0 \text{ mm}$	$\xi = 0.05$
$M_0 = 1.05$	$M_1 = 1.05$

**Table 5.12** Magic Formula Tyre Model (Version 3)  
Parameters

Lateral Force	Aligning Moment
A0 = 0.103370E+01	C0 = 0.235000E+01
A1 = -0.224482E-05	C1 = 0.266333E-05
A2 = 0.132185E+01	C2 = 0.249270E-02
A3 = 0.604035E+05	C3 = -0.159794E-03
A4 = 0.877727E+04	C4 = -0.254777E-01
A5 = 0.0	C5 = 0.142145E-03
A6 = 0.458114E-04	C6 = 0.00
A7 = 0.468222	C7 = 0.197277E-07
A8 = 0.381896E-06	C8 = -0.359537E-03
A9 = 0.516209E-02	C9 = 0.630223
A10 = 0.00	C10 = 0.00
A11 = -0.366375E-01	C11 = 0.120220E-06
A12 = -0.568859E+02	C12 = 0.275062E-02
A13 = 0.00	C13 = 0.00
A14 = 0.00	C14 = -0.172742E-02
A15 = 0.00	C15 = 0.544249E+01
A16 = 0.00	C16 = 0.00
A17 = 0.379913	C17 = 0.00
	C18 = 0.00
	C19 = 0.00
	C20 = 0.00

## 5.9 Case study 6 – comparison of vehicle handling tyre models

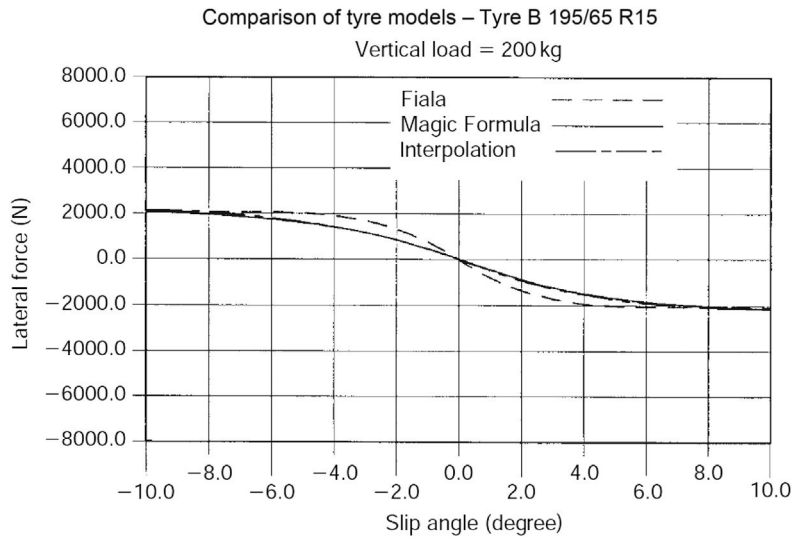
Using the data derived from the tyre tests and the MSC.ADAMS tyre test rig the following plots have been produced here by way of example.

1. Lateral force for a range of plus and minus 10° of slip angle.
2. Lateral force for a range of plus and minus 2° of slip angle (near zero).

For each plot a set of three curves is presented showing the variation in force or moment for each of the three tyre models. The comparisons are made separately at vertical loads of 200, 400, 600 and 800 kg. Using the Interpolation model as a benchmark the results plotted for the Magic Formula and Fiala models are compared in [Figures 5.91 to 5.98](#).

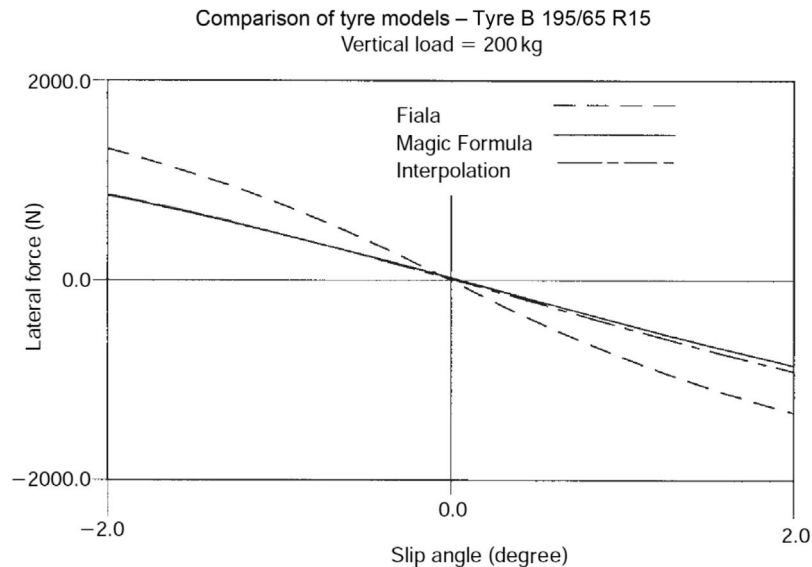
The plots show that the more detailed Magic Formula will produce a better match than the simpler Fiala model when examining the variation in lateral force with slip angle. Comparing the Magic Formula with the Interpolation data there is little to distinguish between the two sets of curves. The plots also indicate that the Magic Formula can accurately represent offsets in lateral force at zero slip angle due to ply-steer and conicity. Looking at the plots for the Fiala model it can be seen that the



**FIGURE 5.91**

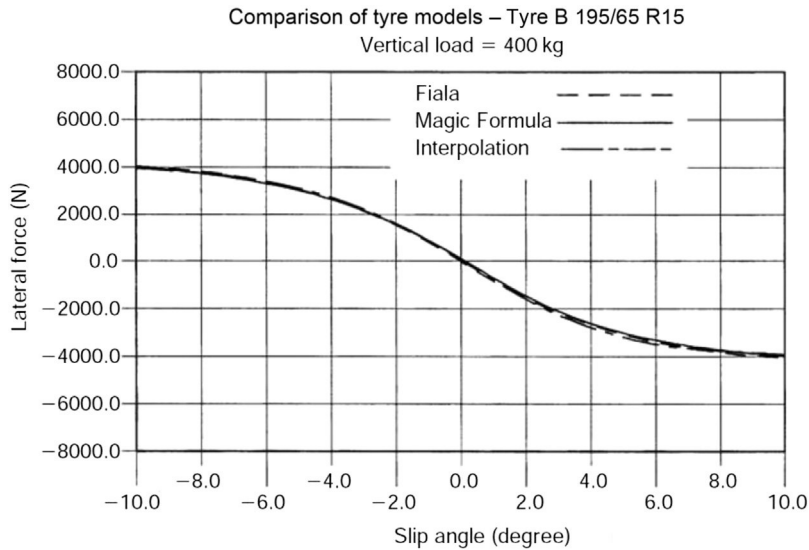
Comparison of tyre models – lateral force with slip angle (200 kg load).

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**FIGURE 5.92**

Comparison of tyre models – lateral force at near zero slip angle (200 kg load).

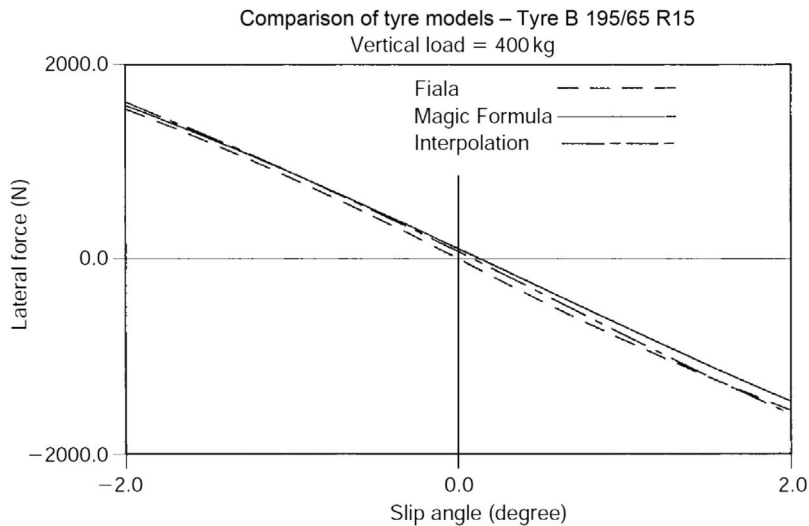
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**FIGURE 5.93**

Comparison of tyre models – lateral force with slip angle (400 kg load).

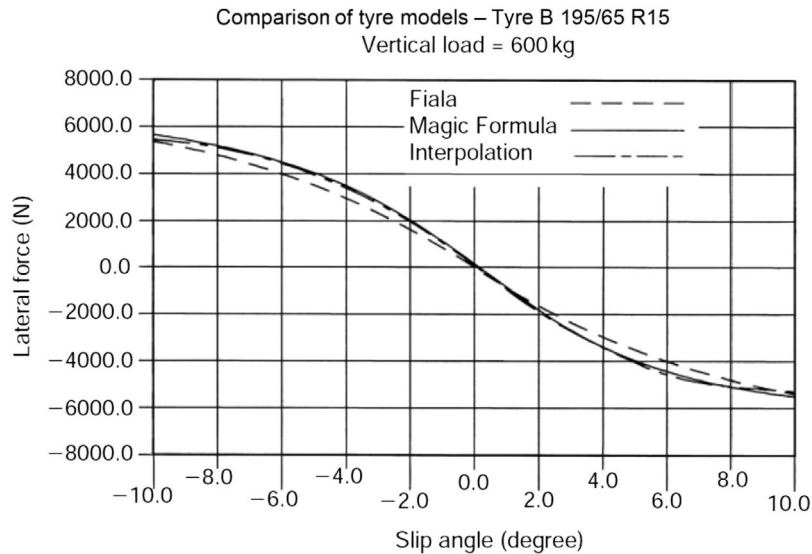
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**FIGURE 5.94**

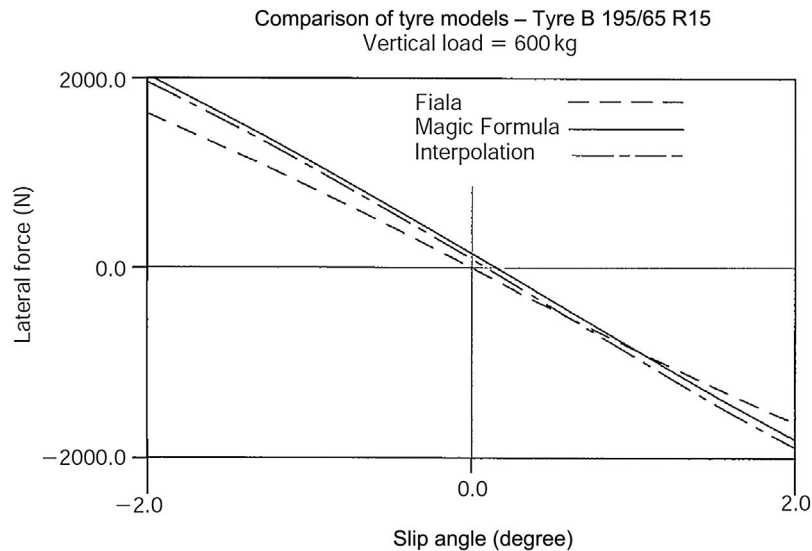
Comparison of tyre models – lateral force at near zero slip angle (400 kg load).

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**FIGURE 5.95**

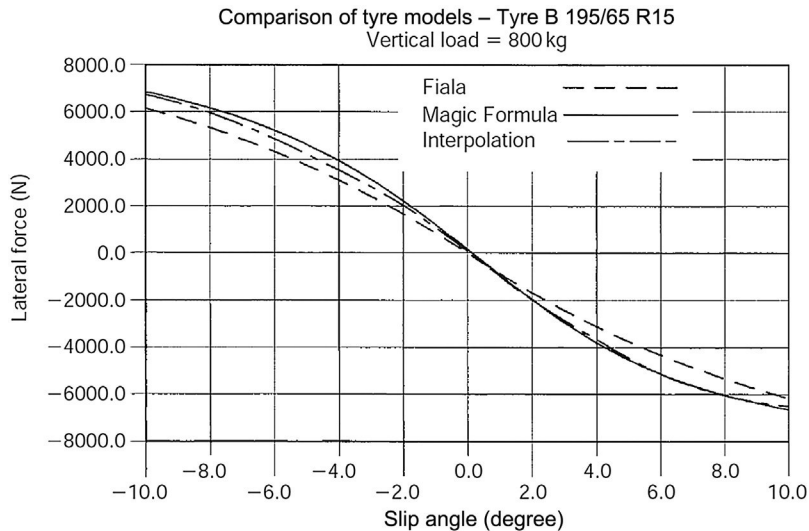
Comparison of tyre models – lateral force with slip angle (600 kg load).

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**FIGURE 5.96**

Comparison of tyre models – lateral force at near zero slip angle (600 kg load).

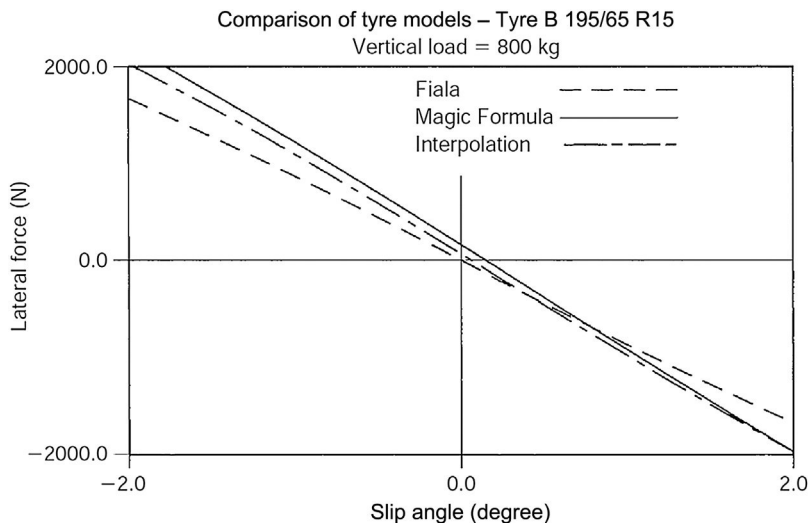
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**FIGURE 5.97**

Comparison of tyre models – lateral force with slip angle (800 kg load).

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**FIGURE 5.98**

Comparison of tyre models – lateral force at near zero slip angle (800 kg load).

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model underestimates lateral force where higher slip angles coincide with higher wheel loads. These plots also confirm that the Fiala model is ignoring lateral force offsets at zero slip angle.

The case study here is presented as an example of good practice to interrogate a tyre model and parameters before incorporation in a vehicle simulation exercise. The comparison of lateral force variation with slip angle and load shown here can be extended using the virtual rig to consider the effects of camber angle, braking, traction and combinations thereof on the significant forces and moments predicted by the model.

# Modelling and Assembly of the Full Vehicle

# 6

*Everything should be as simple as it can be, says Einstein, but not simpler.*

Louis Zukofsky

## 6.1 Introduction

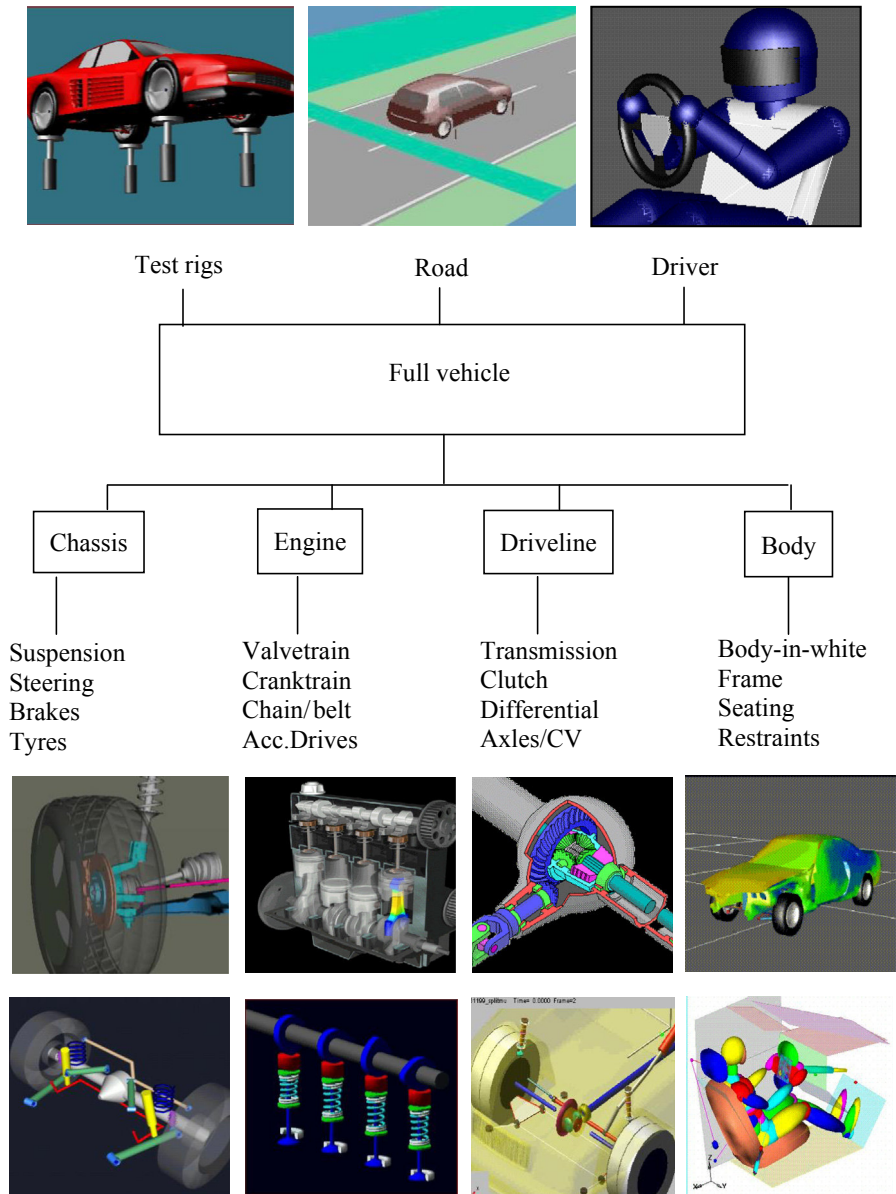
In this chapter we will address the main systems that must be modelled and assembled to create and simulate the dynamics of the full vehicle system. The term ‘full vehicle system’ needs to be understood within the context of this textbook. The use of powerful modern multibody systems (MBS) software allows the modelling and simulation of a range of vehicle subsystems representing the chassis, engine, driveline and body areas of the vehicle. This is illustrated in [Figure 6.1](#) where it can be seen that MBS models for each of these areas are integrated together to provide a detailed ‘literal’ representation of the full vehicle. Note that [Figure 6.1](#) includes the modelling of the driver and road as elements of what is considered to constitute a full vehicle system model.

In this chapter we restrict our discussion of ‘full vehicle system’ modelling to a level appropriate for the simulation of the vehicle dynamics. As such the modelling of the suspension systems, anti-roll bars, steering system, steering inputs, brake system and drive inputs to the road wheels will all be covered. With regard to steering, the modelling of the driver inputs will also be described with a range of driver models.

Note at this stage we do not consider the active elements of vehicle control. Chapter 8 is dedicated to the modelling of active systems.

For the vehicle dynamics task a starting point involving models of less elaborate construction than that suggested in [Figure 6.1](#) will provide useful insights much earlier in the design process. Provided such models correctly distribute load to each tyre and involve a usefully accurate tyre model, such as the Magic Formula described in Chapter 5, good predictions of the vehicle response for typical proving ground manoeuvres can be obtained.

The modelling of the suspension system was considered in detail in Chapter 4. The treatment that follows in this chapter will discuss a range of options that address the representation of the suspension in the full vehicle as either an assembly of linkages or using simpler ‘conceptual’ models. It is necessary here to start with the discussion of suspension representation in the full vehicle to set the scene for following sections dealing with the modelling of springs in simple suspension models or the derivation of roll stiffness. A case study provided at the end of this chapter will



**FIGURE 6.1**  
Integration of subsystems in a full vehicle model.

*Provided courtesy of MSC Software.*

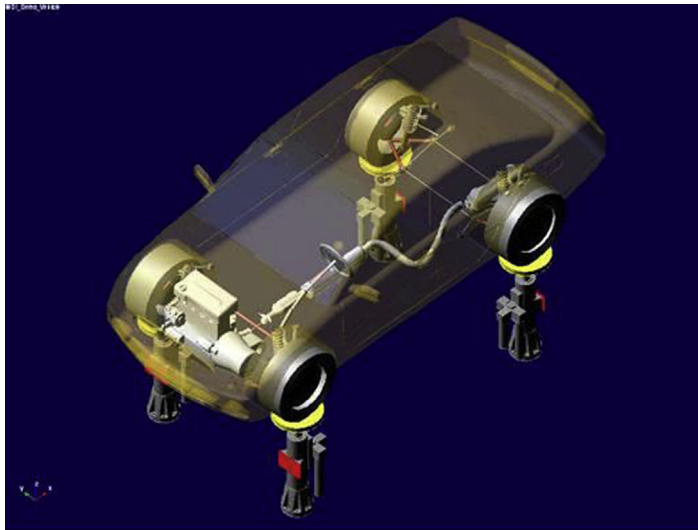
compare the simulated outputs for a simulated vehicle manoeuvre using a range of suspension modelling strategies that are described in the following section.

## 6.2 The vehicle body

For the vehicle dynamics task the mass, centre of mass position and mass moments of inertia of the vehicle body require definition within the multibody data set describing the full vehicle. It is important to note that the body mass data may include not only the structural mass of the body-in-white but also the mass of the engine, exhaust system, fuel tank, vehicle interior, driver, passengers and any other payload. A modern CAD system, or the preprocessing capability for example in ADAMS/View, can combine all these components to provide the analyst with a single lumped mass notwithstanding the cautions raised in Section 3.2.4.

Figure 6.2 shows a detailed representation of a full vehicle model. In a model such as this there are a number of methods that might be used to represent the individual components. Using a model that most closely resembles the actual vehicle, components such as the engine might for example be elastically mounted on the vehicle body using bush elements to represent the engine mounts.

The penalty for this approach will be the addition of 6 degrees of freedom (DOF) for each mass treated in this way. Alternatively a fix joint may be used to rigidly attach the mass to the vehicle body. Although this would not add DOF, the model would be less efficient through the introduction of additional equations representing the extra body and the fix joint constraint. The use of fix joint constraints may also

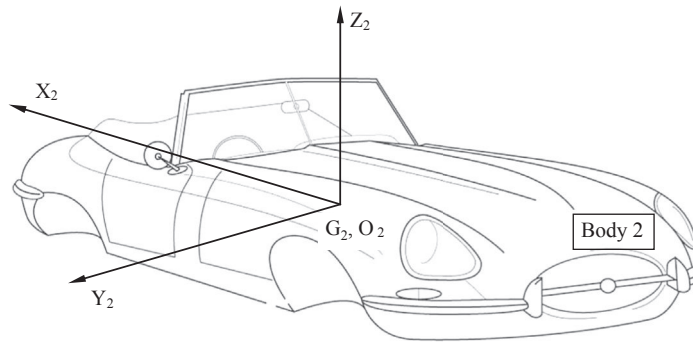


**FIGURE 6.2**

A detailed multi-body systems vehicle model.

*Provided courtesy of MSC Software.*



**FIGURE 6.3**

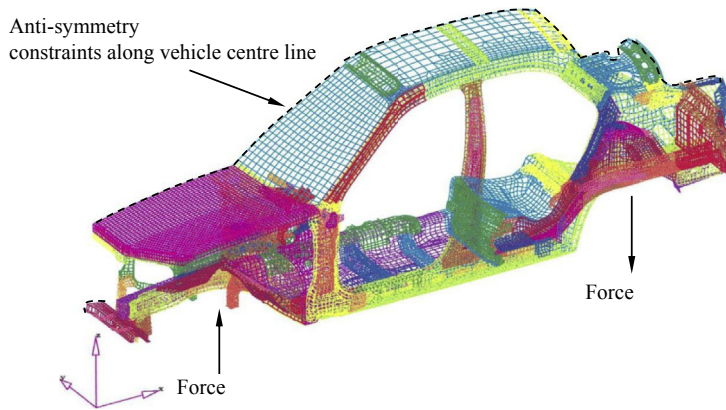
Vehicle body reference frame.

introduce high reaction moments that would not exist in the model when using elastic mounts distributed about the mass.

An example of a vehicle body referenced frame  $O_2$  located at the mass centre  $G_2$  for Body 2 is shown in Figure 6.3. For this model the XZ plane is located on the centre line of the vehicle with gravity acting parallel to the negative  $Z_2$  direction. Using an approach where the body is a single lumped mass representing the summation of the major components, the mass centre position can be found by taking first moments of mass, and the mass moments of inertia can be obtained using the methods described in Chapter 2. From inspection of Figure 6.3 it can be seen that a value would exist for the  $I_{xz}$  cross product of inertia but that  $I_{xy}$  and  $I_{yz}$  should approximate to zero given the symmetry of the vehicle. In reality there may be some asymmetry that results in a CAD system outputting small values for the  $I_{xy}$  and  $I_{yz}$  cross products of inertia.

The dynamics of the actual vehicle are greatly influenced by the yaw moment of inertia,  $I_{zz}$ , of the complete vehicle, to which the body and associated masses will make the dominant contribution. A parameter often discussed is the ratio  $k^2/ab$ , sometimes referred to as the ‘dynamic index’, where  $k$  is the radius of gyration associated with  $I_{zz}$  and  $a$  and  $b$  locate the vehicle mass centre longitudinally relative to the front and rear axles respectively. The significance of this is discussed later in Chapter 7.

The assumption so far has been that the vehicle body is represented as a single rigid body but it is possible to model the torsional stiffness of the vehicle structure if it is felt that this could influence the full vehicle simulations. A simplistic representation of the torsional stiffness of the body may be used (Blundell, 1990) where the vehicle body is modelled as two rigid masses, front and rear half body parts, connected by a revolute joint aligned along the longitudinal axis of the vehicle and located at the mass centre. The relative rotation of the two body masses about the axis of the revolute joint is resisted by a torsional spring with a stiffness corresponding to the torsional stiffness of the vehicle body. Typically, the value of torsional stiffness may be obtained using a finite element (FE) model of the type shown in Figure 6.4. For efficiency, symmetry has been exploited here to model only one half of the vehicle body. This requires the use of anti-symmetry constraints along

**FIGURE 6.4**

Finite element model of body-in-white.

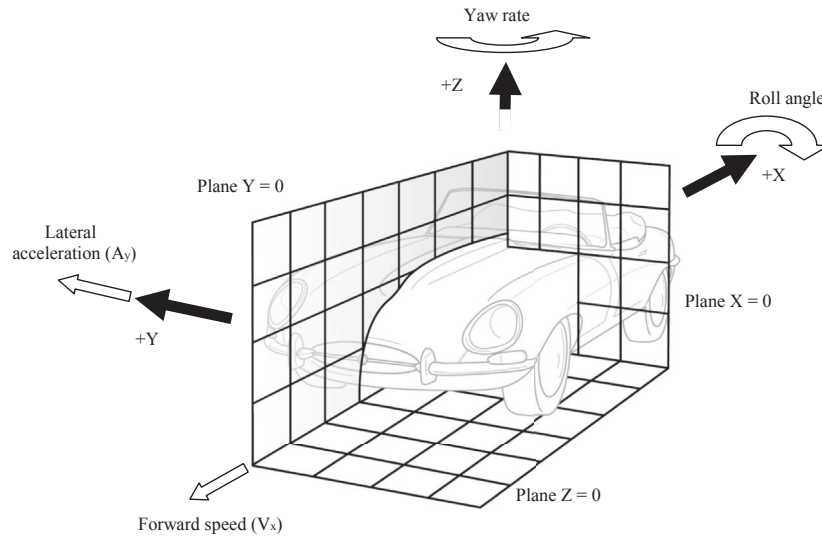
the centre line of the FE model, for all nodes on the plane of geometric symmetry, to carry out the asymmetric torsion case.

It is also possible to incorporate a FE representation of the vehicle body within the MBS full vehicle model. An example of this was shown in Figure 4.3 where the flexibility of a racing kart frame was included in the model. Despite the capability of modern engineering software to include this level of detail it will be seen from the case study at the end of this chapter that a single lumped mass is an efficient and accurate representation of a relatively stiff modern vehicle body for the simulation of a vehicle handling manoeuvre for most vehicles, with commercial vehicles and motorcycles being conspicuous exceptions.

### 6.3 Measured outputs

Before continuing in this chapter to describe the subsystems that describe the full vehicle we need to consider the typical outputs measured on the proving ground and predicted by simulation. The use of instrumented vehicles to investigate handling performance can be traced back to the work of Segal in the early 1950s, which as mentioned in Chapter 1 was the subject of one of the seminal 'IME Papers' (Segal, 1956). Testing was carried out using a 1953 Buick Super, four-door sedan, to investigate steady state behaviour with a fixed steering input at various speeds and also transient response to sudden pulse inputs at the steering wheel. The instrumentation used at that time allowed the measurement of the following:

1. Left front wheel steer
2. Right front wheel steer
3. Steering wheel rotation
4. Lateral acceleration
5. Roll angle

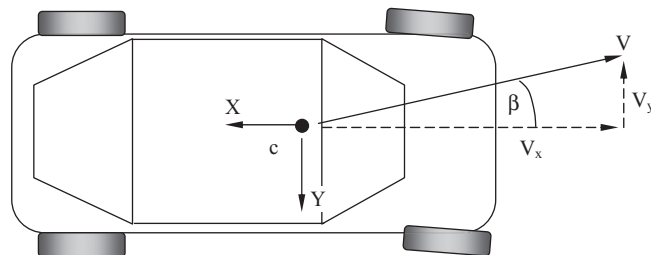
**FIGURE 6.5**

Typical responses measured in vehicle coordinate frame.

6. Pitch angle
7. Yaw rate
8. Roll rate
9. Forward velocity

Some of these responses are shown in Figure 6.5. The trajectory (path) of the vehicle can also be recorded. With simulation this is straightforward but in the past has been difficult to measure on the test track; testers resorting to measuring a trail of dye left by the vehicle on the test track surface. Modern instrumentation has improved on this somewhat.

Another measure often determined during test or simulation is the body slip angle,  $\beta$ . This being the angle of the vehicle velocity vector measured from a longitudinal axis through the vehicle as shown in Figure 6.6. The components of velocity

**FIGURE 6.6**

Body slip angle.

of the vehicle mass centre,  $V_x$  and  $V_y$ , measured in vehicle body reference frame can be used to readily determine this.

## 6.4 Suspension system representation

### 6.4.1 Overview

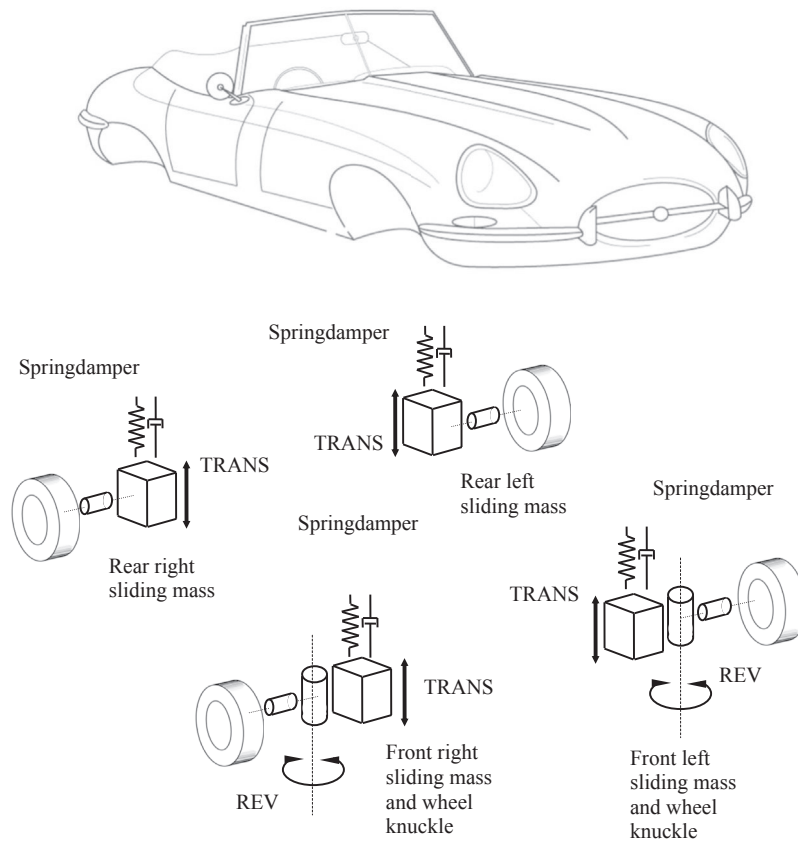
In Chapter 4 the modelling and analysis of the suspension system was considered in isolation. In this section the representation of the suspension as a component of the full vehicle system model will be considered. As stated the use of powerful MBS analysis programs often results in modelling the suspension systems as installed on the actual vehicle. In the following discussion a vehicle modelled with the suspension represented in this manner is referred to as a *linkage model*.

Before the advent of computer simulation, classical vehicle dynamicists needed to simplify the modelling of the vehicle to a level where the formulation of the equations of motion was manageable and the solution was amenable with the computational tools available at the time. Such an approach encouraged efficiency, with the analyst identifying the modelling issues that were important in representing the problem in hand. The use of modern software need not discourage such an approach. The following sections summarise four vehicle models, one of which is based on modelling the suspension linkages with three other models that use alternative simplified implementations. All four models have been used to simulate a double lane change manoeuvre (Blundell, 2000b) and are compared in the case study at the end of this chapter. The four models described here involve levels of evolving detail and elaboration and can be summarised as follows:

1. A *lumped mass model*, where the suspensions are simplified to act as single lumped masses that can only translate in the vertical direction with respect to the vehicle body.
2. An *equivalent roll stiffness model*, where the body rotates about a single roll axis that is fixed and aligned through the front and rear roll centres.
3. A *swing arm model*, where the suspensions are treated as single swing arms that rotate about a pivot point located at the instant centres for each suspension.
4. A *linkage model*, where the suspension linkages and compliant bush connections are modelled in detail in order to recreate as closely as possible the actual assemblies on the vehicle.

### 6.4.2 Lumped mass model

For the lumped mass model the suspension components are considered lumped together to form a single mass. The mass is connected to the vehicle body at the wheel centre by a translational joint that only allows vertical sliding motion with no change in the relative camber angle between the road wheels and the body. The camber angle between the road wheels and the road will therefore be directly

**FIGURE 6.7**

Lumped mass model approach.

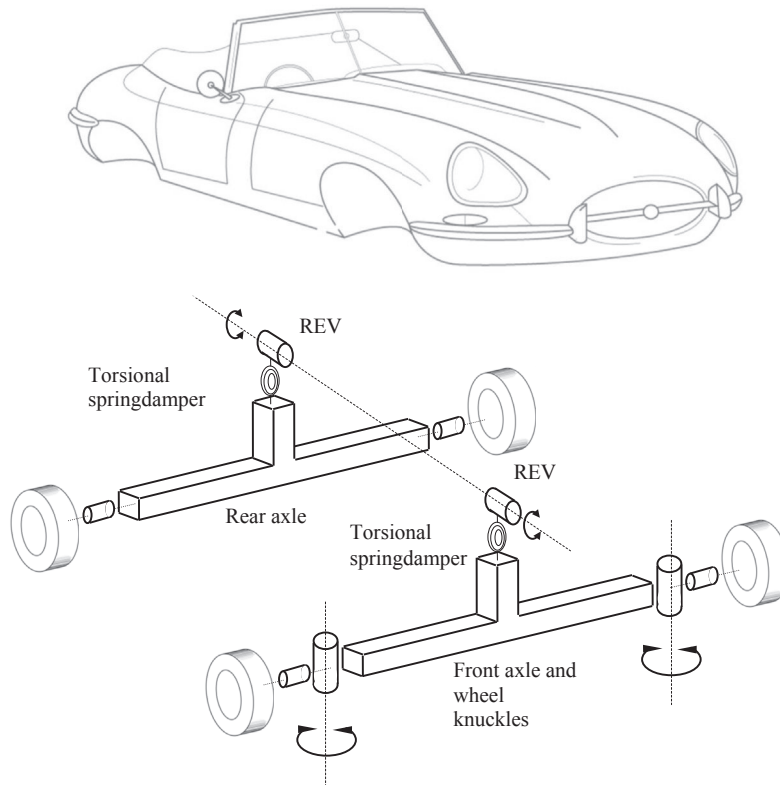
REV, revolute joint; TRANS, translational joint.

related to the roll angle of the vehicle. Spring and damper forces act between the suspensions and the body. Such suspensions have been used on early road vehicles, notably the Lancia Lambda (1908–1927), where it was termed ‘sliding pillar’.

The front wheel knuckles are modelled as separate parts connected to the lumped suspension parts by revolute joints. The steering motion required for each manoeuvre is achieved by applying time-dependent rotational motion inputs about these joints. Each road wheel is modelled as a part connected to the suspension by a revolute joint. The lumped mass model is shown schematically in [Figure 6.7](#).

### 6.4.3 Equivalent roll stiffness model

This model is developed from the lumped mass model by treating the front and rear suspensions as rigid axles connected to the body by revolute joints. The locations of

**FIGURE 6.8**

Equivalent roll stiffness model approach.  
REV, revolute joint.

the joints for the two axles are their respective ‘roll centres’ as described in Chapter 4. A torsional spring is located at the front and rear roll centres to represent the roll stiffness of the vehicle. The determination of the roll stiffness of the front and rear suspensions required an investigation as described in the following section. The equivalent roll stiffness model is shown schematically in [Figure 6.8](#).

Note that this model shows the historical background to much of the current unclear thinking about roll centres and their influence on vehicle behaviour. With beam axles, as were prevalent in the 1920s, this model is a good equivalent for looking at handling behaviour on flat surfaces and ignoring ride inputs. For independent suspensions where the anti-roll geometry remains relatively consistent with respect to the vehicle and where the roll centres are relatively low (i.e. less than around 100 mm for a typical passenger car) – a fairly typical double wishbone setup, for example – then this approximation can be useful despite its systematic inaccuracy. However, drawing general conclusions from such specific circumstances can be

dangerous; vehicles that combine a strut suspension at one end (with very mobile anti-roll geometry) and double wishbone at the other (with relatively constant anti-roll geometry) may not be amenable to such simplifications. With this and all other simplified models, the analyst must consider whether or not the conclusions that are drawn reflect upon the simplification adopted or actually reveal some useful insight. The case study presented at the end of this chapter shows a vehicle that behaves acceptably when modelled in this way.

#### **6.4.4 Swing arm model**

This model is developed from the equivalent roll stiffness mass model by using revolute joints to allow the suspensions for all four wheels to ‘swing’ relative to the vehicle body rather than considering using the suspensions linked on an axle. The revolute joints are located at the instant centres of the actual suspension linkage assembly. These positions are found by modelling the suspensions separately as described in Chapter 4. The swing arm model has an advantage over the roll centre model in that it allows the wheels to change camber angle independently of each other and relative to the vehicle body. The swing arm model is shown schematically in [Figure 6.9](#). Although in the sketch the swing arms are shown with an axis parallel to the vehicle axis, this need not be so in general. Also, although in the sketch the swing arms are shown as a ‘plausible’ mechanical arrangement (i.e. not overlapping) this also need not be so; in general, contact between elements is not modelled for vehicle dynamics studies and in general the instant centres are widely spaced and not necessarily within the physical confines of the vehicle body. The swing arm model has the advantage over the equivalent roll stiffness model in that the heave and pitch ride behaviour can be included.

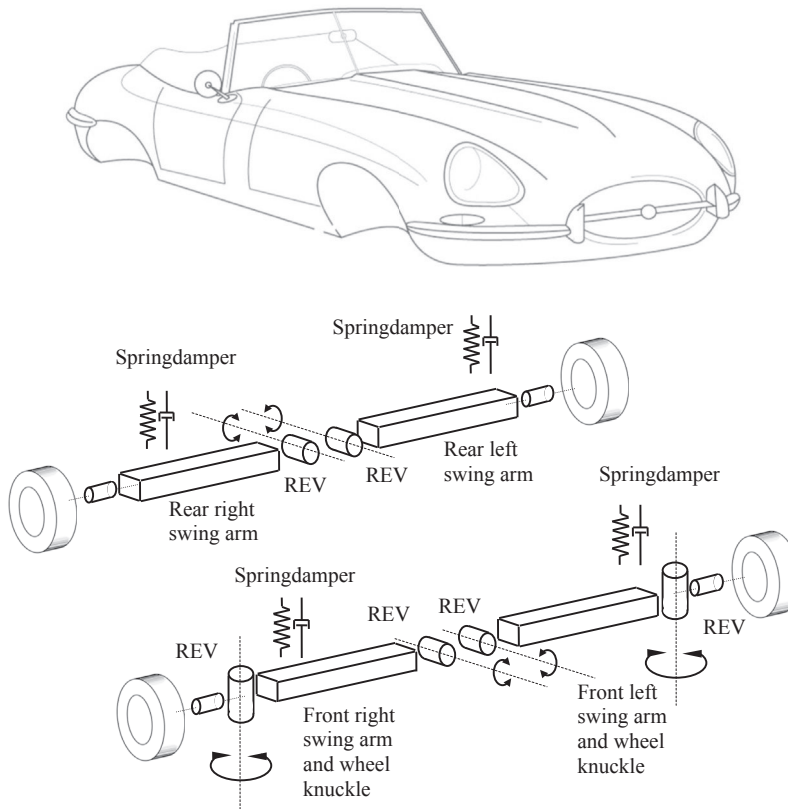
#### **6.4.5 Linkage model**

The model based on linkages as shown in [Figure 6.10](#) is the model that most closely represents the actual vehicle. This sort of vehicle model is the most common approach adopted by MBS software users in the automotive industry, often extending the model definition to include full nonlinear bush characteristics.

A simplification of a model based on linkages is to treat the joints as rigid and generate a kinematic representation of the suspension system. As described in Chapter 4 a double wishbone arrangement is typical of a suspension system that can be modelled in this way and used for handling simulations.

#### **6.4.6 The concept suspension approach**

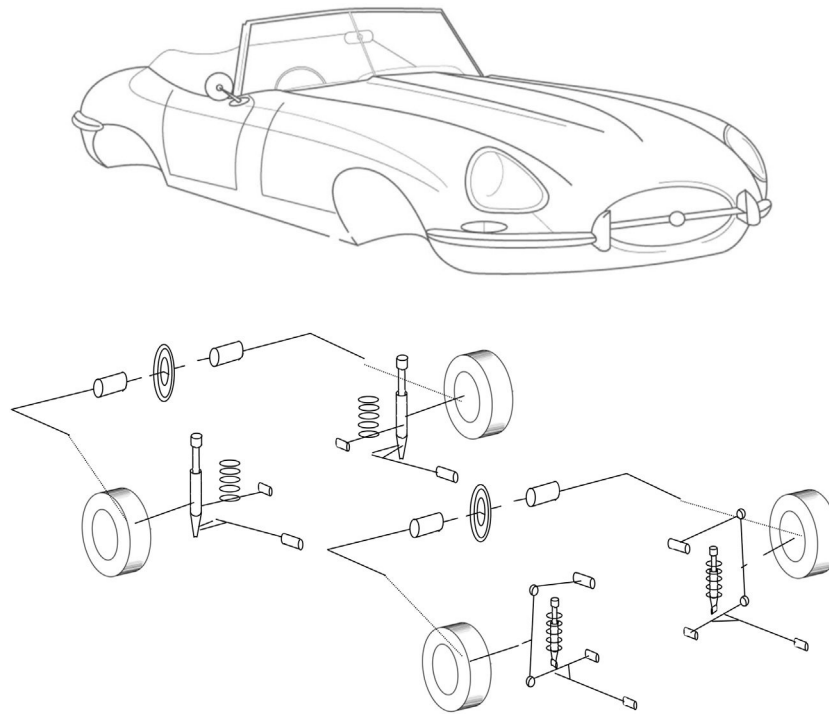
In addition to the four suspension modelling approaches just described, another form of suspension model simplification (Scapaticci et al., 1992) considers an approach where the model contains no elements representing a physical connection between the road wheel and the chassis. Instead the movement of the road wheel with respect to the chassis is described by a functional representation that describes the wheel

**FIGURE 6.9**

Swing arm model approach.  
REV, revolute joint.

centre trajectory and orientation as it moves vertically between full bump and rebound positions. The authors (Scapatucci et al., 1992) describe this approach as the implementation of synthetic wheel trajectories. Such a method has been adopted within MSC ADAMS where the model is referred to as a 'concept suspension' and is the basis of many dedicated vehicle dynamics modelling software tools such as Milliken Research Associate's VDMS, MSC's CarSim, University of Michigan's ArcSim, Leeds University's VDAS. The way in which such a model is applied is summarised in [Figure 6.11](#). In essence the vehicle model containing the concept suspension can be used to investigate the suspension design parameters that can contribute to the delivery of the desired vehicle handling characteristics without modelling of the suspension linkages. In this way, the analyst can gain a clear understanding of the dominant issues affecting some aspect of vehicle dynamics performance. A case study is given in [Section 6.15](#) describing the use of a reduced



**FIGURE 6.10**

Linkage model 'as is' approach.

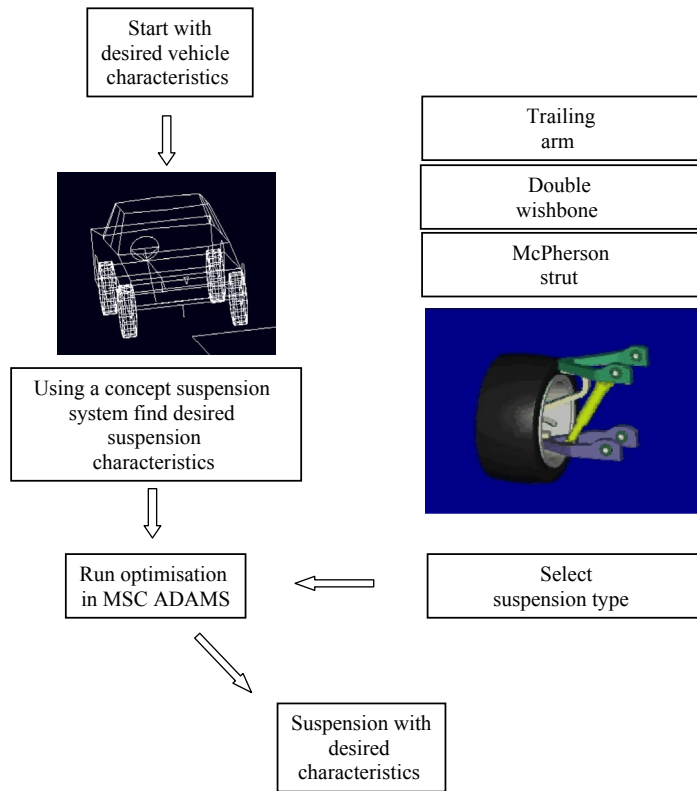
(3 degree-of-freedom) linear model to assess the influence of suspension characteristics on straight line stability. These models belong very firmly in the 'analysis' segment of the overall process diagram described in Chapter 1, Figure 1.6.

The functional representation of the model is based on components that describe effects due to kinematics dependent on suspension geometry and also elastic effects due to compliance within the suspension system. A schematic to support an explanation of the function of this model is provided in [Figure 6.12](#).

If we consider first the kinematic effects due to suspension geometry we can see that there are two variables that provide input to the model:

- $\Delta z$  is the change in wheel centre vertical position (wheel travel)
- $\Delta v$  is the change in steering wheel angle

The magnitude of the wheel travel,  $\Delta z$ , will depend on the deformation of the surface, the load acting vertically through the tyre resulting from weight transfer during a simulated manoeuvre and a representation of the suspension stiffness and damping acting through the wheel centre. The magnitude of the change in steering wheel angle,  $\Delta v$ , will depend on either an open loop fixed time-dependent rotational motion input or a closed loop torque input using a controller to feedback vehicle

**FIGURE 6.11**

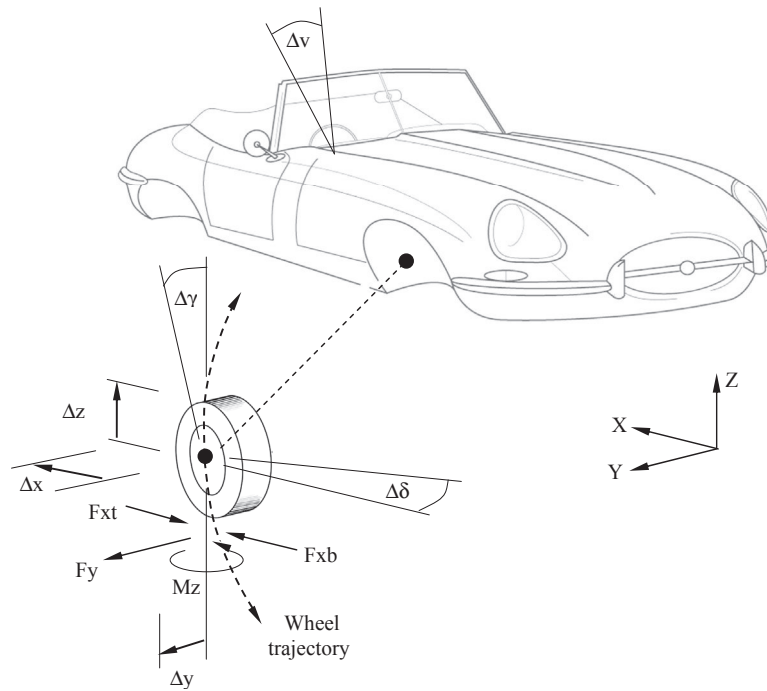
Application of a concept suspension model.

*Provided courtesy of MSC Software.*

position variables so as to steer the vehicle to follow a predefined path. The modelling of steering inputs is discussed in more detail later in this chapter. The dependent variables that dictate the position and orientation of the road wheel are:

- $\Delta x$  is the change in longitudinal position of the wheel
- $\Delta y$  is the change in lateral position (half-track) of the wheel
- $\Delta \delta$  is the change in steer angle (toe in/out) of the wheel
- $\Delta \gamma$  is the change in camber angle of the wheel

The functional dependencies that dictate how the suspension moves with respect to the input variables can be obtained through experimental rig measurements, if the vehicle exists and is to be used as a basis for the model, or by performing simulation with suspension models as described in Chapter 4. For example the dependence of camber angle,  $\Delta \gamma$ , on wheel travel can be derived from the curves plotted for case study 1 in Chapter 4.

**FIGURE 6.12**

Concept suspension system model schematic.

The movement of the suspension due to elastic effects is dependent on the forces acting on the wheel. In their paper (Scapaticci et al., 1992) the authors describe the relationship using the equation shown in (Eqn 6.1) where the functional dependencies due to suspension compliance are defined using the matrix,  $F_E$

$$\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \delta \\ \Delta \gamma \end{bmatrix} = \begin{bmatrix} F_E \end{bmatrix} \begin{bmatrix} F_{xt} \\ F_{xb} \\ F_y \\ M_z \end{bmatrix} \quad (6.1)$$

and the inputs are the forces acting on the tyre:

$F_{xt}$  is the longitudinal tractive force

$F_{xb}$  is the longitudinal braking force

$F_y$  is the lateral force

$M_z$  is the self-aligning moment

Note that the dimensions of the matrix  $F_E$  are such that cross-coupling terms, such as toe change under braking force, can exist. The availability of such data early in the design phase can be difficult but the adoption of such a generalised form

allows the user to speculate on such values and thus use the model to set targets for acceptable behaviour.

## 6.5 Modelling of springs and dampers

### 6.5.1 Treatment in simple models

The treatment of road springs and dampers in a vehicle where the suspensions have been modelled using linkages is generally straightforward. A road spring is often modelled as linear but the damper will usually require a nonlinear representation as discussed in Chapter 3. It is also common for the bump travel limiter to be engaged early and to have both stiffness and damping elements to its behaviour; both those aspects may be modelled using the methods discussed here. The choice of whether to combine them with the road spring and damper forces is entirely one of modelling convenience; the authors generally find the ease of debugging and auditing the model is worth the carriage of two not strictly necessary additional force generating terms.

For the simplified modelling approach used in the lumped mass and swing arm models the road springs cannot be directly installed in the vehicle model as with the linkage model. Consider the lumped mass model when compared with the linkage model as shown in Figure 6.13.

Clearly there is a mechanical advantage effect in the linkage model that is not present in the lumped mass vehicle model. At a given roll angle for the lumped mass model the displacement and hence the force in the spring will be too large when compared with the corresponding situation in the linkage model.

For the swing arm model the instant centre about which the suspension pivots is often on the other side of the vehicle. In this case the displacement in the spring is approximately the same as at the wheel and a similar problem occurs as with the lumped mass model. For all three simplified models this problem can be overcome as shown in Figure 6.14 by using an ‘equivalent’ spring that acts at the wheel centre.

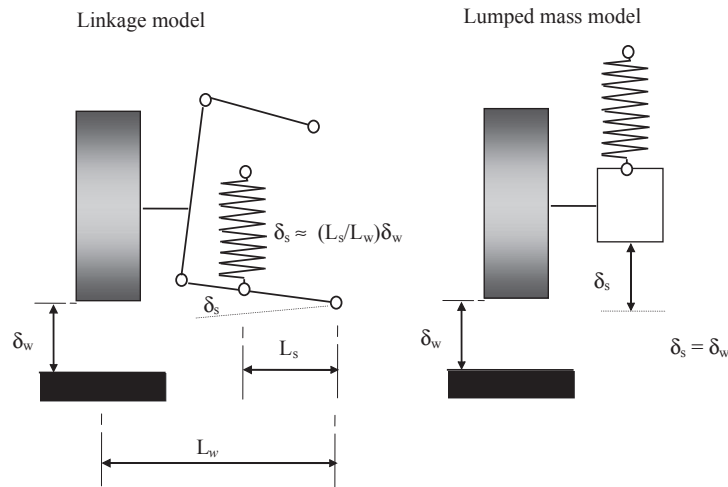
As an approximation, ignoring exact suspension geometry, the expression (Eqn 6.2) can be used to represent the stiffness,  $k_w$ , of the equivalent spring at the wheel

$$k_w = F_w / \delta_w = (L_s / L_w) F_s / (L_w / L_s) \delta_s = (L_s / L_w)^2 k_s \quad (6.2)$$

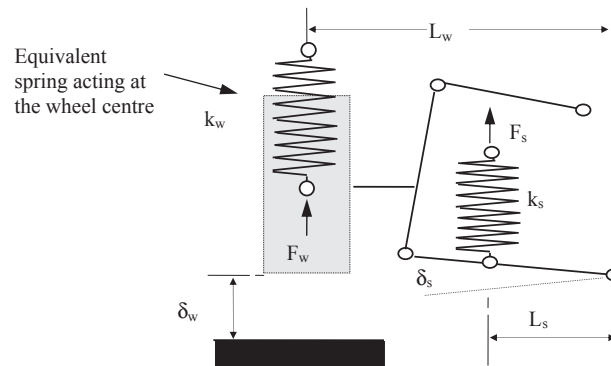
The presence of a square function in the ratio can be considered a combination of both the extra mechanical advantage in moving the definition of spring stiffness to the wheel centre and the extra spring deflection at the wheel centre.

### 6.5.2 Modelling leaf springs

Although the modelling of leaf springs is now rare on passenger cars they are still fitted extensively on light trucks and goods vehicles where they offer the advantage

**FIGURE 6.13**

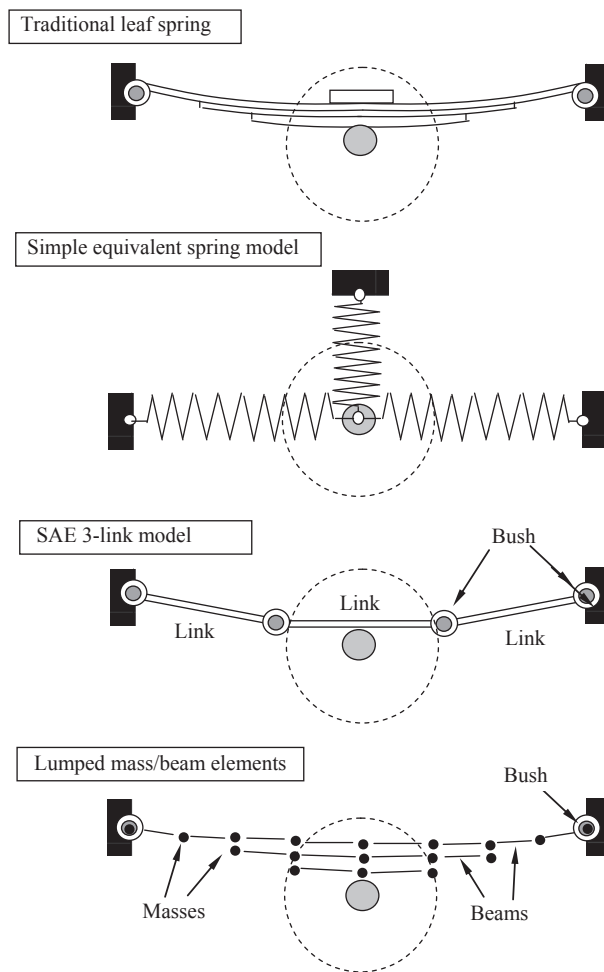
Road spring in linkage and lumped mass models.

**FIGURE 6.14**

Equivalent spring acting at the wheel centre.

of providing relatively constant rates of stiffness for large variations in load at the axle. The modelling of leaf springs has always been more of a challenge in an MBS environment when compared with the relative simplicity of modelling a coil spring. Several approaches may be adopted, the most common of which are shown in Figure 6.15.

Early attempts at modelling leaf springs utilised the simple approach based on equivalent springs to represent the vertical and longitudinal force-displacement characteristic of the leaf spring. On the actual vehicle the leaf springs also contribute to the lateral positioning of the axle, with possible additional support from a panhard

**FIGURE 6.15**

Leaf spring modelling strategies.

rod. Although not shown in [Figure 6.15](#) lateral springs could also be incorporated to represent this.

The next approach is based on modelling the leaf spring as three bodies (SAE 3-link model) interconnected by bushes or revolute joints with an associated torsional stiffness that provides equivalent force-displacement characteristics as found in the actual leaf spring. The last approach shown in [Figure 6.15](#) uses a detailed ‘as is’ approach representing each of the leaves as a series of distributed lumped masses interconnected by beam elements with the correct sectional properties for the leaf. This type of model is also complicated by the need to model the interleaf contact forces between the lumped masses with any associated components

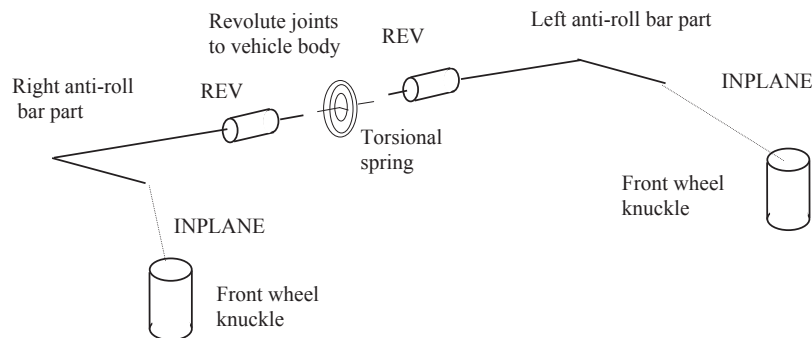
of sliding friction. Assembling such models is greatly eased by the presence of some kind of macro; some software toolkits offer such macros but others leave the user to devise it.

## 6.6 Anti-roll bars

As shown in [Figure 6.16](#) anti-roll bars may be modelled using two parts connected to the vehicle body by revolute joints and connected to each other by a torsional spring located on the centre line of the vehicle. In a more detailed model the analyst could include rubber bush elements rather than the revolute joints shown to connect each side of the anti-roll bar to the vehicle. In this case for a cylindrical bush the torsional stiffness of the bush would be zero to allow rotation about the axis, or could have a value associated with the friction in the joint. In this model the connection of the anti-roll bars to the suspension system is not modelled in detail, rather each anti-roll bar part is connected to the suspension using an inplane joint primitive that allows the vertical motion of the suspension to be transferred to the anti-roll bars and hence produce a relative twisting motion between the two sides.

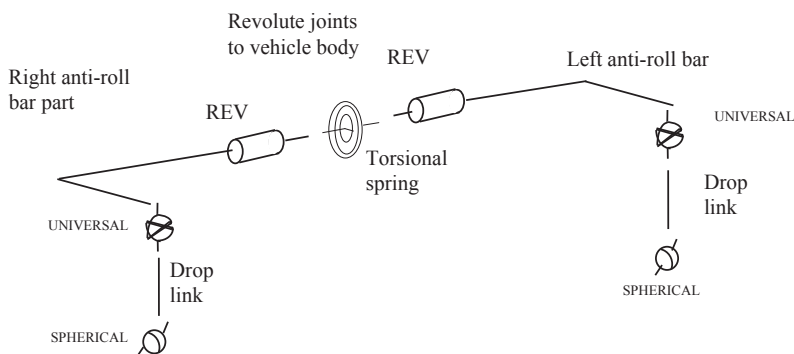
A more detailed approach, shown in [Figure 6.17](#), involves including the drop links to connect each side of the anti-roll bar to the suspension systems. The drop link is connected to the anti-roll bar by a universal joint and is connected to the suspension arm by a spherical joint. This is similar to the modelling of a tie rod as discussed in Chapter 4 where the universal joint is used to constrain the spin of the link about an axis running along its length, this DOF having no influence on the overall behaviour of the model.

The stiffness,  $K_T$ , of the torsional spring can be found directly from fundamental torsion theory for the twisting of bars with a hollow or solid circular cross-section. Assuming here a solid circular bar and units that are consistent with the examples that support this text we have



**FIGURE 6.16**

Modelling the anti-roll bars using joint primitives.  
REV, revolute joint.

**FIGURE 6.17**

Modelling the anti-roll bars using drop links.

REV, revolute joint.

$$K_T = \frac{GJ}{L} \quad (6.3)$$

where

$G$  is the shear modulus of the anti-roll bar material ( $\text{N/mm}^2$ )

$J$  is the polar second moment of area ( $\text{mm}^4$ )

$L$  is the length of the anti-roll bar (mm)

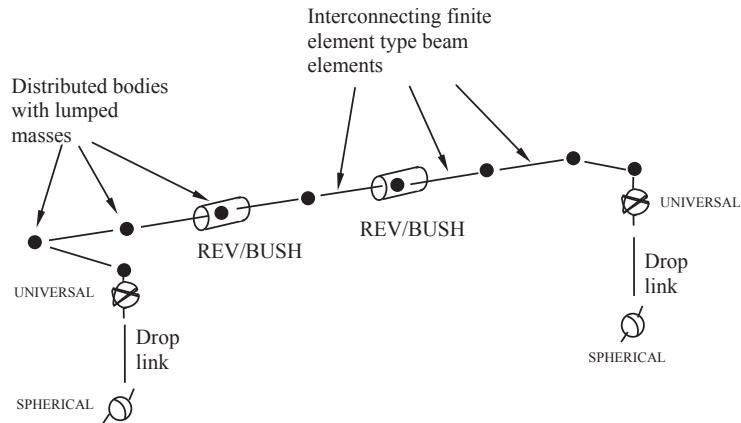
Note that the length  $L$  used in Eqn (6.3) is the length of the bar subject to twisting. For the configuration shown in Figure 6.17 this is the transverse length of the anti-roll bar across the vehicle and does not include the fore-aft lengths of the system that connect to the drop links. These lengths of the bar provide the lever arms to twist the transverse section of bar and are subject to bending rather than torsion. An externally solved FE model could be used to give an equivalent torsional stiffness for a simplified representation such as this.

Given that bending or flexing of the roll bar may have an influence, the next modelling refinement of the anti-roll bar system uses FE beams, of the type described in Chapter 3, to interconnect a series of rigid bodies with lumped masses distributed along the length of the bar. Such sophistication becomes necessary to investigate anti-roll bar interactions with steer torque, or anti-roll bar lateral ‘walking’ problems in the vehicle; in general though, such detail is not required for vehicle behaviour modelling.

Lumped masses must be included where the revolute joints connect the anti-roll bar to the vehicle. Again these joints could be modelled with bushes if needed. A final extension of the model shown in Figure 6.18 would be to model the drop links with lumped masses and beams if the flexibility of these components needed to be modelled.

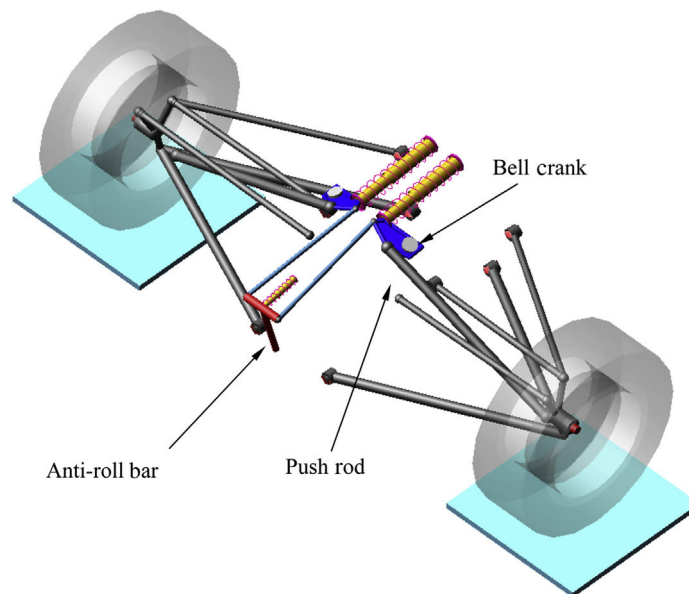
The modelling described so far has been for the modelling of the conventional type of anti-roll bar found on road vehicles. Vehicles with active components in the anti-roll bar system might include actuators in place of the drop links or a coupling device



**FIGURE 6.18**

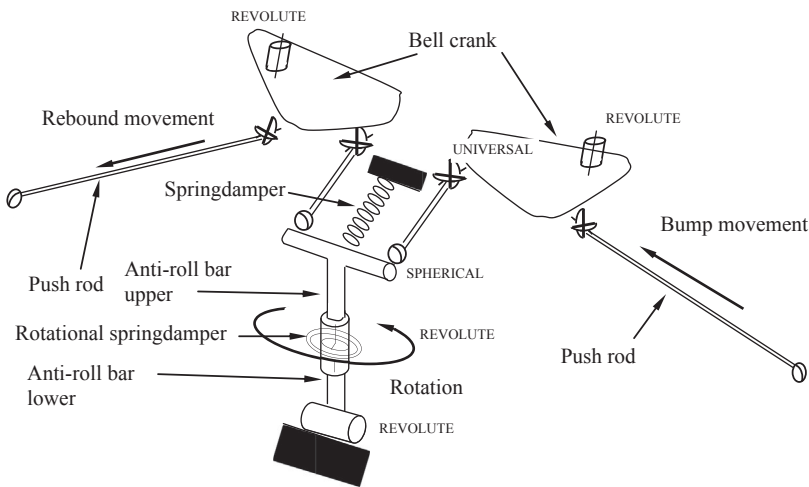
Modelling the anti-roll bars using interconnected finite element beams.  
REV, revolute joint.

connecting the two halves of the system providing variable torsional stiffness at the connection. Space does not permit a description of the modelling of such systems here, but with ever more students becoming involved in motorsport this section will conclude with a description of the type of anti-roll bar model that might be included in a typical student race vehicle. A graphic for the system is shown in [Figure 6.19](#).

**FIGURE 6.19**

Graphic of anti-roll bar in typical student race vehicle.

*Provided courtesy of MSC Software.*

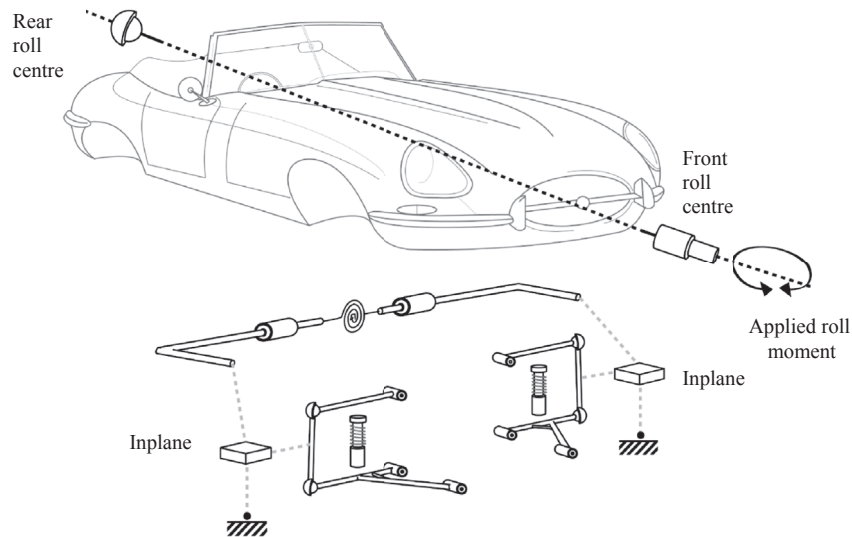
**FIGURE 6.20**

Modelling of anti-roll bar mechanism in student race car.

The modelling of this system is illustrated in the schematic in [Figure 6.20](#) where it can be seen that the anti-roll bar is installed vertically and is connected to the chassis by a revolute joint. The revolute joint allows the anti-roll bar to rock back and forward as the bell cranks rotate during parallel wheel travel but prevents rotation during opposite wheel travel when the body rolls. As the body rolls the torsional stiffness of the anti-roll bar, modelled with the rotational springdamper, resists the pushing motion of one push rod as the suspension moves in bump on one side and the pulling motion as the suspension moves in rebound on the other side. The small springdamper helps to locate the anti roll bar with respect to the vehicle chassis and adds to the heave stiffness and damping. Alternative linkage designs are possible that allow the use of a translational spring element and hence allow independent control of damping in roll compared to damping in heave. Such ‘three spring’ systems are common in higher formula motorsports events when allowed by the rules.

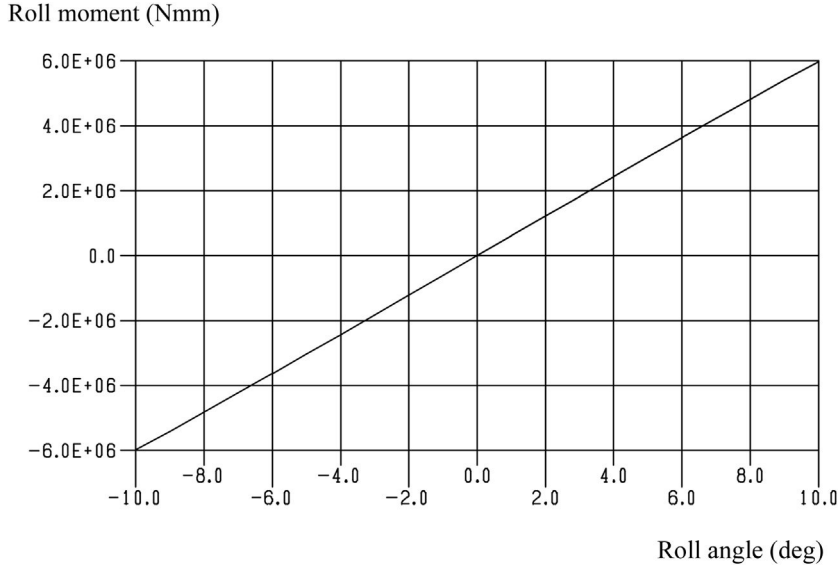
## 6.7 Determination of roll stiffness for the equivalent roll stiffness model

In order to develop a full vehicle model based on roll stiffness it is necessary to determine the roll stiffness and damping of the front and rear suspension elements separately. The estimation of roll damping is obtained by assuming an equivalent linear damping and using the positions of the dampers relative to the roll centres to calculate the required coefficients. If a detailed vehicle model is available, the

**FIGURE 6.21**

Determination of front end roll stiffness.

procedure used to find the roll stiffness for the front suspension elements involves the development of a model as shown in Figure 6.21. This model includes the vehicle body, this being constrained to rotate about an axis aligned through the front and rear roll centres. The roll centre positions can be found using the methods described in Chapter 4. The vehicle body is attached to the ground part by a cylindrical joint located at the front roll centre and aligned with the rear roll centre. The rear roll centre is attached to the ground by a spherical joint in order to prevent the vehicle sliding along the roll axis. A motion input is applied at the cylindrical joint to rotate the body through a given angle. By requesting the resulting torque acting about the axis of the joint it is possible to calculate the roll stiffness associated with the front end of the vehicle. The road wheel parts are not included nor are the tyre properties. The tyre compliance is represented separately by a tyre model and should not be included in the determination of roll stiffness. The wheel centres on either side are constrained to remain in a horizontal plane using inplane joint primitives. Although the damper force elements can be retained in the suspension models they have no contribution to this calculation as the roll stiffness is determined using static analysis. The steering system, although not shown in Figure 6.21, may also be included in the model. If the steering system is present, a zero-motion constraint is needed to lock it in the straight-ahead position during the roll simulation.

**FIGURE 6.22**

Front end roll simulation.

For the rear end of the vehicle the approach is essentially the same as for the front end, with in this case a cylindrical joint located at the rear roll centre and a spherical joint located at the front roll centre.

For both the front and rear models the vehicle body can be rotated through an appropriate angle either side of the vertical. For the example vehicle used in this text the body was rotated  $10^\circ$  each way. The results for the front end model are plotted in Figure 6.22. The gradient at the origin can be used to obtain the value for roll stiffness used in the equivalent roll stiffness model described earlier.

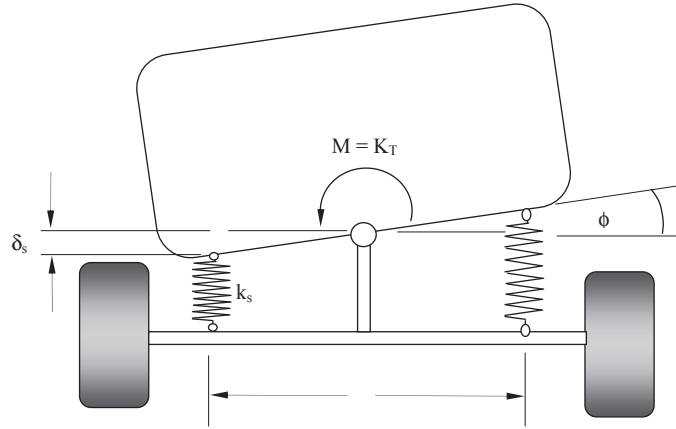
In the absence of an existing vehicle model that can be used for the analysis described in the preceding section, calculations can be performed to estimate the roll stiffness. In reality this will have contributions from the road springs, anti-roll bars and possibly the suspension bushes. Figure 6.23 provides the basis for a calculation of the road spring contribution for the simplified arrangement shown. In this case the inclination of the road springs is ignored and has a separation across the vehicle given by  $L_s$ .

As the vehicle rolls through an angle  $\phi$  the springs on each side are deformed with a displacement,  $\delta_s$ , given by

$$\delta_s = \phi L_s / 2 \quad (6.4)$$

The forces generated in the springs  $F_s$  produce an equivalent roll moment  $M_s$  given by

$$M_s = F_s L_s = k_s \delta_s L_s = k_s \phi L_s^2 / 2 \quad (6.5)$$

**FIGURE 6.23**

Calculation of roll stiffness due to road springs.

The roll stiffness contribution due to the road springs  $K_{Ts}$  at the end of the vehicle under consideration is given by

$$K_{Ts} = M_s / \phi = k_s L_s^2 / 2 \quad (6.6)$$

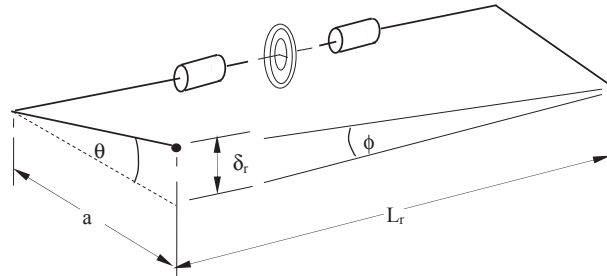
In a similar manner the contribution to the roll stiffness at one end of the vehicle due to an anti-roll bar can be determined as follows (Figure 6.24).

In this case if the ends of the anti-roll bar are separated by a distance  $L_r$  and the vehicle rolls through an angle  $\phi$ , the relative deflection of one end of the anti-roll bar to the other,  $\delta_r$ , is given by

$$\delta_r = a \theta = \phi L_r \quad (6.7)$$

The angle of twist in the roll bar is given by

$$\theta = \frac{TL_r}{GJ} \quad (6.8)$$

**FIGURE 6.24**

Calculation of roll stiffness due to the anti-roll bar.

where, as discussed earlier,  $G$  is the shear modulus of the anti-roll bar material,  $J$  is the polar second moment of area and  $T$  is the torque acting about the transverse section of the anti-roll bar. Note that in this analysis we are ignoring the contribution due to bending. The forces acting at the ends of the anti-roll bar,  $F_r$ , produce an equivalent roll moment  $M_r$  given by

$$M_r = F_r L_r = T L_r / a = \theta GJ/a = \phi L_r GJ/a^2 \quad (6.9)$$

The roll stiffness contribution due to the anti-roll bar,  $K_{Tr}$ , at the end of the vehicle under consideration is given by

$$K_{Tr} = M_r/\phi = L_r GJ/a^2 \quad (6.10)$$

The contribution of both the road springs and the anti-roll bar can then be added, ignoring suspension bushes here, to give the roll stiffness,  $K_T$

$$K_T = K_{Ts} + K_{Tr} \quad (6.11)$$

Note that current practice in vehicles is to have relatively soft springs and fit stiffer anti-roll bars than was the norm some years ago. If vehicles achieve a large proportion of their roll stiffness from anti-roll bars, the subjective phenomenon of ‘roll rock’ (also known as ‘lateral head toss’) becomes problematic. A rule of thumb is that such phenomena begin to emerge when the anti-roll bars form more than about one-third of the overall roll stiffness — in other words if  $K_{Tr}$  is greater than  $0.5 K_{Ts}$ .

---

## 6.8 Aerodynamic effects

Some treatment of aerodynamics is generally given in existing textbooks (Milliken and Milliken, 1998; Gillespie, 1992) dealing with vehicle dynamics. Other textbooks are dedicated to the subject (Hucho, 1998). The flow of air over the body of a vehicle produces forces and moments acting on the body resulting from the pressure distribution (form) and friction between the air and surface of the body. The forces and moments are considered using a body-centred reference frame where longitudinal forces (drag), lateral forces, and vertical forces (lift or down thrust) will arise. The aerodynamic moments will be associated with roll, pitch and yaw rotations about the corresponding axes.

Current practice is generally to ignore aerodynamic forces for the simulation of most proving ground manoeuvres but for some applications and classes of vehicles this is clearly not representative of the vehicle dynamics in the real world, for example winged vehicles. Prior to rules limiting downforce, it was often said that for some vehicles of this type the down thrust is so great that this could overcome the weight of a vehicle, allowing it for example to drive upside down through a tunnel, although this has never been demonstrated.

The lack of speed limits on certain autobahns in Germany also means that a vehicle manufacturer selling a high-performance vehicle to that market will need

to test the vehicle at speeds well over twice the legal UK limit. The possibility of aerodynamic forces at these high speeds destabilising the vehicle needs to be investigated and where physical testing is to be done, equivalent computer simulation is also desirable. Other effects such as side gusting are also tested for and have been simulated by vehicle dynamicists in the past.

An approach that has been commonly used is to apply forces and moments to the vehicle body using measured results from wind tunnel testing, in look-up tables. As the vehicle speed and the attitude of the body changes during the simulation, the forces and moments are interpolated from the measured data and applied to the vehicle body. A difficulty with such an approach is that the measured results are for steady state in each condition and that transient effects are not included in the simulation. Consideration has been given to the use of a computational fluid dynamics (CFD) programme to calculate aerodynamic forces and moments in parallel with (co-simulation) an MBS programme solving the vehicle equations of motion. The problem at the current time with this approach is the mismatch in the computation time for both methods. MBS models of a complete vehicle can simulate vehicle handling manoeuvres in seconds, or even real time, whereas complex CFD models can involve simulation times running to days. CFD methods can now handle aerodynamic transient effects (e.g. vortex shedding) but the timescale mismatch remains. Thus there is no realistic prospect of the practical use of the interaction of transient aerodynamics effects and vehicle dynamics being modelled in the near future. However, genuine transient aerodynamic effects, such as those involved in so-called ‘aeroelastic flutter’ — an unsteady aerodynamic flow working in sympathy with a structural resonance — are extremely rare in ground vehicles.

In order to introduce readers to the fundamentals, consider a starting point where it is intended only to formulate an aerodynamic drag force acting on the vehicle body.

The drag force,  $F_D$ , can be considered to act along a line of action parallel to the ground and in the vehicle plane of symmetry having the following formulation:

$$F_D = \frac{1}{2} \frac{\rho V^2 C_D A}{GC} \quad (6.12)$$

where

$C_D$  = the aerodynamic drag coefficient

$\rho$  = the density of air

$A$  = the frontal area of the vehicle (projected onto a yz plane)

$V$  = the velocity of the vehicle in the direction of travel

$GC$  = a gravitational constant.

The gravitational constant is included in Eqn (6.12) to remind readers that this is a dynamic force. If the model units are SI then  $GC$  is equal to 1. If as commonly used the model units for length are in millimetres then  $GC$  is equal to 1000. When formulating the aerodynamic drag force more generally it should be considered that the overall force acts along a line that does not typically pass through the centre of

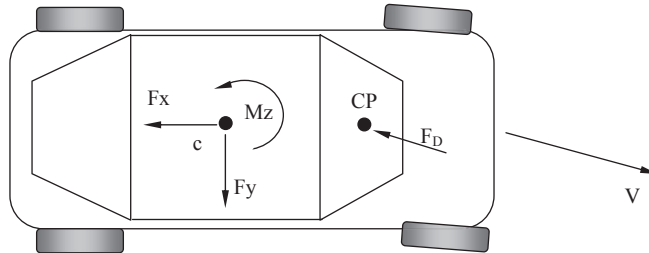
gravity and therefore moments as well as forces are necessary to capture to the correct effects on the vehicle. In the straight line condition generally a pitch moment is all that is needed to reposition the drag vector correctly. It is also generally true that the forces and moments change as the vehicle aerodynamic attitude changes, and one possible way of capturing this is to have an attitude-sensitive formulation for the aerodynamic coefficients. Within MBS models it is difficult to capture effects such as the apparent curvature of the flow when a car is travelling on a tight arc in still air, but tight arcs are typically not possible at anything except the lowest speeds and so the effect is not important in terms of the overall behaviour of the system. At higher speeds, in still air, the body slip angle may be considered a useful surrogate for the aerodynamic yaw angle. In this case, a relatively simple formulation can capture the increase in, for example, drag coefficient with aerodynamic yaw angle:

$$F_D = \frac{1}{2} \frac{\rho V^2 (C_{D0} + C_{D\beta} \cdot \beta) A}{GC} \quad (6.13)$$

where

- $C_{D0}$  = drag coefficient at zero aerodynamic yaw angle
- $C_{D\beta}$  = drag coefficient sensitivity to aerodynamic yaw angle
- $\beta$  = aerodynamic yaw angle (or body slip angle surrogate)

This formulation avoids the need for a knowledge of changing aerodynamic frontal area with attitude changes, which can be hard to obtain, and can be calculated from a fairly ordinary set of wind tunnel results. For the position shown in [Figure 6.25](#) it is clear that for anything other than straight line motion it is going to be necessary to model the forces as components in the body-centred axis system. If we consider the vehicle moving only in the xy plane then this is going to require at least the formulation of a longitudinal force,  $F_x$ , a lateral force,  $F_y$  and a yawing moment,  $M_z$ , all in a body-centred axis system, usually located at the mass centre. Wind tunnel testing or CFD analysis is able to yield coefficients for all six possible forces and moments acting on the body, referred back to the mass centre. Note that for passenger vehicles it is typical that the aerodynamic yaw moment is as shown in the diagram, i.e. is such to make the vehicle turn away from the wind. For other



**FIGURE 6.25**

Application of aerodynamic drag force.



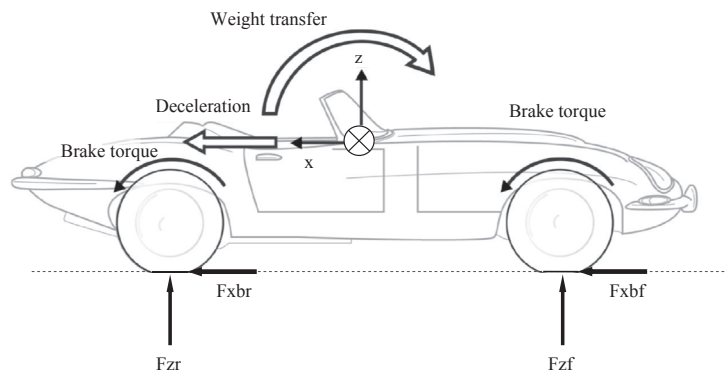
vehicles this may not be true and individual research on the vehicle in question is needed. Note also that over body slip angles of interest, the characteristics are often observed to be usefully linear and so a simple formulation of the type proposed in Eqn (6.13) is sufficient for all the aerodynamic forces and moments; this presumption should be verified when the vehicle has an unusual configuration or operates at extremely large slip angles.

## 6.9 Modelling of vehicle braking

In Chapter 5 the force and moment generating characteristics of the tyre were discussed and it was shown how the braking force generated at the tyre contact patch depends on the value of the slip ratio which varies from zero for a free rolling wheel to unity for a braked and fully locked wheel. In this section we are not so much concerned with the tyre, given that we would be using a tyre model interfaced with our full vehicle model to represent his behaviour. Rather we now address the modelling of the mechanisms used to apply a braking torque acting about the spin axis of the road wheel that produces the change in slip ratio and subsequent braking force.

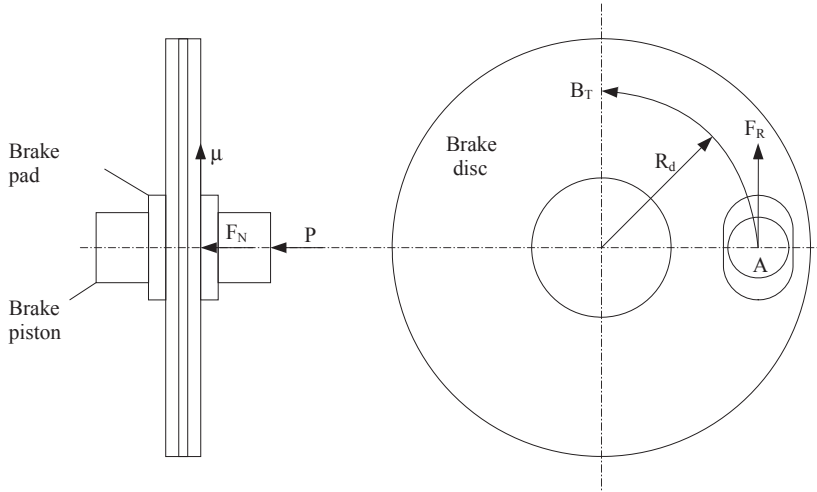
Clearly as the vehicle brakes, as shown in Figure 6.26, there is weight transfer from the rear to the front of the vehicle. Given what we know about the tyre behaviour the change in the vertical loads acting through the tyres will influence the braking forces generated. As such the braking model may need to account for real effects such as proportioning the braking pressures to the front and rear wheels or the implementation of ABS. Before any consideration of this we need to address the mechanism to model a braking torque acting on a single road wheel.

If we consider a basic arrangement, the mechanical formulation of a braking torque, based on a known brake pressure, acting on the piston can be derived from Figure 6.27.



**FIGURE 6.26**

Braking of a full vehicle.



**FIGURE 6.27**

Braking mechanism.

The braking torque  $B_T$  is given by

$$B_T = n \mu p A R_d \quad (6.14)$$

where

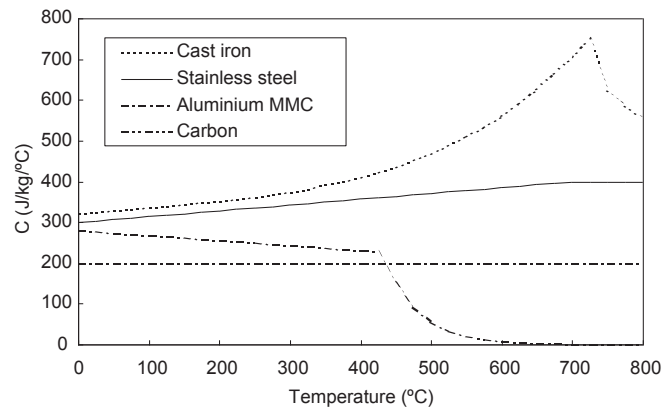
- $n$  = the number of friction surfaces (pads)
- $\mu$  = the coefficient of friction between the pads and the disc
- $p$  = the brake pressure
- $A$  = the brake piston area
- $R_d$  = the radius to the centre of the pad

Note that depending on the sophistication of the model, the coefficient of friction  $\mu$  may be constant or defined as a run-time variable as a function of brake rotor temperature. Note also that brake pad/rotor friction is substantially more Newtonian than tyre friction and thus subject to much less complication than the rubber friction previously discussed. Some kind of friction model is still needed to prevent the direct application of a brake torque spinning the wheel backwards, which does not happen in practice. Figure 6.28 shows typical brake friction vs temperature characteristics for different brake rotor materials. Brake rotor temperature,  $T$ , can be calculated using the differential equation

$$T = T_0 + \frac{1}{mc} \int_0^t [B_T(t) \omega(t) - hA_c (T(t) - T_{env})] dt \quad (6.15)$$

where

- $T_0$  = Initial brake rotor temperature (K)
- $\omega(t)$  = Brake rotor spin velocity ( $\text{rads s}^{-1}$ )

**FIGURE 6.28**

Specific heat capacity,  $c$ , versus temperature,  $T$  (Farr, 1999).

$t$  = time (s)

$h$  = Brake rotor convection coefficient ( $\text{W m}^{-2} \text{K}^{-1}$ )

$A_c$  = Convective area of brake disc ( $\text{m}^2$ )

$T_{\text{env}}$  = Environmental temperature (K)

$m$  = mass of brake rotor (kg)

$c$  = specific heat capacity of brake rotor ( $\text{J kg}^{-1} \text{K}^{-1}$ )

$BT(t)$  = Brake Torque (Nm)

For the most common brake rotor material, cast iron, the specific heat versus temperature characteristic can be approximated in the working range (0 to 730 °C) by the expression

$$c = 320 + 0.15T + 1.164 \times 10^{-9} T^4 \quad (6.16)$$

Note that in the above expression, temperature  $T$  is in centigrade (Celsius) and not Kelvin. The brake torque and temperature models may be used easily within MBS models using a combination of design variables and run-time variables as shown in Table 6.1 where we are using an input format that corresponds to a command language used in MSC ADAMS. Note the need for an explicit iteration since the temperature depends on the heat capacity and the heat capacity depends on the temperature. When modelling such behaviour in a spreadsheet, it is sufficient to refer to the temperature of the preceding time step. Although this is possible within many MBS packages, it can be awkward to implement and can also lead to models with some degree of numerical delicacy. Output from the model during a single braking event is shown in Figure 6.29.

Note also that it is common practice within brake manufacturers to separate the brake energising event from the brake cooling event for initial design calculations, leading to a systematic overestimation of the temperature during fade/recovery testing. This conservative approach is unsurprising given the consequences of brake

**Table 6.1** A Brake Rotor Temperature Model Based on Brake Torque

```

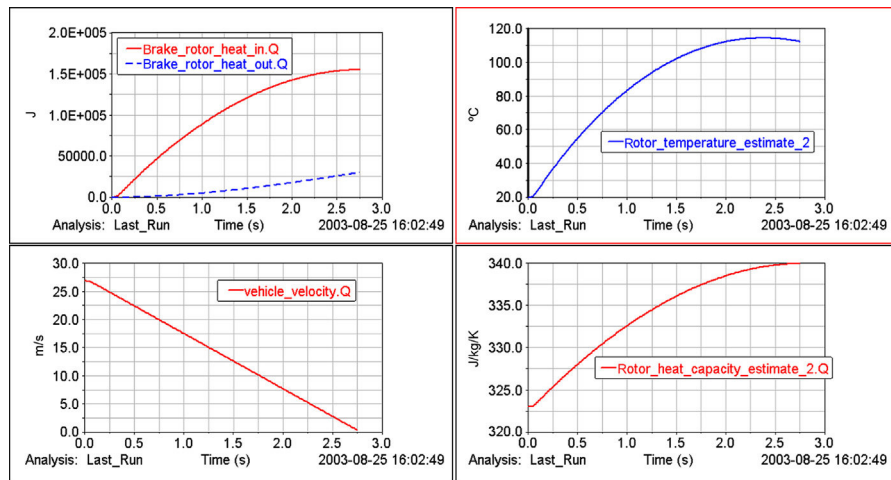
!----- Function definitions -----!
!
part create equation differential_equation &
  differential_equation_name = .model_1.brake_heating_integral &
  adams_id = 2 &
  comments = "Brake Heat Input Integral" &
  initial_condition = 0.0 &
  function = "VARVAL(Brake_Torque)*VARVAL(vehicle_velocity)/0.3" &
  implicit = off &
  static_hold = off

data_element create variable &
  adams_id = 102 &
  variable_name = brake_rotor_heat_in &
  function = "DIF(2)"
!
data_element create variable &
  variable_name = "rotor_temperature_kelvin_estimate_1" &
  function = "T_env + VARVAL(brake_rotor_heat_in)/(rotor_mass*350)"
!
data_element create variable &
  variable_name = "rotor_temperature_estimate_1" &
  function = "VARVAL(rotor_temperature_kelvin_estimate_1)-273"
!
data_element create variable &
  variable_name = "rotor_heat_capacity_estimate_2" &
  function = "320 + 0.15*VARVAL(rotor_temperature_estimate_1)", &
  " + 1.164E-9*VARVAL(rotor_temperature_estimate_1)**4"
!
part create equation differential_equation &
  differential_equation_name = .model_1.brake_cooling_integral &
  adams_id = 3 &
  comments = "Brake Heat Rejection Integral" &
  initial_condition = 0.0 &
  function = "hAc*(VARVAL(rotor_temperature_kelvin_estimate_1)-T_env)" &
  implicit = off &
  static_hold = off

data_element create variable &
  adams_id = 103 &
  variable_name = brake_rotor_heat_out &
  function = "DIF(3)"
!
data_element create variable &
  variable_name = "rotor_temperature_kelvin_estimate_2" &
  function = "T_env +", &
  " (VARVAL(brake_rotor_heat_in) - VARVAL(brake_rotor_heat_out) )/", &
  "(rotor_mass*VARVAL(rotor_heat_capacity_estimate_2)+0.001)"
!
data_element create variable &
  variable_name = "rotor_temperature_estimate_2" &
  function = "VARVAL(rotor_temperature_kelvin_estimate_2)-273"

```

system under design. The example given is a relatively simple one, with convection characteristics that are independent of vehicle velocity and no variation of brake friction with brake temperature. Although in practice these simplifications render the results slightly inaccurate, they are useful when used for comparative purposes — for example, if the brake temperature model is used with an ESP algorithm it can rank control strategies in terms of the energy added to individual brake rotors.

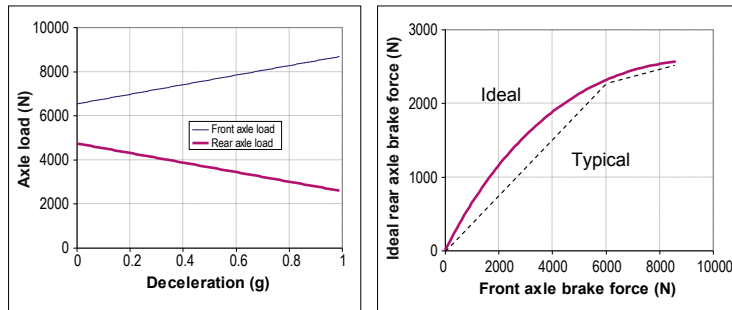
**FIGURE 6.29**

Output from the brake temperature model shown in Table 6.1 during a 60 mph – 0 stop.

Similar modelling is of course possible for other frictional systems within the vehicle, such as drive or transmission clutches. Typical values of the convection constant,  $hAc$ , are around 150 W/K for a front disc brake installation, around 80 W/K for a rear disc brake installation and as low as 20 W/K for a rear drum brake.

A further key factor in modelling brake performance is the distribution of brake torques around the vehicle. While decelerating, the vertical loads on the axles change as described in Section 4.8.1 due to the fact that the mass centre of the vehicle is above the ground.

It may be presumed that for ideal braking, the longitudinal forces should be distributed according to the vertical forces. Using the above expressions, the graphs in Figure 6.30 can be calculated for vertical axle load versus deceleration. Knowing the total force necessary to decelerate the vehicle it is possible to calculate the horizontal forces for ‘ideal’ (i.e. matched to vertical load distribution) deceleration. Plotting rear force against front force leads to the characteristic curve shown in Figure 6.30. However, in general it is not possible to arrange for such a distribution of force and so the typical installed force distribution is something like that shown by the dashed line in the figure. Note that the ideal distribution of braking force varies with loading condition and so many vehicles have a brake force distribution that varies with vehicle loading condition. For more detailed information on brake system performance and design, Limpert (1999) gives a detailed breakdown of performance characteristics and behaviour, all of which may be incorporated within a MBS model of the vehicle using an approach similar to that shown in Table 6.1 if desired – which is to say the formulation of state variables and their manipulation into quantities of interest, followed by their use to modify applied forces in the model.

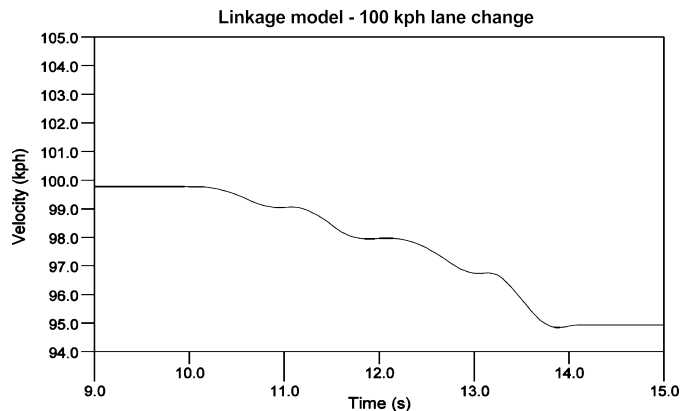
**FIGURE 6.30**

Force distribution for ideal and typical braking events.

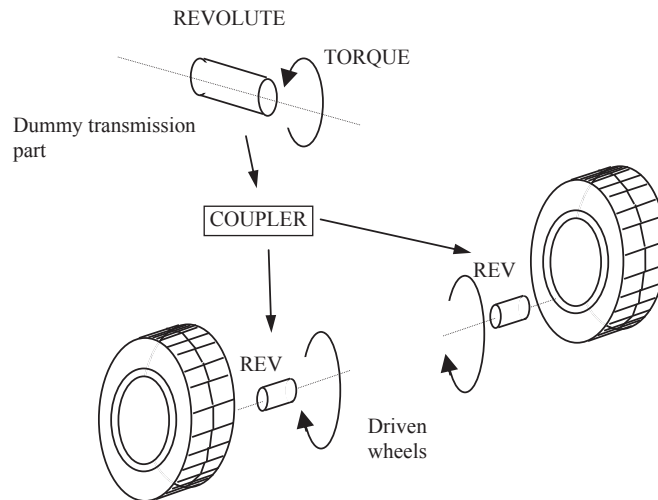
## 6.10 Modelling traction

For some simulations it is necessary to maintain the vehicle at a constant velocity. Without some form of driving torque the vehicle will 'coast' through the manoeuvre using the momentum available from the velocity defined with the initial conditions for the analysis. Ignoring rolling resistance and aerodynamic drag will reduce losses but the vehicle will still lose momentum during the manoeuvre due to the 'drag' components of tyre-cornering forces generated during the manoeuvre. An example is provided in [Figure 6.31](#) where for a vehicle lane change manoeuvre it can be seen that during the 5 s taken to complete the manoeuvre the vehicle loses about 5 km/h in the absence of any tractive forces at the tyres.

As discussed in Chapter 3 single-purpose general-purpose programs such as ADAMS/Car include a driveline model as part of the full vehicle as a means to impart

**FIGURE 6.31**

Loss in velocity as vehicle 'coasts' through the lane change manoeuvre.

**FIGURE 6.32**

Simple drive torque model.  
REV, revolute joint.

torques to the road wheels and hence generate tractive driving forces at the tyres. Space does not permit a detailed consideration of driveline modelling here but as a start a simple method of imparting torque to the driven wheels is shown in Figure 6.32.

The rotation of the front wheels is coupled to the rotation of the dummy transmission part as shown in Figure 6.32. The coupler introduces the following constraint equation:

$$s_1 \cdot r_1 + s_2 \cdot r_2 + s_3 \cdot r_3 = 0 \quad (6.17)$$

where  $s_1$ ,  $s_2$  and  $s_3$  are the scale factors for the three revolute joints and  $r_1$ ,  $r_2$  and  $r_3$  are the rotations. In this example suffix 1 is for the driven joint and suffixes 2 and 3 are for the front wheel joints. The scale factors used are  $s_1 = 1$ ,  $s_2 = 0.5$  and  $s_3 = 0.5$  on the basis that 50% of the torque from the driven joint is distributed to each of the wheel joints. This gives a constraint equation linking the rotation of the three joints:

$$r_1 = 0.5 r_2 + 0.5 r_3 \quad (6.18)$$

Note that this equation is not determinant. For a given input rotation  $r_1$ , there are two unknowns  $r_2$  and  $r_3$  but only the single equation. In order to solve  $r_2$  and  $r_3$  this equation must be solved simultaneously with all the other equations representing the motion of the vehicle. This is important particularly during cornering where the inner and outer wheels must be able to rotate at different speeds; the constraint equation thus reflects a perfect open differential. A Coulomb style frictional force may also be introduced to reflect the imperfections in real differentials; it may be scaled with applied driveline torque to reflect the function of a Torsen-style ('torque biasing') differential that may or may not have symmetric operation; it may be

scaled in response to some externally calculated solution variable to reflect the operation of some electronically controlled proportional clutch across the differential. Alternatively a torque may be applied that is proportional to the speed difference between the two output shafts and thus represents a viscous style element. All these representations may be combined in a single model to represent the components present in the real vehicle.

## 6.11 Other driveline components

The control of vehicle speed is significantly easier than the control of vehicle path inside a vehicle dynamics model. In the real vehicle, speed is influenced by the engine torque, brakes and aerodynamic drag. As discussed earlier these are relatively simple devices to represent in an MBS model, with the exception of torque converters and turbochargers. Even these latter components can be represented using differential equations of the form:

$$T_{\text{BOOST}} = T_2 \cdot \hat{T}_{\text{BOOST}} \quad (6.19)$$

$$\frac{d}{dt}(T_2) = \frac{T_1}{k_2} \cdot (t_{\text{boost}} - T_2) \quad (6.20)$$

$$\frac{d}{dt}(T_1) = k_1 \cdot (t_{\text{boost}} - T_1) \quad (6.21)$$

where  $\hat{T}_{\text{BOOST}}$  is the maximum possible torque available,  $t_{\text{boost}}$  is the throttle setting to be applied to the boost torque (which may be different to the throttle setting applied to the normally aspirated torque to model the rapid collapse of boost off-throttle) and  $k_{1,2}$  are mapped, state-dependent values to calibrate the behaviour of the engine (i.e. large delays at low engine speed, reducing delays with rising engine speed). An example of the statements required to model the resulting torque is shown in [Table 6.2](#).

In this example the variable throttle runs from  $-0.3$  to  $1.0$  to simulate overrun torque. The variable `boost_throttle` is a clipped version from  $0$  to  $1.0$  since no turbocharger boost is available on overrun. The variable `throttle_derivative` is the first time derivative of throttle. All the other variables (varvals) are retrieved from the relevant curves (splines) plotted in [Figure 6.33](#).

The delays inherent in a torque converter are amenable to such modelling techniques using typical torque converter characteristic data in a similar empirical manner.

Once the physical elements of the system are modelled, the task of modelling the driver behaviour is largely similar to that for path following described later. In order to represent, for example, the effect of a driver using the throttle to maintain a steady velocity through a manoeuvre, a controller can be developed to generate the torque shown in [Figure 6.33](#).



**Table 6.2** Example MSC ADAMS Command Statements for an Empirical Mean-State Turbocharger

```

!      -- First First Order Differential Equation --
part create equation differential_equation &
    differential_equation_name = turbo_lag_equation_1 &
    adams_id = 12 &
    comments = "Lag Equation 1 - Explicit" &
    initial_condition = 0.0 &
    function = "varval(K1_now) * ( varval(boost_throttle)*100-DIF(12) )" &
    implicit = off &
data_element create variable &
    variable_name=K2 &
    function="STEP(varval(throttle_derivative),", &
        "-10, 100.0,", &
        "-1, (DIF(12))/varval(K2_divisor_now)", &
        ")"

!      -- Second First Order Differential Equation --
part create equation differential_equation &
    differential_equation_name = turbo_lag_equation_2 &
    adams_id = 13 &
    comments = "Lag Equation 2 - Explicit" &
    function = "varval(K2) * ( varval(boost_throttle)*100-DIF(13) )" &
    implicit = off &
data_element create variable &
    variable_name = boost_torque_scaling &
    function="DIF(13)/100"

!      -- Sum both normally aspirated and turbocharged (delayed) component
data_element create variable &
    variable_name = prop_torque &
    function = "(", &
        " VARVAL(na_engine_torque)*VARVAL(throttle)*1000", &
        "+VARVAL(boosted_engine_torque)*VARVAL(boost_torque_scaling)*1000", &
        ")"

```

A simple but workable solution is to model the driving torque  $T$ , with the following formulation:

$$T = K * (V_s - V_a) * \text{STEP}(\text{Time}, 0, 0, 1, 1) \quad (6.22)$$

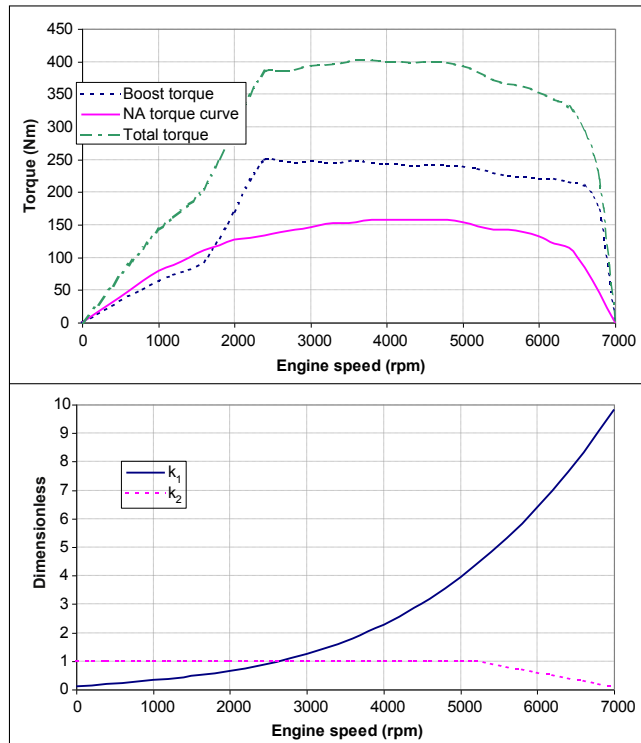
where:

$K$  = a proportional gain that is tuned to stabilize the torque

$V_s$  = the desired velocity for the simulation

$V_a$  = the forward velocity of the vehicle, which can be obtained using a system variable

It can be seen that the expression delivers a drive torque proportional to the speed error. The purpose of the STEP FUNCTION is to define a change of state in the expression that is continuous.

**FIGURE 6.33**

Empirical mean-state turbocharger model.

The step function can be used to factor a force function by ramping it on over a set time period. In this case the driving torque is being switched on between time = 0 and time = 1 s. This is important because it is common to perform an initial static analysis of the vehicle at time = 0 when  $V_a = 0$  and the torque must not act.

As can be seen a 'reference' (desired) state is needed, an error term is defined by the difference between the current state and the reference state and finally, responses to that in terms of throttle or brake application to adjust the speed back towards the reference value. There are two possible approaches; the simplest provides a speed 'map' for the track, similar to the curvature map description of it. More elaborately, it is possible to examine the path curvature map locally and decide (through knowledge of the ultimate capabilities of the vehicle, perhaps) whether or not the current speed is excessive, appropriate or insufficient for the local curvature and use brakes or engine appropriately. For the development of vehicles, open-loop throttle or brake inputs may be preferable and are sometimes mandated in defined test manoeuvres, rendering the whole issue of speed control moot.

In many ways the skill of the competition driver lies entirely in this ability to judge speed and adjust it appropriately. It is also a key skill to cultivate for limit

handling development and arguably for road driving too, so as not to arrive at hazards too rapidly to maintain control of the vehicle. For this functionality, some form of preview is essential. It is both plausible and reasonable to run a ‘here and now at the front axle’ model for the path follower and a ‘previewing’ speed controller within the same model.

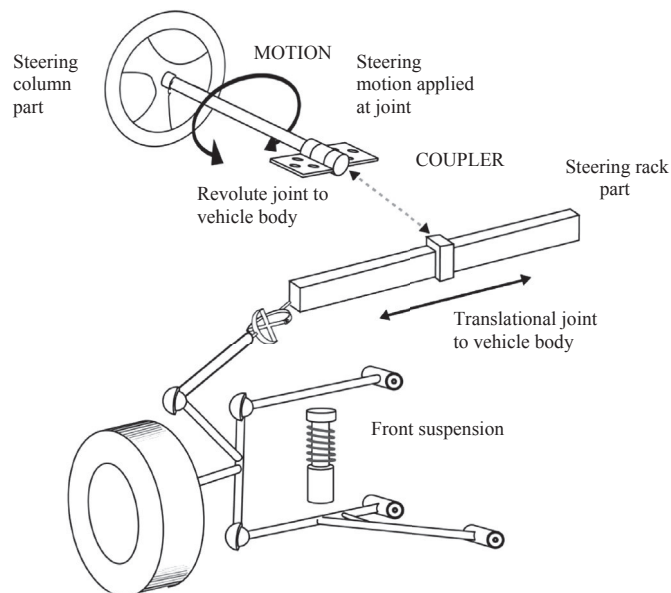
## 6.12 The steering system

### 6.12.1 Modelling the steering mechanism

There are a number of steering system configurations available for cars and trucks based on linkages and steering gearboxes. The treatment in the following sections is limited to a traditional rack and pinion system.

For the simple full vehicle models discussed earlier, such as that modelled with lumped mass suspensions, there are problems when trying to incorporate the steering system. Consider first the arrangement of the steering system on the actual vehicle and the way this can be modelled on the detailed linkage model as shown in Figure 6.34. In this case only the suspension on the right hand side is shown for clarity.

The steering column is represented as a part connected to the vehicle body by a revolute joint with its axis aligned along the line of the column. The steering inputs



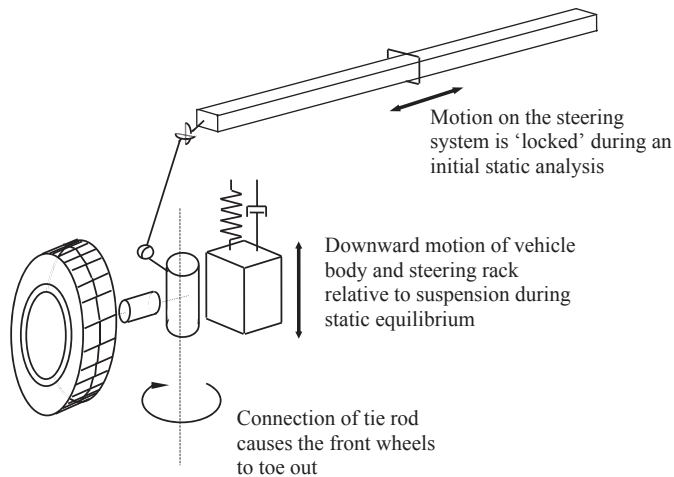
**FIGURE 6.34**

Modelling the steering system.

required to manoeuvre the vehicle are applied as motion or torque inputs at this joint. The steering rack part is connected to the vehicle body by a translational joint and connected to the tie rod by a universal joint. The translation of the rack is related to the rotation of the steering column by some kind of coupling statement that defines the ratio; such constructs are common to most general purpose software packages.

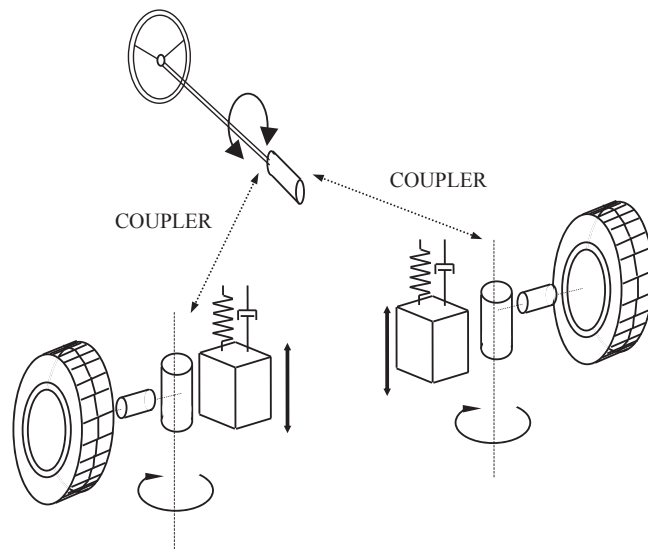
Attempts to incorporate the steering system into the simple models using lumped masses, swing arms and roll stiffness will be met with a problem when connecting the steering rack to the actual suspension part. This is best explained by considering the situation shown in [Figure 6.35](#).

The geometry of the tie rod, essentially the locations of the two ends, is designed with the suspension linkage layout and will work if implemented in an ‘as is’ model of the vehicle including all the suspension linkages. Physically connecting the tie rod to the simple suspensions does not work. During an initial static analysis of the full vehicle, to settle at kerb height, the rack moves down with the vehicle body relative to the suspension system. This has a pulling effect, or pushing according to the rack position, on the tie rod that causes the front wheels to steer during the initial static analysis. The solution to this is to establish the relationship between the steering column rotation and the steer change in the front wheels and to model this as a direct ratio using two coupler statements to link the rotation between the steering column and each of the front wheel joints as shown in [Figure 6.36](#). Advanced steer-by-wire research platforms, such as the ‘P1’ vehicle in use at Stanford University, use this layout in a physical vehicle with a software controller solving the constraint equation by driving electric motors to move the steered road wheels.



**FIGURE 6.35**

Toe change in front wheels at static equilibrium for simple models.

**FIGURE 6.36**

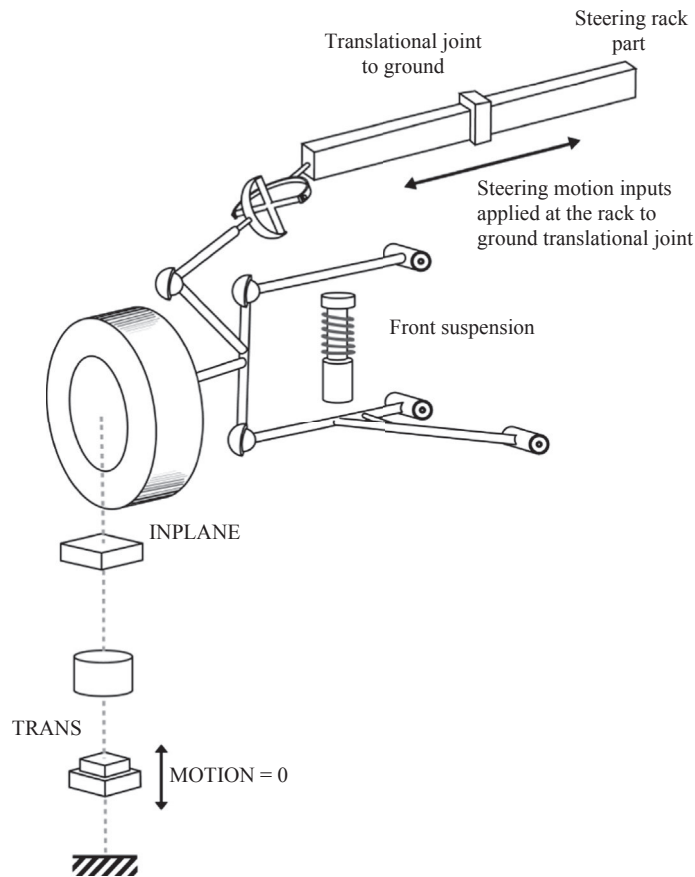
Coupled steering system model.

### 6.12.2 Steering ratio

In order to implement the ratios used in the couplers shown in [Figure 6.36](#), linking the rotation of the steering column with the steer change at the road wheels, it is necessary to know the steering ratio. At the start of a vehicle dynamics study the steering ratio can be a model design parameter. In the examples here a ratio of  $20^\circ$  of handwheel rotation to  $1^\circ$  of road wheel steer is used. On some vehicles this may be lower and on trucks or commercial vehicles it may be higher. To treat steering ratio as linear is a simplification of the situation on a modern vehicle. For example the steering ratio may vary between a lower value on centre to a higher value towards the limits of rack travel. This could promote a feeling of stability for smaller handwheel movements at higher motorway speeds and assist lower speed car park manoeuvres. The opposite arrangement — a higher value on centre and a lower level towards the limits could be argued as desirable to increase agility at speed and reduce the burden on power steering systems when parking. The authors suspect the advantages and disadvantages are probably larger in the minds of the steering system vendors than of the customers.

Using the MBS approach the steering ratio can be investigated through a separate study carried out using the front suspension system connected to the ground part instead of the vehicle body. The modelling of these two subsystems, with only the suspension on the right side shown, is illustrated in [Figure 6.37](#).

The approach of using a direct ratio to couple the rotation between the steering column and the steer angle of the road wheels is common practice in simpler

**FIGURE 6.37**

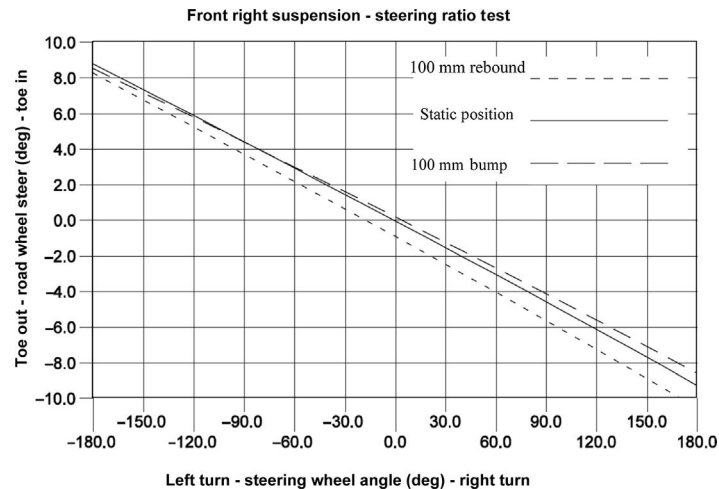
Front suspension steering ratio test model.

models but may have other limitations in addition to the treatment of the ratio as linear:

1. In the real vehicle and the linkage model the ratio between the column rotation and the steer angle at the road wheels would vary as the vehicle rolls and the road wheels move in bump and rebound.
2. For either wheel the ratio of toe out or toe in as a ratio of left or right handwheel rotation would not be exactly symmetric.

Modelling the suspension with linkages will capture these effects. Although this may influence the modelling of low-speed turning they have little effect for handling manoeuvres with comparatively small steer motions.

With simpler vehicle models, not including suspension linkages, the ratio would need to be functionally dependent on the vertical movement of the suspension and

**FIGURE 6.38**

Results of steering ratio test for MSC ADAMS front right suspension model.

direction of handwheel rotation if the behaviour is to be modelled. It should also be noted that compliance in the steering rack or rotational compliance in the steering column could be incorporated if it adds value to the analysis.

In the following example the geometric ratio between the rotation of the steering column and the travel of the rack is already known, so it is possible to apply a motion input at the rack to ground joint that is equivalent to handwheel rotations either side of the straight ahead position. The jack part shown in [Figure 6.37](#) can be used to set the suspension height during a steering test simulation. Typical output is shown in [Figure 6.38](#) where the steering wheel angle is plotted on the x-axis and the road wheel angle is plotted on the y-axis. The three lines plotted represent the steering ratio test for the suspension in the static (initial model set up here), bump and rebound positions.

Having decided on the suspension modelling strategy and how to manage the relationship between the handwheel rotation and steer change at the road wheels, the steering inputs from the driver and the manoeuvre to be performed need to be considered.

### 6.12.3 Steering inputs for vehicle handling manoeuvres

The modelling of steering inputs suggests for the first time some representation of the driver as part of the full vehicle system model. Inputs to the handwheel are generally referred to as 'open loop' or 'closed loop'.

An open loop steering input requires a time-dependent rotation to be applied to the part representing a steering column or handwheel in the simulation model. In the absence of these bodies an equivalent translational input can be applied to the joint connecting a rack part to the vehicle body or chassis, assuming a suspension linkage modelling approach has been used.

We can consider an example of an open loop manoeuvre for a steering input where we want to ramp a steering input of  $90^\circ$  on between 1 and 1.5 s of simulation time.

The function applied to the steering motion would be:

```
FUNCTION = STEP (TIME, 1, 0, 1.5, 90D)
```

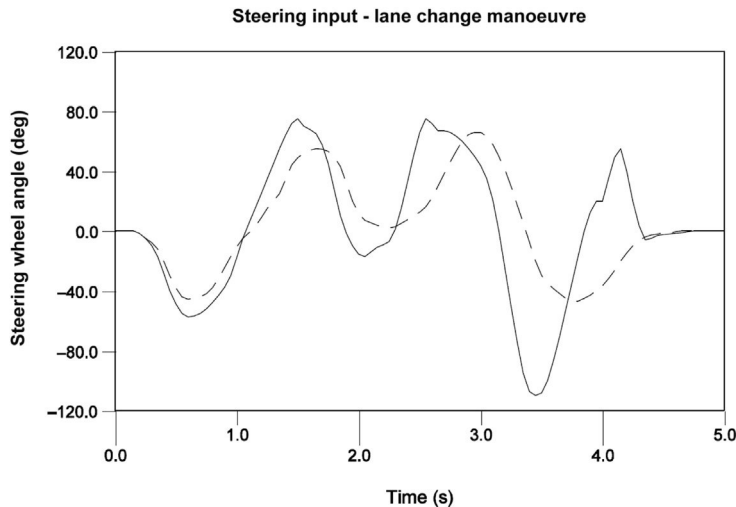
In a similar manner if we wanted to apply a sinusoidal steering input with an amplitude of  $30^\circ$  and a frequency of 0.5 Hz we could use:

```
FUNCTION = 30D * SIN (TIME*180D)
```

For the lane change manoeuvre described earlier the measured steering wheel angles from a test vehicle can be extracted and input as a set of XY pairs that can be interpolated using a cubic spline fit. A time history plot for the steering inputs is shown in [Figure 6.39](#) for lane change manoeuvres at 70 and 100 km/h.

By way of example, the MSC ADAMS statements that apply the steering motion to the steering column to body revolute joint and the spline data are shown in [Table 6.3](#) for a 100 km/h lane change. The x values are points in time and the y values are the steering inputs in degrees. In the absence of measured data it is possible to construct an open loop single or double lane change manoeuvre using a combination of nested arithmetic ‘If’ functions with embedded step functions with some planning and care over syntax. Note that for a fixed steering input a change in vehicle configuration will produce a change in response so that the vehicle fails to follow a path.

The term “closed loop” is generally associated with some kind of controlled system. Any controlled system can be considered to consist of three elements — the ‘plant’ (the item to be controlled), the input to the plant and the output from the plant ([Figure 6.40](#)).



**FIGURE 6.39**

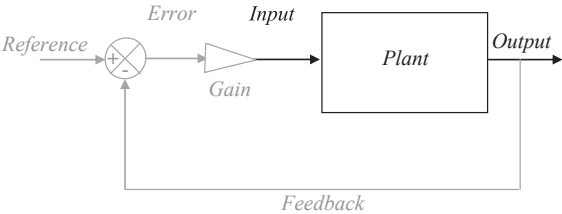
Steering input for the lane change manoeuvre at 70 km/h (dashed line) and 100 km/h (solid line).



**Table 6.3**    MSC ADAMS Statements for Lane Change Steering Inputs

```
MOTION/502,JOINT=502,ROT
,FUNC=(PI/180)*CUBSPL(TIME,0,1000)

SPLINE/1000
,X=0,1,2,3,4,5,6,7,8,9
,9.1,9.2,9.3,9.4,9.5,9.6,9.7
,9.8,9.9,10,10.1,10.2,10.3,10.4,10.5,10.6,10.7,10.8,10.9,11
,11.1,11.2,11.25,11.3,11.4,11.5,11.6,11.7,11.8,11.9,12,12.1
,12.2,12.3,12.4,12.5,12.6,12.7,12.8,12.9,13,13.1,13.2,13.3
,13.4,13.5,13.6,13.7,13.75,13.8,13.9,14,14.1,14.2,14.3,14.4,14.5
,14.6,14.7,14.8,14.9,15
,Y=0,0,0,0,0,0,0,0,0,0
,0,0,0,0,0,0,0
,0,0,-5,-17,-40,-55,-57,-52,-43,-30,-5,15,35,55,72,75,70,65,45,10
,-10,-17,-11,-7,15,50,75,67,66,60,50,35,0,-50,-95,-110,-100,-70,
-35,0
,20,20,35,55,20,-6,-3,-2,-1,0,0,0,0,0,0
```



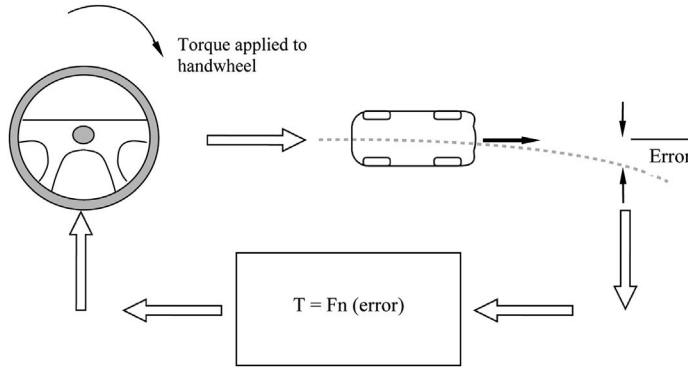
**FIGURE 6.40**

A generic control system.

When a controller is added to the system, its goal is to allow the input to the plant to be adjusted so as to produce the desired output. The desired output is referred to as the ‘reference’ state; a difference between the actual output and the reference is referred to as an ‘error’ state. The goal of the control system is to drive the error to zero.

As a start we can consider an example of a closed loop steering input that requires a torque to be applied to the handwheel or steering column such that the vehicle will follow a predetermined path during the simulation. As such a mechanism must be modelled to measure the deviation of the vehicle from the path and process this in a manner that feeds back to the applied steering torque. As the simulation progresses, the torque is constantly modified based on the observed path of the vehicle and the desired trajectory. Such an input is referred to as closed loop, as the response is observed and fed back to the input thus closing the control loop.

For a closed loop steering manoeuvre a column torque or rack force is applied, which will vary during the simulation so as to maintain the vehicle on a predefined

**FIGURE 6.41**

Principle of a closed loop steering controller.

path. This requires a steering controller to process feedback of the observed deviation from the path (error) and to modify the torque accordingly, as illustrated in Figure 6.41.

Evaluating the path error for the vehicle is somewhat less straightforward than it might appear. Odhams (2006) proposes a method using intrinsic coordinates.

A known trajectory is presumed, consisting of a compass heading angle,  $\Psi_t$ , that is defined as a function of trajectory distance,  $S_t$ . An absolute compass heading  $\Psi$  is calculated using the velocity vector, which may vary from the compass angle of the plane of symmetry of the vehicle  $\theta$  due to the presence of a body slip angle  $\beta$ . For reasons of habit, East is declared as zero, which corresponds to the X-axis in many schemes:

$$\psi = \arctan\left(\frac{V_y}{V_x}\right) + \beta \quad (6.23)$$

Note that the 2-argument arctangent function ATAN2 may be preferred over the simpler formulation since the trajectory can go in any direction. For trajectories that describe a loop, some care is needed as the calculations wrap through  $180^\circ$ . Distance travelled is simply formulated as the integral of the velocity vector. Noting that the orientation of the vector varies with time, the actual path on the ground can be calculated as X and Y:

$$S(t) = \int_0^t V(t) dt \quad (6.24)$$

$$X(t) = X_0 + \int_0^t V(t) \cos\psi(t) dt \quad (6.25)$$

$$Y(t) = Y_0 + \int_0^t V(t) \sin\psi(t) dt \quad (6.26)$$

For a vehicle travelling with an actual compass heading angle  $\Psi$  at an actual distance travelled of  $S$ , from Figure 6.42, presuming small angle theory applies:

$$\delta S_t = R \delta \delta_t \quad (6.27)$$

$$\delta n = (R + Y_{\text{err}}) \delta \psi_t = \delta S_t + Y_{\text{err}} \delta \psi_t = \delta S \cos(\psi_{\text{err}}) \quad (6.28)$$

$$\delta S = \sqrt{V_y^2 + V_x^2} \cdot \delta t \quad (6.29)$$

$$\psi_{\text{err}} = \psi - \psi_t \quad (6.30)$$

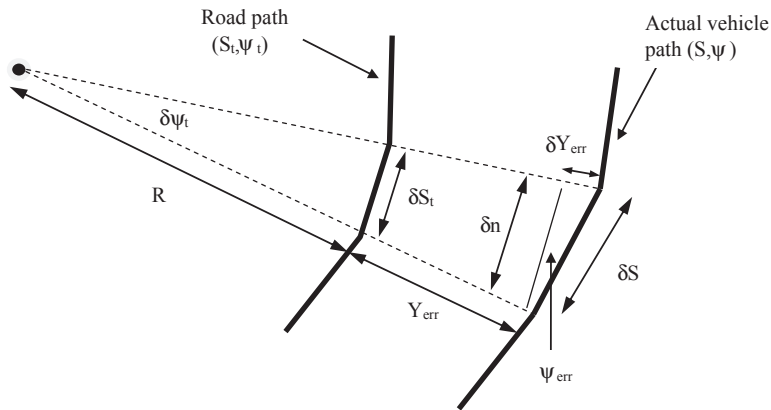
Rearranging to find  $\delta S$ :

$$\delta S = \frac{\delta S_t + Y_{\text{err}} \delta \psi_t}{\cos(\psi_{\text{err}})} = \frac{\delta S_t + Y_{\text{err}} \frac{\delta \psi_t}{\delta S_t} \delta S_t}{\cos(\psi_{\text{err}})} \quad (6.31)$$

Rearranging further:

$$\frac{\delta S}{\delta S_t} = \frac{1 + Y_{\text{err}} \frac{\delta \psi_t}{\delta S_t}}{\cos(\psi_{\text{err}})} \quad (6.32)$$

$$\frac{\delta S_t}{\delta S} = \frac{\cos(\psi_{\text{err}})}{1 + Y_{\text{err}} \frac{\delta \psi_t}{\delta S_t}} \quad (6.33)$$



**FIGURE 6.42**

Intrinsic coordinate calculation proposed.

*Reproduced from Odhams (2006).*

$$\frac{\delta S_t}{\delta S} = \frac{\cos(\psi - \psi_t)}{1 + Y_{\text{err}} \frac{\delta \psi_t}{\delta S_t}} \quad (6.34)$$

$$\frac{\delta S_t}{\sqrt{V_y^2 + V_x^2} \cdot \delta t} = \frac{\cos(\psi - \psi_t)}{1 + Y_{\text{err}} \frac{\delta \psi_t}{\delta S_t}} \quad (6.35)$$

$$\frac{\delta S_t}{\delta t} = \frac{\cos(\psi - \psi_t) \sqrt{V_y^2 + V_x^2}}{1 + Y_{\text{err}} \frac{\delta \psi_t}{\delta S_t}} \quad (6.36)$$

In the limit, as  $\delta t$  tends to zero:

$$\frac{dS_t}{dt} = \frac{\cos(\psi_{\text{err}}) \sqrt{V_y^2 + V_x^2}}{1 + Y_{\text{err}} \frac{d\psi_t}{dS_t}} \quad (6.37)$$

Similarly, it can be written by inspection that

$$\delta Y_{\text{err}} = \delta S \sin(\psi_{\text{err}}) \quad (6.38)$$

Using the previous relationships:

$$\delta Y_{\text{err}} = \delta S \sin(\psi - \psi_t) \quad (6.39)$$

$$\frac{\delta Y_{\text{err}}}{\delta S} = \sin(\psi - \psi_t) \quad (6.40)$$

$$\frac{\delta Y_{\text{err}}}{\sqrt{V_y^2 + V_x^2} \cdot \delta t} = \sin(\psi - \psi_t) \quad (6.41)$$

$$\frac{\delta Y_{\text{err}}}{\delta t} = \sin(\psi - \psi_t) \sqrt{V_y^2 + V_x^2} \quad (6.42)$$

In the limit, as  $\delta t$  tends to zero:

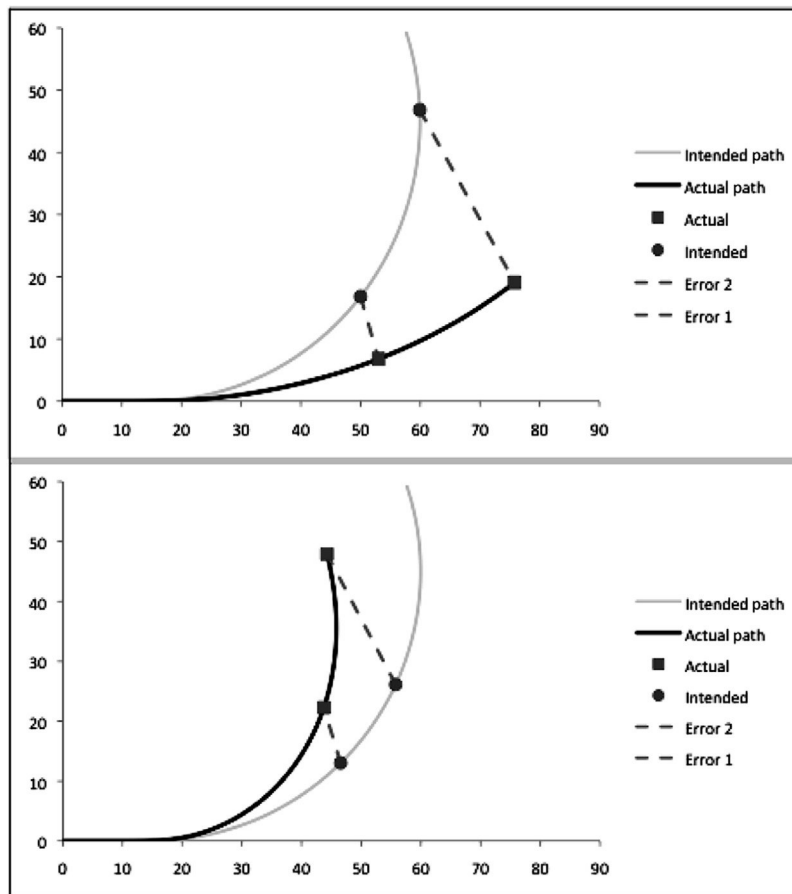
$$\frac{dY_{\text{err}}}{dt} = \sin(\psi - \psi_t) \sqrt{V_y^2 + V_x^2} \quad (6.43)$$

In principle, the values for  $\Psi_t$ , the derivative with respect to distance along the trajectory  $\frac{d\psi_t}{dS_t}$  must be determined from the model and the differential equations integrated to give current values for actual distance travelled and path error.

When this method is implemented it appears satisfactory but on closer inspection there are some reservations about it. In order to illustrate them, a simple intended path was generated consisting of a straight line followed by a cosine ramp to a constant curvature (constant radius). A vehicle trajectory was generated by presuming a constant value for longitudinal speed and scaling the heading of the vehicle by some factor compared to the intended path. A scaling of unity returns a path error of zero when the calculations are functioning correctly.

Scaling the vehicle path by less than unity results in 'running wide' of the turn. Implementing the calculations as described above with a simple trapezoidal integration returns the results shown in [Figure 6.43](#).

The calculations can be seen to return an error normal to the vehicle when the vehicle is running wide but the results are less obvious when the vehicle turns



**FIGURE 6.43**

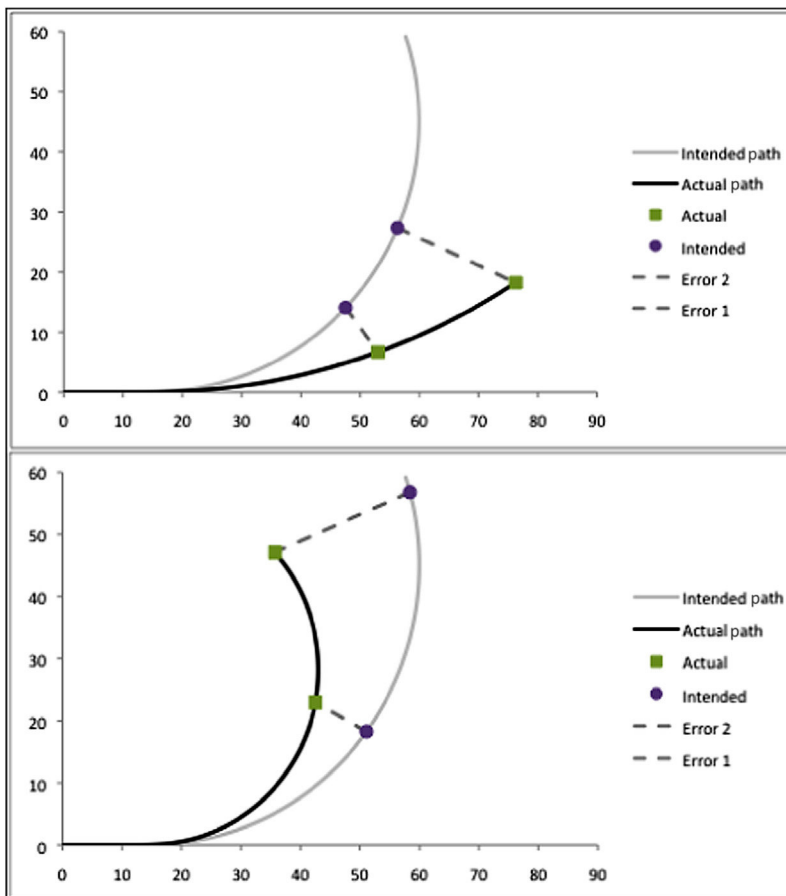
Odham's calculation for trajectory error.

more tightly than the intended trajectory. Returning an error normal to the vehicle can be seen to risk undefined results (or at least poorly conditioned results) when the vehicle trajectory is substantially perpendicular to the intended path.

A different formulation is possible, that returns the path error measured normal from the path, which gives the results shown in Figure 6.44 for the same two scenarios.

The result in Figure 6.44 is suggested as more intuitively reasonable as it finds the part of the path closest to the current position, but still in the context of a larger journey — which is to say it would not get confused by a figure-of-eight since it looks for a local minimum.

The process is simpler than the Odhams method but has the disadvantage of being iterative. Using similar notation to above, velocity, heading and, distance travelled and



**FIGURE 6.44**

An alternative path error calculation scheme returns error normal to the intended path.

current location are calculated as before. Using the distance travelled  $S$ , the intended location of the vehicle can also be calculated or retrieved as a lookup operation:

$$X_t(S) = X_0 + S \int_0^S \cos(\psi_t(S)) dS \quad (6.44)$$

$$Y_t(S) = Y_0 + S \int_0^S \sin(\psi_t(S)) dS \quad (6.45)$$

A first estimate of the path error can be calculated simply:

$$e_1 = \sqrt{(X_t(S) - X(S))^2 + (Y_t(S) - Y(S))^2} \quad (6.46)$$

Also, a relative angle between the error vector and the local trajectory heading,  $\psi_t(S)$  can be calculated:

$$\varphi_{\text{err}}(S) = \arctan\left(\frac{Y_t(S) - Y(S)}{X_t(S) - X(S)}\right) - \psi_t(S) \quad (6.47)$$

An assessment can be made of how close this is to  $90^\circ$  and, in the general case an improved estimate  $S_i$  can be made for the trajectory length that should be considered in computing the path error based on the previous iteration step:

$$S_i = S_{i-1} - e_{i-1} \cdot \cos(\varphi_{\text{err}}(S_{i-1})) \quad (6.48)$$

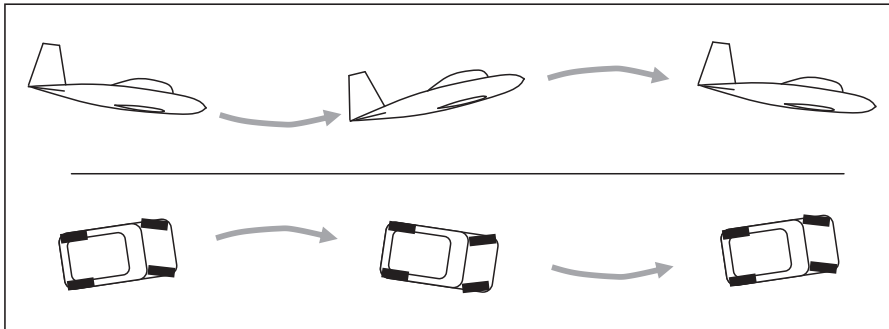
This scheme can be iterated until subsequent estimates converge to some convenient amount. Some care is needed in handling the wrapping of trigonometric function and with iteration convergence, as with all numerical approaches.

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## 6.13 Driver behaviour

It becomes inevitable with any form of vehicle dynamics modelling that the interaction of the operator with the vehicle is a source of both input and disturbance. In flight dynamics, the phenomenon of ‘PIO’ — pilot-induced oscillation — is widely known. This occurs when inexperienced pilots, working purely visually and suffering from some anxiety, find their inputs are somewhat excessive and causing the aircraft to, for example, pitch rhythmically instead of holding a constant altitude (Figure 6.45).

PIO is caused when the operator is unable to recognise the effects of small control inputs and therefore increases those inputs, before realising they were excessive and reversing them through a similar process. It is analogous to the ‘excess proportional control gain oscillation’ discussed in classical control theory (Leva et al, 2002). For road vehicles, drivers most likely to induce PIO in steering tend to be inexperienced or anxious drivers travelling at a speed with which they are uncomfortable. This type of PIO is not to be confused with the typical experience of drivers

**FIGURE 6.45**

Pilot-Induced Oscillation (PIO) – not exclusively an aeronautical phenomenon.

of skidding vehicles when the initial skid is corrected but the vehicle subsequently ‘fishtails’ or simply departs in the opposite direction – this is a ‘phasing at resonance’ control error; the driver fails to apply a ‘feed forward’ (open loop, knowledge-based) correction in advance of the vehicle’s response to compensate for the delay in vehicle response.

PIO also occurs in tractive (i.e. throttle) control inputs and is the reason even experienced drivers are incapable of travelling at a constant speed on highways; perception of changes in following distance is universally poor. If too little attention is spent on the driving task or if insufficient following distance is left, these PIOs become successively amplified by following drivers until the speed variation results in a ‘shunt’ accident. Radar-based cruise control systems will alleviate this risk but are no substitute for attentive driving while anything less than the whole vehicle fleet is fitted with it.

### 6.13.1 Steering controllers

There are a variety of controller models suitable for modelling driver behaviour in existence. Some are very complete while others, such as the two-loop feedback control model used by the authors, are simpler. The analyst must consider the needs of the simulation (and the financial constraints of the company) and choose the most appropriate level of modelling to achieve the task at hand. Driver models in general fall into two categories:

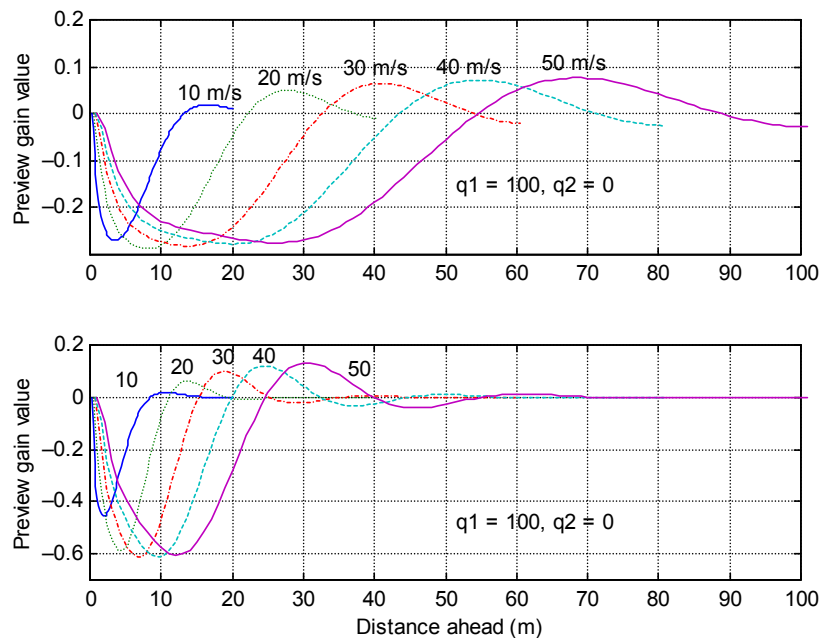
#### 6.13.1.1 Optimum control models

Optimum control models use some form of ‘penalty function’ – a measure used to assess the quality of control achieved. For example, for a vehicle steering model the appropriate variable might be lateral deviation from the intended path. Optimum control models use repeated simulations of the event and numerical optimisation methods to ‘tune’ the parameters for a control system to minimise the value(s) of the penalty function(s) over the duration of the event of interest. For learnt events,



such as circuit driving, these methods are excellent to produce a prediction of likely driver behaviour. However, some care must be exercised with their use. For road vehicles, drivers are generally unskilled and so the application of modelling techniques in which repeated solutions are used to discover the ‘best’ way of achieving a manoeuvre may not be appropriate when simulating a manoeuvre that the driver has only one attempt at completing — for example, emergency evasive manoeuvres. For race vehicle simulation, some care must also be exercised lest extended calculations result in the proof that the driver can adapt to a remarkable variety of vehicle changes — without any real insight into which will improve performance in a competition environment.

The science of optimum control driver models is changing at a pace and it seems likely that these will become the architecture for future autonomous driving products, since they can adapt to changing circumstances such as vehicle payload. A very promising thread appears to be appearing with Linear Quadratic Regulators (Sharp, 2005) with a previewing controller. By examining the properties of the vehicle, an optimised set of gains for a given speed and preview distance length can be calculated. The actual control input is the combined influence of all the path errors at multiple preview distances multiplied by their optimised gains. The form of the optimised gains reflects some kind of projection of the ‘inverse dynamics’ of the car in front of it and is interesting in its own right, as can be seen in Figure 6.46.



**FIGURE 6.46**

Preview gains for two different classes of car (Sharp, 2005).

The two vehicles are labelled ‘Buick’ and ‘Ferrari’ by Sharp and a number of similarities and differences are immediately apparent between the vehicles. Similar is the fact that a first lobe with a strongly negative character dominates the sequence. Were the controller imagined as a single point previewer, it is quite likely that the distance ahead for the most negative point would reflect a useful preview at different vehicle speeds. The negative sign is a function of the sign convention in use and means that the steering must be turned in opposition to the error in order to reduce it — the familiar ‘negative feedback’ mechanism.

An obvious difference is that the gain sequence is appreciably longer for the Buick in the top set of gains, requiring over 100 m of preview when travelling at 50 m/s before the gains settle to zero. (It is noted in Sharp’s example that the gain sequence does not in fact settle to zero, being truncated before that happens; however, a sound knowledge of vehicle dynamics suggests that nothing particularly unusual is likely to happen to the gain sequence and it will probably converge in some manner similar to the Ferrari in the lower trace.)

The significance of the zero settling of the gain sequence can be imagined in the following way. Imagine a road profile that has an arbitrary discontinuity at some point ahead. Before the discontinuity, the road profile is a straight line described by  $y(x) = 0$ . After the discontinuity, the road profile is a straight line described by  $y(x) = 1$ . If the gain sequence is thought of as ‘extending’ in front of the car, then if the discontinuity is past the zero settling point the sum of all the inputs is zero, and the driver model does nothing. As the nonzero points reach the discontinuity, the driver begins to steer by the same amount as the gain sequence (since the discontinuity is of unit magnitude). It can be seen this will result in a small movement *in opposition to what would be expected to negotiate the discontinuity*. As the gain sequence crosses further, the reversal of the gain sequence will overpower the initial out-of-the-corner input and produce a steer input into the corner. A short while later the steering will reverse again and will undergo yet a further reversal before the final decisive steer input to negotiate the discontinuity. This apparently baffling course of events will be familiar to anyone who has ever enjoyed watching classic rally cars — it is a so-called ‘Scandinavian Flick’.

The gain sequence can thus be seen as some kind of measure of how far in advance of any intended path change the driver has to plan, or to view it from the other end, how late the driver can leave things before making a decision. In general, the later a vehicle can adapt to changing circumstances, the better we might declare that elusive thing called ‘handling’. What is particularly intriguing about Sharp’s approach is that the zero settling point is some kind of invariant property of the vehicle at a given speed; no actual manoeuvre simulations need to be carried out to evaluate whether or not an improvement has been made — a shorter distance to zero settling allows later adaptation; the Ferrari is handsomely better than the Buick.

These types of calculations are not currently implemented in any of the typical MBS general purpose software tools available at the time of writing. However, they are well documented enough in the literature to be implemented with typical commercial tools; there is no ‘magic bullet’ in the formulation.

### 6.13.1.2 *Moment-by-moment feedback models*<sup>1</sup>

Such models are really a subset of the optimum control models described above — the optimum control method repeatedly uses feedback models in order to discern the best state of tune for the controller. When the feedback controllers are used alone, the analyst must set their tuning. Although the absence of ‘automated’ correlation makes them less appropriate for circuit racing, it also adds clarity in the sense that the parameters, once set, remain constant and so changes in the vehicle behaviour and/or driver inputs can be readily understood.

In general, the driver behaves as the most generic form of loop-closing controller. There are several attractive control technologies represented in the literature and some of their proponents believe they represent a ‘one size fits all’ solution for the task of applying control to any system. The competing technologies are outlined for comparison.

#### 6.13.1.2.1 Logic controller

A logic controller produces output that has only certain possible values. For example, if a driver model was implemented using logic, the logic might be ‘if the vehicle is to the left of the intended path, steer right and vice versa’. With a logic controller, the amount of steer is fixed and so any control of the vehicle would be achieved as a series of jerks, oscillating about the intended path. While probably functional it would be unlikely to represent any normal sort of driver.

#### 6.13.1.2.2 PID controller

PID stands for ‘proportional, integral and derivative’. The error state is used in three ways; used directly, a control effort is applied in proportion to (and opposition to) the error — this is the ‘P’, proportional, element of the control. The fact that the control effort is in opposition to the error is important, since otherwise the control effort would increase the error instead of reducing it. For this reason, such systems are often referred to as ‘negative feedback’ systems. The error can also be integrated and differentiated, with control forces applied proportional to the integral and the differential — these are the ‘I’ and the ‘D’ terms in the controller. One or more of the terms may not be used at all in any particular controller. An analogy for PID controllers can be found in vehicle suspensions. If the ride height is thought of as the desired output, then individual components of the suspension behave as parts of a control system. The springs produce a force proportional to the change in ride height and the dampers produce a force proportional to the derivative of ride height. Real dampers are often nonlinear in performance, and there is nothing to stop nonlinear

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<sup>1</sup>Feedback models ought to be known as ‘instantaneous feedback’ models but the word instantaneous has become slightly muddled in recent times. It should be used unambiguously to mean ‘existing for a moment in time’ but has become sadly confused with ‘instant’, meaning immediate. Instant feedback would imply the inability to represent transport delays and the like in the controller model, which is incorrect. The use of moment-by-moment is therefore preferred although it introduces confusion with moment in the sense of torque.

gains being used for any of the control terms. The D term has the effect of introducing damping into the control system. An analogy for the I term is a little harder to come by. The best analogy is that of a self-levelling unit fitted to the suspension that applies a restoring force related to the length of time the vehicle has been at the wrong ride height and how wrong the ride height is. (This is an imperfect analogy for many reasons but allows the notion to be understood at least.) In real systems, when the output is *nearly* the same as the reference state it is frequently the case that the control forces become too small to influence the system, either because of mechanical hysteresis or sensor resolution or some similar issue. One important measure of the quality of any control system is the accuracy with which it achieves its goals. Such an offset characterises an inaccurate system; an integral term ‘winds up’ from a small error until powerful enough to restore the system to the reference state. Thus for classical control, integral terms are important for accuracy. However, since they take some time to act they can introduce delays into the system. In general PID controllers have the advantage that they produce ‘continuous’ output — that is to say all the derivatives are finite, the output has no steps — which is quite like the behaviour of real people.

#### 6.13.1.2.3 Fuzzy logic

Fuzzy logic was first described in the 1960s but found favour in the 1980s as a fashionable ‘new’ technology. Notions of ‘true’ and ‘false’ govern ‘logic’ in computer algorithms. Simple control systems assess a set of conditions and make a decision based on whether or not such variables are true or false. Fuzzy logic simply defines ‘degrees of truth’ by using numbers between 0 and 1 such that the actions taken are some blend of actions that would be taken were something completely true and other actions that would be taken were something completely false. Fuzzy logic is most applicable to control systems where actions taken are dependent on circumstance and where a simple PID controller is unable to produce the correct output in every circumstance. For example, throttle demand in a rear-wheel drive vehicle model might be controlled with a PID controller to balance understeer; however, too much throttle would cause oversteer and some more sophisticated blend of steer and throttle input would be required to retain control under these circumstances. A fuzzy logic model could conceivably manage the transition between the two strategies.

#### 6.13.1.2.4 Neural networks

Where the system of interest is highly nonlinear and a lot of data exists that describes desired outputs of the system for many different combinations of inputs, it is possible to use a neural network to ‘learn’ the patterns inherently present in the data. A neural network is quite simply a network of devices that are ‘neuron-like’. Neurons are the brain’s building blocks and are switches with multiple inputs and some threshold to decide when they switch. In general, neural networks are run on transistor devices or in computer simulations. They require a period of ‘training’ when they learn what settings need to be made for individual neurons in order to produce the required outputs. Once trained, they are extremely rapid in operation since there is very little

‘processing’ as such, simply a cascade of voltage switching through the transistor network. If the network is implemented as semiconductor transistors then it works at a speed governed only by the latency of the semiconductor medium — extremely fast indeed. Neural networks are extremely useful for controlling highly nonlinear systems for which it is too difficult to code a traditional algorithm. However, the requirement for a large amount of data can make the learning exercise a difficult one. Recent advances in the field reduce the need for precise data sets of input and corresponding outputs; input data and ‘desirable outcome’ definitions allow neural networks to learn how to produce a desirable outcome by identifying patterns in the incoming data. Such networks are extremely slow in comparison to the more traditional types of network during the learning phase. In general, for driver modelling there is little applicability for neural networks at present due to the lack of fully populated data sets with which to teach them. It is also worth commenting that for any input range that was not encountered during the learning phase, the outputs are unknown and may not prove desirable. This latter feature is not dissimilar to real people; drivers who have never experienced a skid are very unlikely to control it at the first attempt. Also worth mentioning is that if the characteristics of the plant model change once the training period is complete then the controller may not perform well. This is also quite like real drivers; many drivers underapply the brakes in laden vehicles if the loading condition is infrequent.

#### 6.13.1.2.5 System identification

System identification is a useful technique, not dissimilar in concept to neural networking. A large amount of data is passed through one of several algorithms that produce an empirical mathematical formulation that will produce outputs like the real thing when given the same set of inputs. The formulation is more mathematical than neural networking and so the resulting equations are amenable to inspection — although the terms and parameters may lack any immediately obvious significance if the system is highly nonlinear. System identification methods select the level of mathematical complexity required to represent the system of interest (the ‘order’ of the model) and generate parameters to tune a generic representation to the specific system of interest. As with neural networks, the representation of the system for inputs that are beyond the bounds of the original inputs (used to identify the model) is undefined. System identification is useful as a generic modelling technique and so has been successfully applied to components such as dampers as well as control system and plant modelling. System identification is generally faster to apply than neural network learning but the finished model cannot work as quickly. The same data set availability problems for neural networking also mean system identification is not currently applicable to driver modelling.

#### 6.13.1.2.6 Adaptive controllers

Adaptive control is a generic term to describe the ability of a control system to react to changes in circumstances. In general, people are adaptive in their behaviour and so it would seem at first glance that adaptive control is an appropriate tool for

modelling driver behaviour. Optimum control models, described above, generally use some form of adaptive control to optimise the performance of a given controller architecture to the system being controlled and the task at hand. Adaptation is a problem in real-world testing since it obscures differences in performance; equally it can obscure performance changes and so adaptive modelling of driver behaviour is not preferred except for circuit driving. Several techniques come under the headline of adaptive control; the simplest is to change the control parameters in a predetermined fashion according to the operating regime, an operation referred to as ‘gain scheduling’. Gain is the term used for any treatment given to an error state before it is fed to an input — thus the PID controller described above has a P-gain, and I-gain and a D-gain. It might be, for example, that under conditions of opposite lock the P-gain is increased since the driver needs to work quickly to retain control, or under conditions of increasing speed the P-gain is reduced since slower inputs are good for stability at higher speeds. A more complex method is to carry a model of the plant on board in the controller and to use it to better inform some form of gain scheduling, perhaps using information that cannot readily be discerned from on-board instrumentation — such as body slip angle. This is referred to as a Model Reference Adaptive Scheme (MRAS). A further variation on the theme is to use the controller to calculate model parameters using system or parameter identification methods (described above). The control system parameters can be modified based on this information — in effect there is an ongoing redesign of the control system using a classical deterministic method, based on the reference state and the plant characteristics according to the latest estimate. This is referred to as a ‘self-tuning regulator’ and is useful for unpredictably varying systems. Finally, a method known as ‘dual control’ intentionally disturbs the system in order to learn its characteristics, while simultaneously controlling it towards a reference state. In many ways this is similar to a top-level rally driver stabbing the brakes in order to assess friction levels while disturbing the overall speed of the vehicle as little as possible; the knowledge gained allows the driver to tune their braking behaviour according to recently learnt characteristics. Such behaviour is in marked contrast to circuit drivers, who concentrate on learnt braking points and sometimes have difficulty adapting to changing weather conditions. With the exception of the simplest gain scheduling methods, in general adaptive control techniques are unsuitable for the modelling of driver behaviour as part of any practicable process although they may well have a place in future autonomous systems. Once again the variation in simulation output cannot readily be traced to any particular aspect of the system and hence the success or otherwise of an intended modification is difficult to interpret.

In the light of the preceding description, the authors believe a PID controller, with some form of simple gain scheduling, is most appropriate for the modelling of driver behaviour in order to usefully influence the design of a vehicle in an MBS context. Note that for other outcomes — for example, developing a full authority digital autonomous control system — the preferred method will probably be different. The art of implementing a successful model is in selecting the state variables within the model to use with the controller.

### 6.13.2 A path following controller model

The first hurdle to be crossed is the availability of suitable state variables and the use of gain terms to apply to them. Typically in an MBS model, many more variables are available than in a real vehicle. Within the model, these variables can be the subject of differential equations in order to have available integral and differential terms. Table 6.4 shows a portion of a command file from MSC ADAMS implementing those terms for yaw rate. While it is a working example, no claim is made that it is in any sense optimum.

Such variables can usually be manipulated within the model using the programming syntax provided with the code being used. For general purpose software

**Table 6.4** A Portion of an MSC ADAMS Command File Showing the Implementation of Differential Equations to Retrieve and Use Integral and Derivative Terms for a State Variable

```
! -- Derivative Term - not generally used --
part create equation differential_equation &
  differential_equation_name = .test.yaw_rate_error_equation_1 &
  adams_id = 3 &
  comments = "Yaw Rate Error Equation - Implicit" &
  initial_condition = 0.0 &
  function = "DIF(3)-varval(yaw_rate_error)" &
  implicit = on &
  static_hold = off
data_element create variable &
  variable_name = yaw_rate_error_derivative &
  function="DIF1(3)"

! -- Integral Term --
part create equation differential_equation &
  differential_equation_name = .test.yaw_rate_error_equation_2 &
  adams_id = 4 &
  comments = "Yaw Rate Error Equation - Explicit" &
  initial_condition = 0.0 &
  function = "varval(yaw_rate_error)" &
  implicit = off &
  static_hold = off
data_element create variable &
  variable_name = yaw_rate_error_integral &
  function="(DIF(4))"

! Steer input torque in response to path error.
force create direct single_component_force &
  single_component_force_name=yaw_rate_handwheel_torque &
  type_of_freedom=rotational &
  action_only = on &
  i_marker_name = .hand_wheel_column.m_wheel_column &
  j_marker_name = .hand_wheel_column.m_wheel_column &
  function="(", &
    "    VARVAL(yaw_rate_error)           * VARVAL(yp_gain) ", &
    "    + VARVAL(yaw_rate_error_integral) * VARVAL(yi_gain) ", &
    "    + VARVAL(yaw_rate_error_derivative) * VARVAL(yd_gain) ", &
    ")", &
    "** STEP(TIME,0.0,0.0,1.0,1.0)"
```

tools, the format of such calculations can appear a little clumsy but this soon disappears with familiarity. For codes such as MATLAB/Simulink the implementation of control systems is arguably easier since they are written with the prime objective of control system modelling. However, the modelling of the vehicle as a plant is more difficult within these systems and so there is an element of swings and roundabouts if choosing between the codes. General purpose software tools have a history in very accurate simulation of mechanical systems and can be coerced into representing control systems. Codes like MATLAB and MATLAB/Simulink are the reverse; they have a history in very detailed control system simulation and can be coerced into representing mechanical systems. For this reason, a recent development suggests using each code to perform the tasks at which it is best; this is often referred to as 'co-simulation'. Since the first edition of the book, the state of the art has advanced substantially and a recent development has been the development of a software interfacing protocol known as the 'functional mock-up interface' that looks set to ease this difficulty, although its implementation is very incomplete at present. The authors' experiences to date have been universally disappointing for entirely prosaic reasons — the speed of execution remains extremely poor and the robustness of the software suppliers in dealing with different releases of each other's product remains somewhat inconsistent. The effort required to persuade the relevant software to work in an area where it is weak is usually made only once and in any case the additional understanding gained is almost always worthwhile for the analyst involved. Until the performance and robustness of the software improves, the authors do not favour co-simulation except for the most detailed software verification exercises.

The next hurdle to be crossed is the representation of the intended behaviour of the vehicle — the 'reference' states. Competition-developed lap simulation tools use a 'track map' based on distance travelled and path curvature. This representation allows the reference path to be of any form at all and allows for circular or crossing paths (e.g. figures of eight) to be represented without the one-to-many mapping difficulties that would be encountered with any sort of  $y$  versus  $x$  mapping. Integrating the longitudinal velocity for the vehicle gives a distance-travelled measure that shows itself to be tolerably robust against drifting within simulation models. Using this measure, the path curvature can be surveyed in the vicinity of the model.

Some authors favour the use of a preview distance for controlling the path of the vehicle, with an error based on lateral deviation from the intended path, which has already been discussed.

An alternative method, used by the authors with some success for a variety of extreme manoeuvres, is to focus on the behaviour of the front axle. This model fits with the author's experience (Harty) of driving at or near the handling limit, particularly on surfaces such as snow where large body slip angles highlight the mechanisms used in the driver's mind. High-performance driving coaches (Palmer, 1999) rightly concentrate on the use of a 'model' the driver needs in order to retain control in what would otherwise become stressful circumstances of nonlinear vehicle behaviour and multiple requirements for control — typically



vehicle orientation (body slip angle) and velocity (path control). Useful learning occurs on low-grip environments that can be readily transferred across to high grip. In low grip environments, the extreme nonlinearity of response of the vehicle can be explored at low speeds and with low stress levels, allowing the driver to piece together a model to be used within their own heads; it is then a matter of practice to transfer the lessons to a high-grip environment. The same concepts can be used to explore the behaviour of a driver model within an MBS software environment.

The formulation used is described below. All subscripts x and y are in the vehicle reference frame. The ground plane velocity  $V_g$  is given from the components  $V_x$  and  $V_y$  using

$$V_g = \sqrt{V_x^2 + V_y^2} \quad (6.49)$$

The demanded yaw rate  $\omega_d$  is found from the forward velocity  $V_x$  and path curvature  $k$  using

$$\omega_d = V_g k \quad (6.50)$$

The body slip angle  $\beta$  is found from the velocities  $V_y$  and  $V_g$  using

$$\beta = \arcsin\left(\frac{V_y}{V_g}\right) \quad (6.51)$$

The centripetal acceleration  $A^p$  is given from the components of acceleration  $A_x$  and  $A_y$  using

$$A^p = A_y \cos(\beta) + A_x \sin(\beta) \quad (6.52)$$

The front axle no-slip yaw rate  $\omega_{fNS}$  is found from the centripetal acceleration  $A^p$ , the yaw acceleration  $\alpha_z$ , the distance,  $a$ , from the mass centre to the front axle and the ground plane velocity  $V_g$  using

$$\omega_{fNS} = \frac{A^p - \alpha_z a}{V_g} \quad (6.53)$$

The yaw error  $\omega_{err}$  is then found from the demanded yaw rate  $\omega_d$  and the front axle no-slip yaw rate  $\omega_{fNS}$  using

$$\omega_{err} = \omega_d - \omega_{fNS} \quad (6.54)$$

The implementation of Eqns (6.49)–(6.54) is illustrated, using again an example of the MSC ADAMS command file format, in Table 6.5, representative of real vehicle and driver behaviour (Figure 6.47). The simulated driver and vehicle behaviour for a post-limit turn-in event is compared here to a real vehicle. Note the free-wheeling analytical model (of a significantly different vehicle) displays greater body slip angle, while the real vehicle displays greater oversteer.

For a variety of events, this formulation produces good driver/vehicle behaviour.

**Table 6.5** Command File Sample for 'Front Axle Control' Driver Model

```

data_element create variable &
  variable_name = ground_plane_velocity &
  function = "(
    (", &
    "VX(m_body_CG,base)**2 +", &
    "VY(m_body_CG,base)**2", &
    ")**0.5 ) / 1000"

data_element create variable &
  variable_name = demanded_yaw_rate &
  function = "varval(ground_plane_velocity) *", &
    "AKISPL(varval(path_length),0,path_curvature_spline)"

data_element create variable &
  variable_name = beta &
  function="ASIN(VY(m_body_CG,base,m_body_CG)/", &
    "(varval(ground_plane_velocity)+0.00001))"

data_element create variable &
  variable_name = centacc &
  function="(VARVAL(latacc)*COS(VARVAL(beta)))+", &
    "(VARVAL(longacc)*SIN(VARVAL(beta)))"

data_element create variable &
  variable_name = front_axle_no_slip_yaw &
  function="-(", &
    "VARVAL(centacc) ", &
    "- WDTZ(m_body_CG,base,m_body_CG)", &
    "* DX(m_body_CG,mfr_upright_wheel_centre,m_body_CG)", &
    " )", &
    " /", &
    " ( varval(ground_plane_velocity) + 0.00001 )"

data_element create variable &
  variable_name = yaw_rate_error &
  function = " varval(demanded_yaw_rate) - varval(front_axle_no_slip_yaw)"

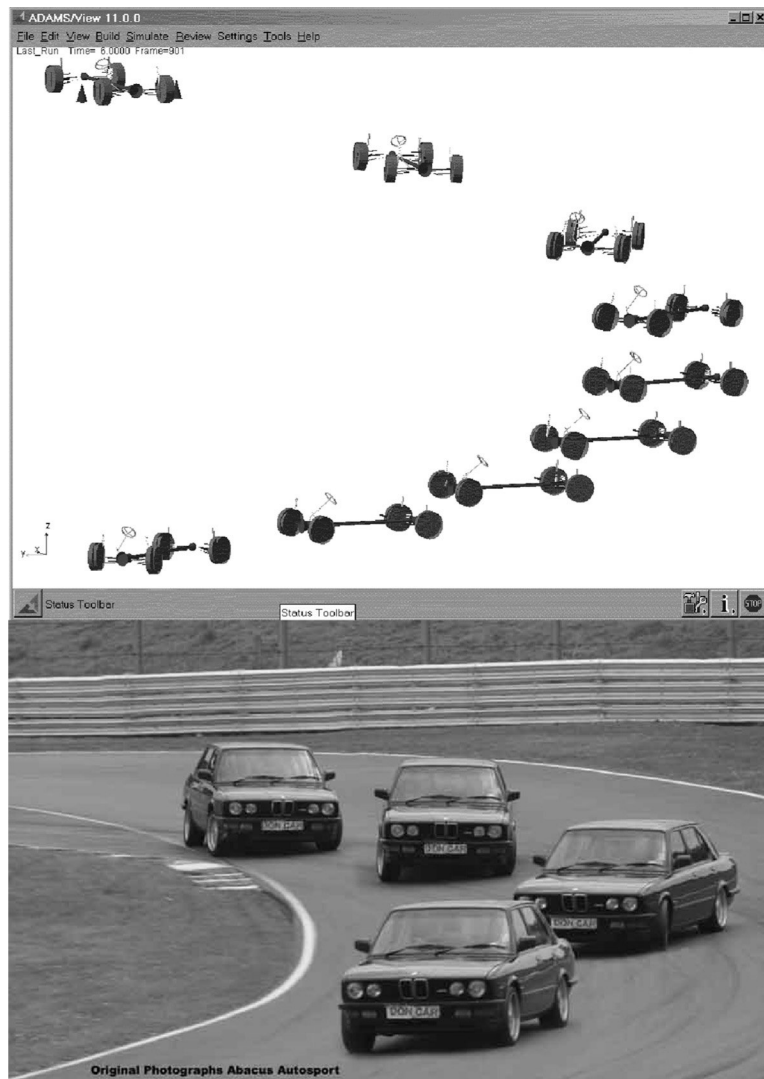
```

### 6.13.3 Body slip angle control

Skilled drivers, particularly rally drivers, frequently operate at large body slip angles. Colloquially, there is much talk of body slip angles being in excess of  $45^\circ$  but recorded data suggest this is not the case despite appearances. Large body slip angles generally slow progress; although some of the yaw transients are rapid, in general the actual body slip angles are comparatively small (Figure 6.48). In the authors' experience, drivers greatly overestimate body slip angle subjectively.

The steering system on a vehicle has only around  $30^\circ$  of lock and so realistically, control beyond these body slip angles is unlikely without very large amounts of space indeed (Figure 6.49).

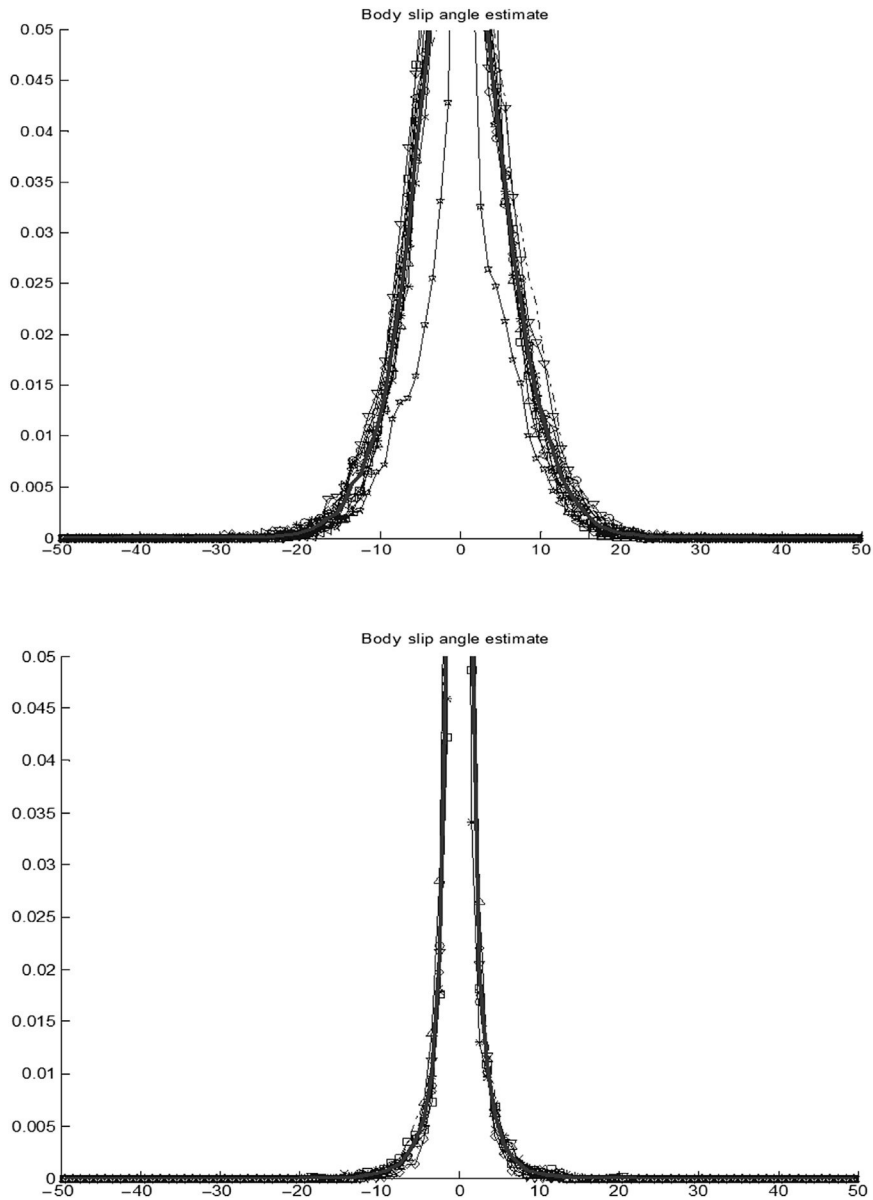
Most drivers are acutely sensitive to the rate of change of body slip angle, albeit they do not always respond correctly to it. Instead a 'threshold' behaviour appears common, with drivers neglecting body slip angle until either the angle becomes large or its rate of change becomes large. For road cars, our goals are to have a road car manage its own body slip angle so as not to put pressure on drivers in an area where,

**FIGURE 6.47**

Driver and vehicle behaviour for a post-limit turn-in event.

Photograph courtesy of Don Palmer, [www.donpalmer.co.uk](http://www.donpalmer.co.uk).

in general, skill is lacking. For driver modelling purposes, a separate body slip angle control loop is desirable to catch spins. It need not be terribly sophisticated since if it is invoked then we have to some extent failed in our goal of having the car manage its own body slip angle. Such behaviour is desirable in the real vehicle too and is the goal of active intervention systems such as brake-based stability control systems; however, the robust sensing of body slip angle still proves elusive in a cost-effective manner

**FIGURE 6.48**

Probability density for body slip angle estimates — Greece 2002 (top) and Germany 2002 (bottom) for Petter Solberg, Subaru World Rally Team.

**FIGURE 6.49**

Large body slip angles are unavailable to normal drivers except as part of an accident. Subaru WRC, Greece 2002.

*Courtesy of Prodrive.*

despite its apparent simplicity. Since the first edition of this book, a frisson of excitement was generated by a proposed Doppler laser sensor that appeared robust on a wide range of surfaces and used low-cost components similar to many computer mice that work on the same principles; however development of such a device appears to have been halted, disappointingly.

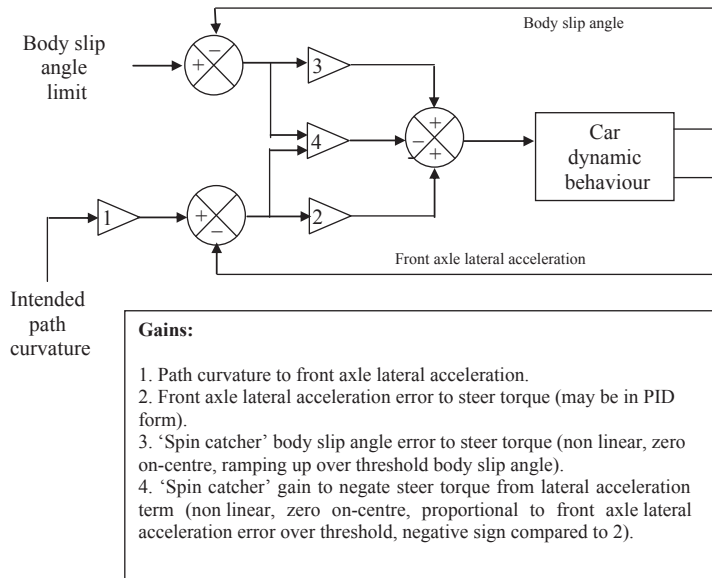
#### 6.13.4 Two-loop driver model

For general use, the authors favour a simple and robust two-loop driver model comprising a path follower and a spin catcher, with a separate speed control as appropriate to the task at hand. [Figure 6.50](#) shows such a model.

### 6.14 Case study 7 – trajectory preparation for a NATO lane change

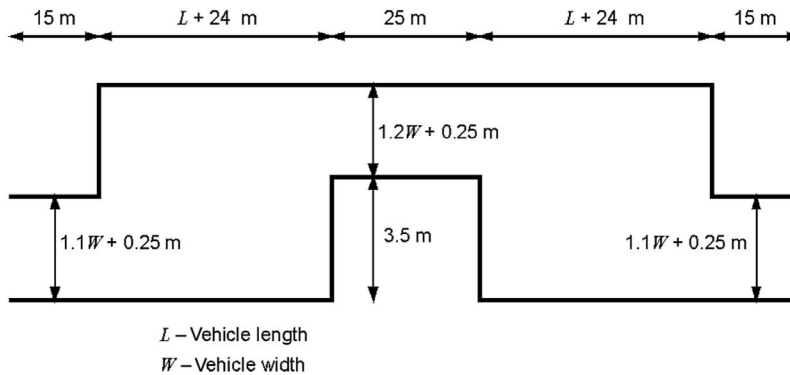
The NATO lane change is essentially a scaled version of the ISO3888 lane change in its geometry, only the test protocol differs slightly. It consists of a defined zone in which the vehicle can take any path and reflects a real-world avoidance manoeuvre within a finite width road. The vehicle must displace laterally by around 3.5 m minimum and then return to its original path. Where ISO3888 is quite short, the NATO double lane change is intended to reflect maximum effort manoeuvring at highway speeds for larger vehicles and is thus commensurately longer. The length and width of the manoeuvring zone are related to the vehicle proportions, as shown in [Figure 6.51](#).

It can be seen in [Figure 6.51](#) that the course is entirely symmetric. For the NATO lane change the vehicle is driven up to the manoeuvre start and the throttle released;



**FIGURE 6.50**

A two-loop driver model.



**FIGURE 6.51**

NATO (AVTP 03-160) lane change test course.

the overrun condition is typically the most difficult in terms of stability. Although the vehicle loses speed as it enters the second of the two manoeuvres, it is typically unsettled dynamically and so the second transition is often the more problematic of the two.

For the subject vehicle, the dimensions used are 6 m length and 2.4 m width.

With real vehicles the performance of a given vehicle through the lane change can be substantially modified with driver skill levels and so when comparing

vehicles it is preferable to use a panel of drivers or at least a consistent driver. In the analysis the driver model is consistent and repeatable, although there is no guarantee it accurately reflects real drivers.

In contrast to other driver models it has no preview. It has been used for a number of vehicle classes and gives satisfactory performance. By controlling only the front axle it places particular emphasis on vehicle stability. While it is undoubtedly some way short of a full description of a skilled driver, nevertheless it exercises the vehicle in a representative and consistent manner — for example the yaw overshoot following the steer reversal is countered with a little opposite lock.

When real drivers complete the manoeuvre they experiment noticeably with the exact trajectory over several attempts. This simulation model is unable to learn from its previous experiences but some modification of the intended trajectory is worth mentioning. For simplicity, only the first half of the manoeuvre is considered in the following discussions (Figure 6.52).

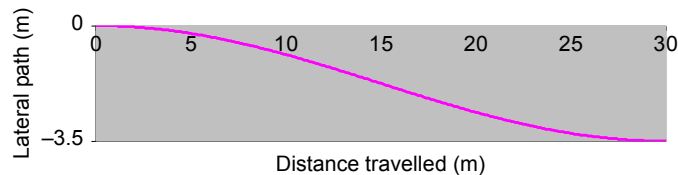
It is tempting to draw the lane change path as a series of connected arcs (O'Hara, 2005) but this does not reflect the reality of a driver's inputs, which are observed to be somewhat fluid and continuous during a high-effort lane change, without the dwell periods a series of arcs would imply. An improvement might be to view the path as a 'cosine ramp' of the form:

$$Y = \frac{3.5}{2} \left( 1 - \cos \left( \frac{X\pi}{24 + L} \right) \right) \quad (6.55)$$

It is presumed that  $Y$  is constant before 0 m and after 30 m. It is a trivial matter to differentiate the curve once for 'compass heading' and again for curvature. Curvature is  $1/\text{radius}$  and thus some notional centripetal acceleration can be computed with no knowledge of vehicle characteristics:

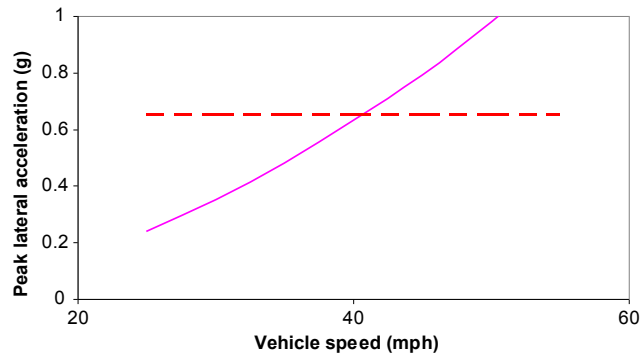
$$A_y = \frac{d^2 Y}{dX^2} V \quad (6.56)$$

With due regard for the lateral acceleration limit imposed by the centre of mass height of the vehicle, it can be seen in Figure 6.53 that using this path geometry will limit the vehicle to something around 40 mph, which is well below reported values for this class of vehicle.



**FIGURE 6.52**

A cosine ramp trajectory for a lane change.

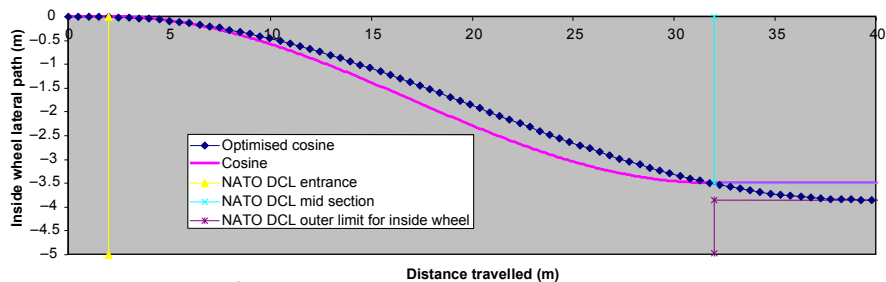
**FIGURE 6.53**

A comparison of trajectory centripetal acceleration (solid) with the centre of mass height-imposed limit (dashed) for the cosine ramp path.

The intended line across the ground is not continuously differentiable. Some further thought suggests a route across the ground made up of several segments – a turn-in segment represented by a cosine ramp up to maximum curvature, a reversal segment from maximum curvature in one direction to maximum curvature in the other and finally a turn-out segment represented by a third cosine ramp. This function is continuously differentiable and the relative length of the turn-in, reversal and turn-out segments can be tuned.

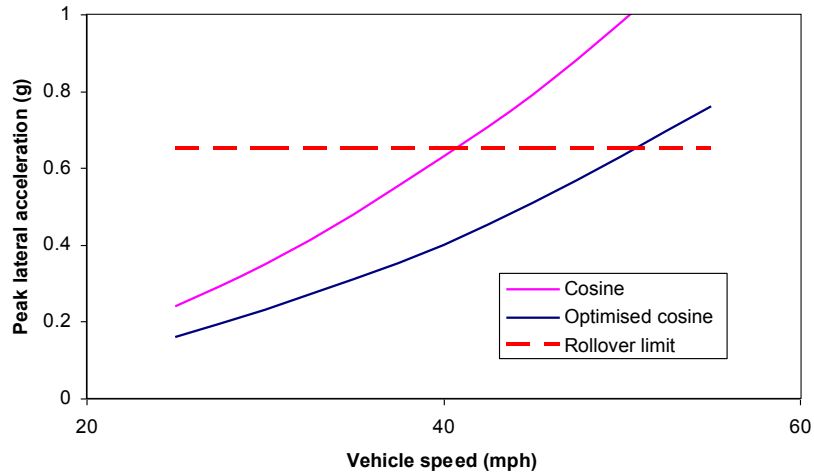
When considering the task in detail, it is noted that the midsection is wider than the vehicle. By positioning the vehicle fractionally away from the very edge of the entrance gate, by tuning the relative length of the turn-in and turn-out sections and by running out to the very widest part of the path available, the length for the manoeuvre is extended from a nominal 30 to 42.5 m, as shown in Figure 6.54.

If this ‘optimised cosine’ form is compared with the initial cosine form, a substantial reduction in centripetal acceleration is realised, as shown in Figure 6.55.

**FIGURE 6.54**

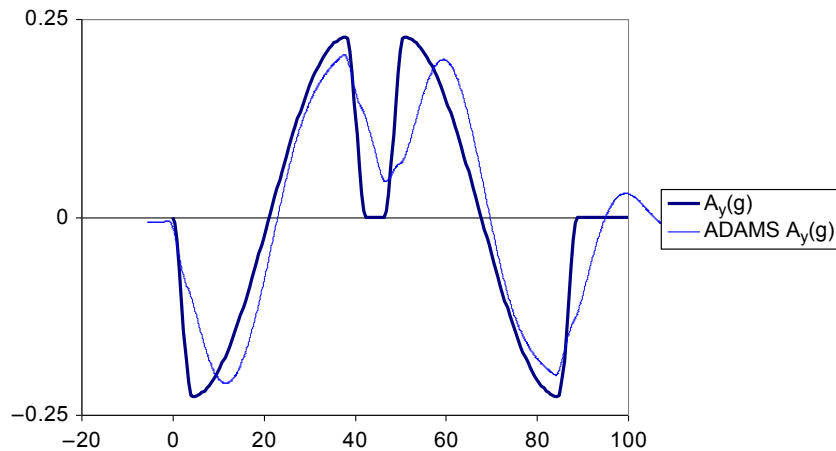
An improved lane change path that stays within NATO constraints and is more typical of real driver behaviour.



**FIGURE 6.55**

A comparison between the optimised cosine form and the previous cosine ramp in comparison within the rollover limit.

A comparison can be made between the base vehicle simulation and the trajectory-only values at 30 mph and shows that even at 30 mph the response is being coloured and limited by the dynamics of the vehicle, but that nevertheless there is broad agreement in [Figure 6.56](#).

**FIGURE 6.56**

A comparison between the trajectory-predicted lateral acceleration and the results achieved with a detailed multibody system model at 30 mph.

Thus it can be seen that in order to deliver the best possible performance from a given model, the preparation of the trajectory is of some importance, and in fact the trajectory for a given manoeuvre shows substantial scope for optimisation — this can easily be overlooked in the pursuit of model detail and accuracy.

## 6.15 Case study 8 – comparison of full vehicle handling models

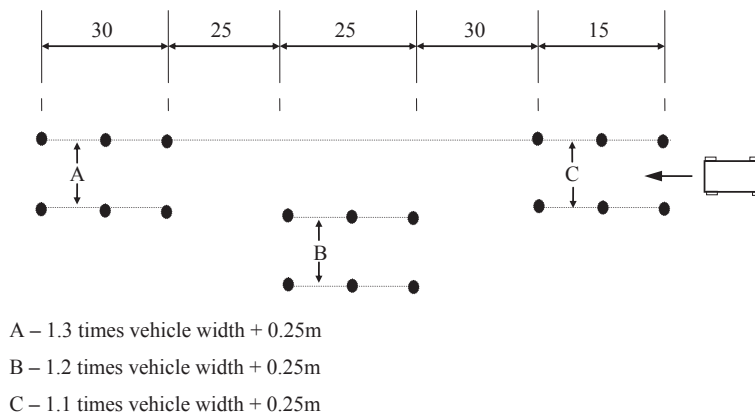
As mentioned at the start of this chapter the use of modern MBS software provides users with the capability to develop a model of a full vehicle that incorporates all the major vehicle subsystems. Clearly the development of such a model is dependent on the stage of vehicle design and the availability of the data needed to model all the subsystems. For the vehicle dynamics task, however, the automotive engineer will want to carry out simulations before the design has progressed to such an advanced state.

In the following case study the level of vehicle modelling detail required to simulate a ‘full vehicle’ handling manoeuvre will be explored. The types of manoeuvres performed on the proving ground are discussed in the next chapter but as a start we will consider a 100 km/h double lane change manoeuvre. The test procedure for the double lane change manoeuvre is shown schematically in Figure 6.57.

For the simulations performed in the following case study the measured steering wheel inputs from a test vehicle have been extracted and applied as a time-dependent handwheel rotation (Figure 6.58) as described in Section 6.12.3.

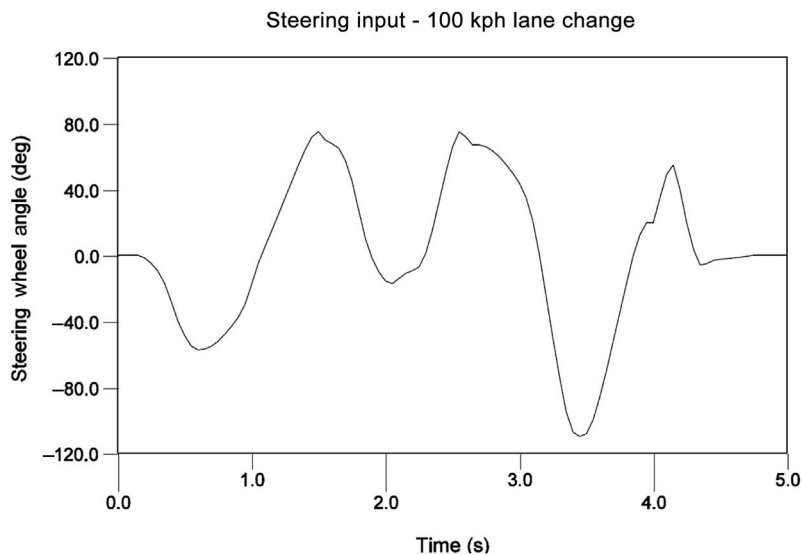
To appreciate the use of computer simulations to represent this manoeuvre, an example of the superimposed animated wireframe graphical outputs for this simulation is given in Figure 6.59.

In this study the influence of suspension modelling on the accuracy of the simulation outputs is initially discussed based on results obtained using the four vehicle

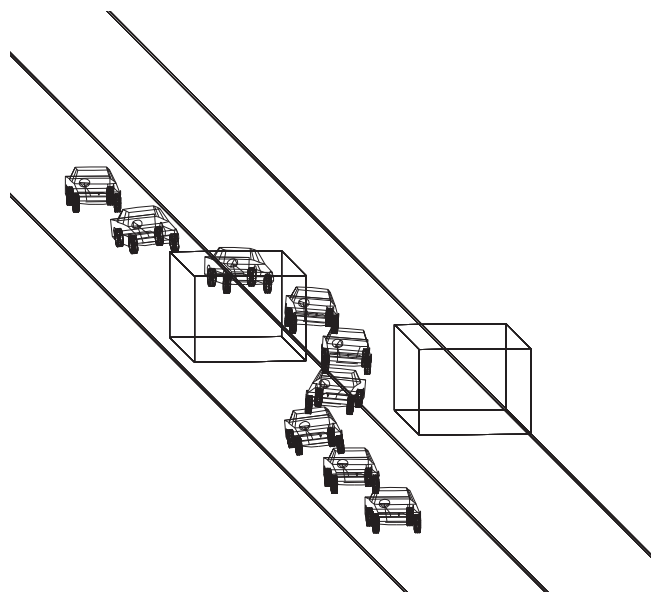


**FIGURE 6.57**

Lane change test procedure.

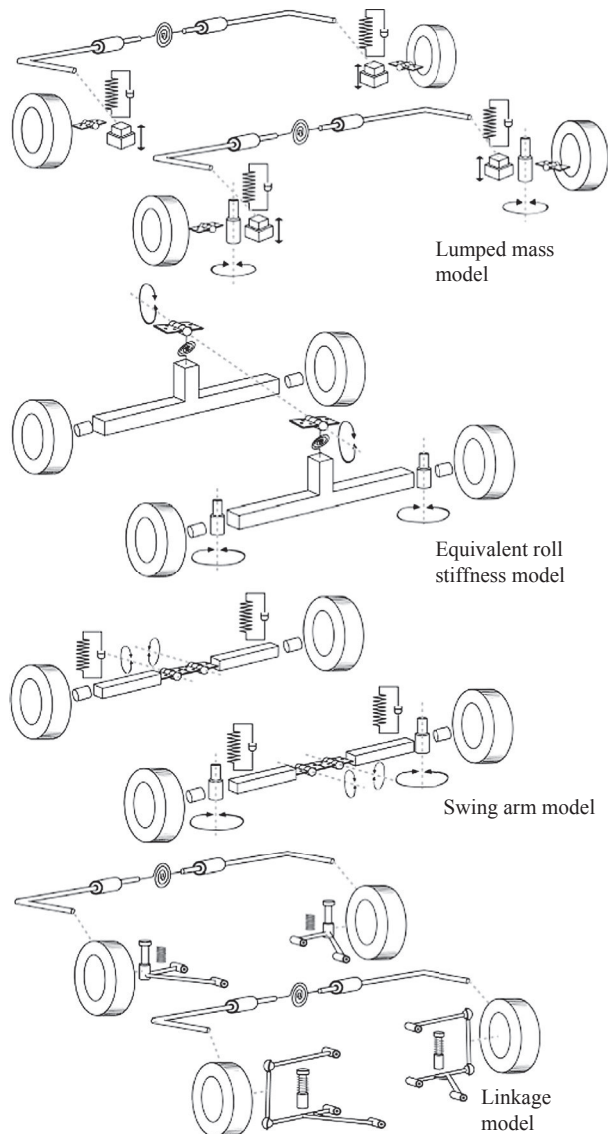


**FIGURE 6.58**  
Steering input for the lane change manoeuvre.



**FIGURE 6.59**  
Superimposed graphical animation of a double lane change manoeuvre.

models described in Section 6.4 and summarised schematically again here in Figure 6.60. The models shown can be thought of as a set of models with evolving levels of elaboration leading to the final linkage model that involves the modelling of the suspension linkages and the bushes.



**FIGURE 6.60**

Modelling of suspension systems.

For each of the vehicle models described here it is possible to estimate the model size in terms of DOF in the model and the number of equations that MSC ADAMS uses to formulate a solution. The calculation of DOF in a system is based on the Gruebler equation given in Chapter 3. It is therefore possible for any of the vehicle models to calculate DOF in the model. An example is provided here for the equivalent roll stiffness model where DOF can be calculated as follows:

$$\text{Parts} \quad 9 \times 6 = 54$$

$$\text{Rev} \quad 8 \times -5 = -40$$

$$\text{Motion} \quad 2 \times -1 = -2$$

$$\sum \text{DOF} = 12$$

In physical terms it is more meaningful to describe these DOF in relative terms as follows. The vehicle body part has 6 DOF. The two axle parts each have one rotational DOF relative to the body. Each of the four road wheel parts has one spin DOF relative to the axles making a total of 12 DOF for the model.

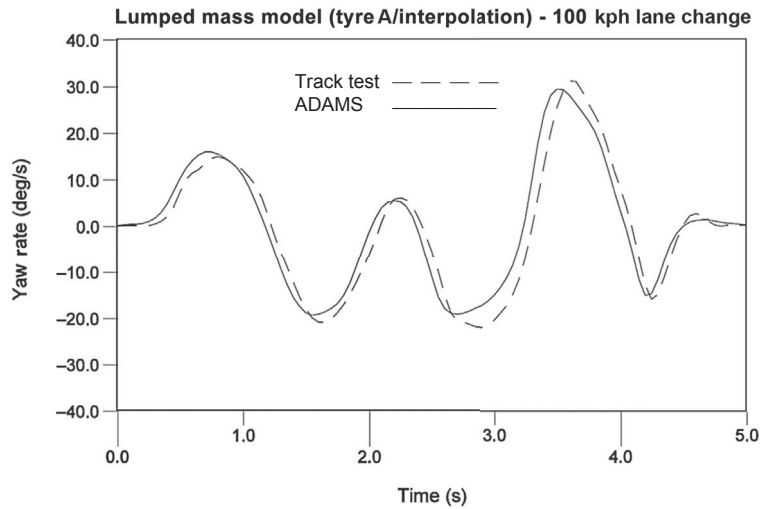
When a simulation is run in MSC ADAMS the programme will also report the number of equations in the model. As discussed in Chapter 3 the software will formulate 15 equations for each part in the model and additional equations representing the constraints and forces in the model. On this basis the size of all the models is summarised in [Table 6.6](#).

The size of the model and the number of equations is not the only issue when considering efficiency in vehicle modelling. Of perhaps more importance is the engineering significance of the model parameters. The roll stiffness model, for example, may be preferable to the lumped mass model. It is not only a simpler model but is also based on parameters such as roll stiffness that will have relevance to the practicing vehicle dynamicists. The roll stiffness can be measured on an actual vehicle or estimated during vehicle design. This model does, however, incorporate rigid axles eliminating the independent suspension characteristics. Note that in this case study an interpolation tyre model of the type described in Chapter 5 has been used with each vehicle model.

Measured outputs including lateral acceleration, roll angle and yaw rate can be compared with measurements taken from the vehicle during the same manoeuvre on the proving ground to assess the accuracy of the models. By way of example the yaw rate predicted by simulation with all four models is compared with measured track test data in [Figures 6.61–6.64](#).

**Table 6.6** Vehicle Model Sizes

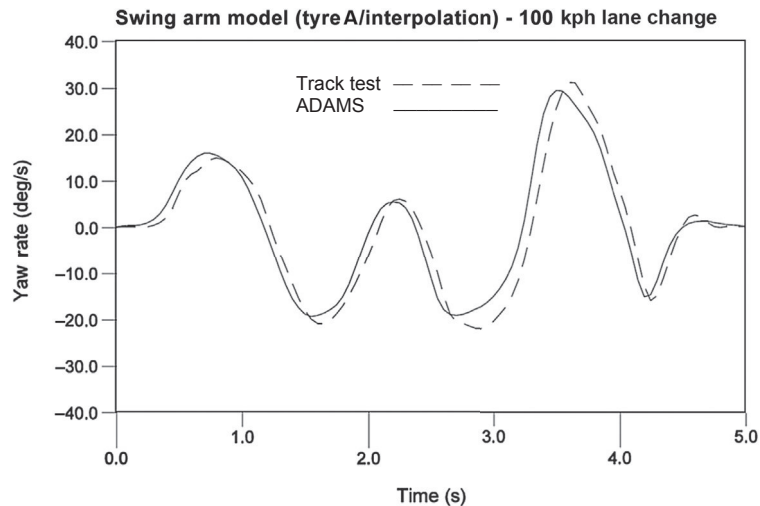
Model	Degrees of Freedom	Number of Equations
Linkage	78	961
Lumped mass	14	429
Swing arm	14	429
Roll stiffness	12	265

**FIGURE 6.61**

Yaw rate comparison – lumped mass model and test.

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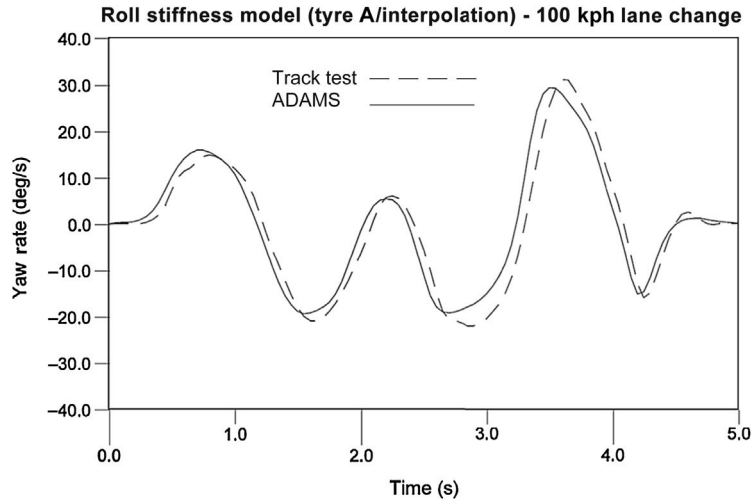
*'The modelling and simulation of vehicle handling, Part 4: handling simulation', M.V. Blundell, page 80, by permission of the Council of the Institution of Mechanical Engineers.*

**FIGURE 6.62**

Yaw rate comparison – swing arm model and test.

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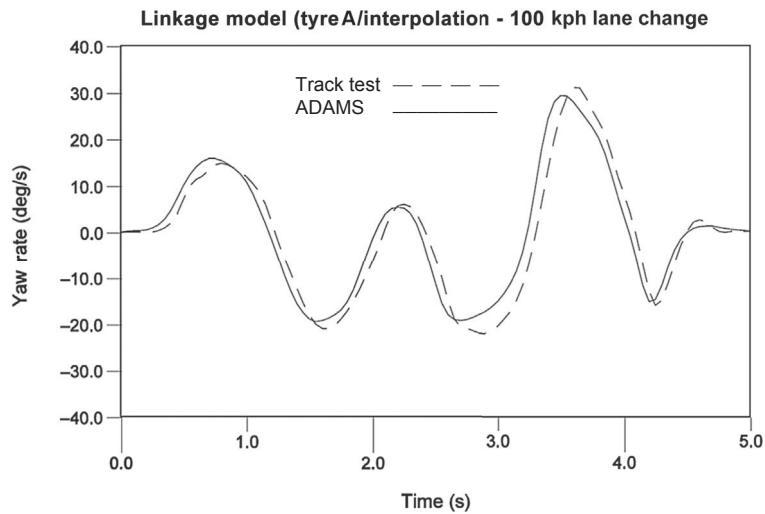
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**FIGURE 6.63**

Yaw rate comparison — roll stiffness model and test.

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**FIGURE 6.64**

Yaw rate comparison — linkage model and test.

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Examination of the traces in [Figures 6.61–6.64](#) raises the question as to how an objective assessment of the accuracy of the simulations may be made. Accuracy is not a ‘yes/no’ quantity, but instead a varying absence of difference exists between predicted (calculated) behaviour and measured behaviour. Such a ‘difference’ is commonly referred to as an ‘error’. This definition neatly sidesteps two other difficulties:

- is the measured data what actually happens in the absence of measurement?
- is the measured data what actually happens during service?

For example, the mass-loading effect of accelerometers may introduce inaccuracies at high frequencies and could mean that the system of interest behaves differently when being measured to when not. The accuracy of controlled measurements in discerning the behaviour of the system when in normal uncontrolled use is another matter entirely. Both topics are far from trivial.

In this case other questions arise such as:

- does the model data accurately represent the vehicle conditions on the day of the test?
- does the tyre test data obtained on a tyre test machine accurately represent the condition of the test surface and tyres used on the day of the test?
- how repeatable are the experimental test results used to make an assessment of model accuracy?
- is there a model data input error common to all the models?

Comparing the performance of the equivalent roll stiffness model with that of the linkage model in [Figures 6.63 and 6.64](#) it is possible to look, for example, at the error measured between the experimental and simulated results for the peaks in the response or to sum the overall error from start to finish. On that basis it may seem desirable to somehow ‘score’ the models giving say the linkage model 8/10 and the roll stiffness model 7/10. In light of the above questions the validity of such an objective measure is debatable and it is probably more appropriate to simply state:

‘For this vehicle, this manoeuvre, the model data, and the available benchmark test data, the equivalent roll stiffness model provides reliable predictions when compared with the linkage model for considerably less investment in model elaboration.’

Clearly it is also possible to use an understanding of the physics of the problem to aid the interpretation of model performance. An important aspect of the predictive models is whether the simplified suspension models correctly distribute load to each tyre and model the tyre position and orientation in a way that will allow a good tyre model to determine forces in the tyre contact patch that impart motion to the vehicle and produce the desired response. Taking this a step further we can see that if we use the equivalent roll stiffness and linkage models as the basis for further comparison it is possible in [Figures 6.57 and 6.58](#) to compare the vertical force in for example the front right and left tyres. The plots indicate the performance

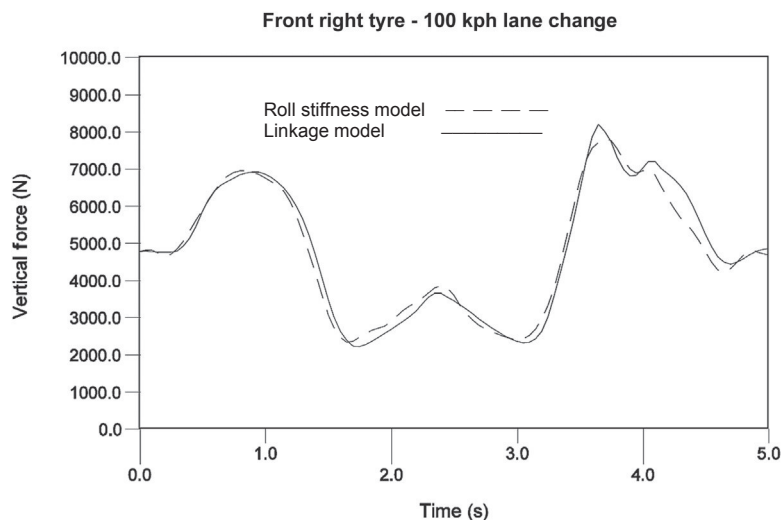


of the simple equivalent roll stiffness model in distributing the load during the manoeuvre. The weight transfer across the vehicle is also evident as is the fact that tyre contact with the ground is maintained throughout. It should also be noticed that in determining the load transfer to each wheel the equivalent roll stiffness model does not include DOF that would allow the body to heave or pitch relative to the suspension systems.

In Figures 6.65–6.68 a similar comparison between the two models is made this time considering, for example, the slip and camber angles predicted in the front right tyre.

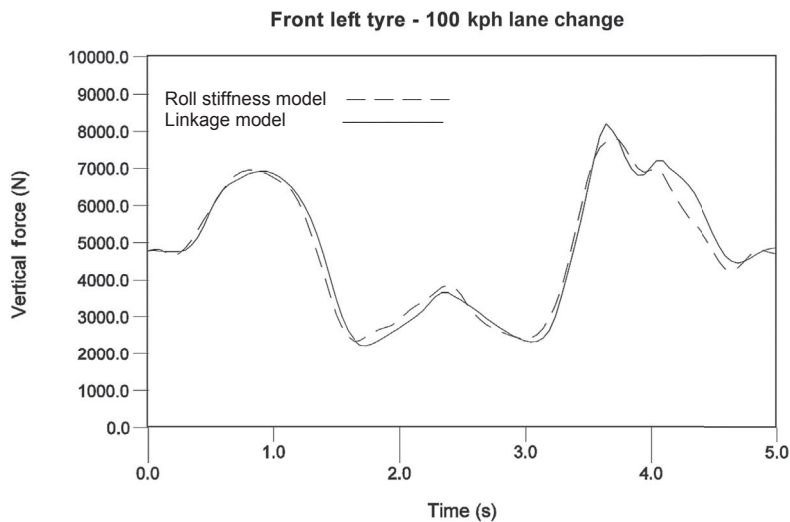
Although the prediction of slip angle agrees well it can be seen in Figure 6.60 that the equivalent roll stiffness model with a maximum value of about  $1.5^\circ$  underestimates the amount of camber angle produced during the simulation when compared with the linkage model where the camber angle approaches  $5^\circ$ . Clearly the wheels in the effective roll stiffness model do not have a camber DOF relative to the rigid axle parts and the camber angle produced here is purely due to tyre deflection.

As discussed in Chapter 5 it is perhaps fortuitous in this case that for a passenger car of the type used here the lateral tyre force produced due to slip angle is considerably more significant than that arising due to camber between the tyre and road surface. Further investigations can be carried out to establish the significance of a poor camber angle prediction input to the tyre model. In Figure 6.69 the linkage model has been run using an interpolation tyre model where it has been possible to deactivate the generation of lateral force arising from camber angle. In this plot

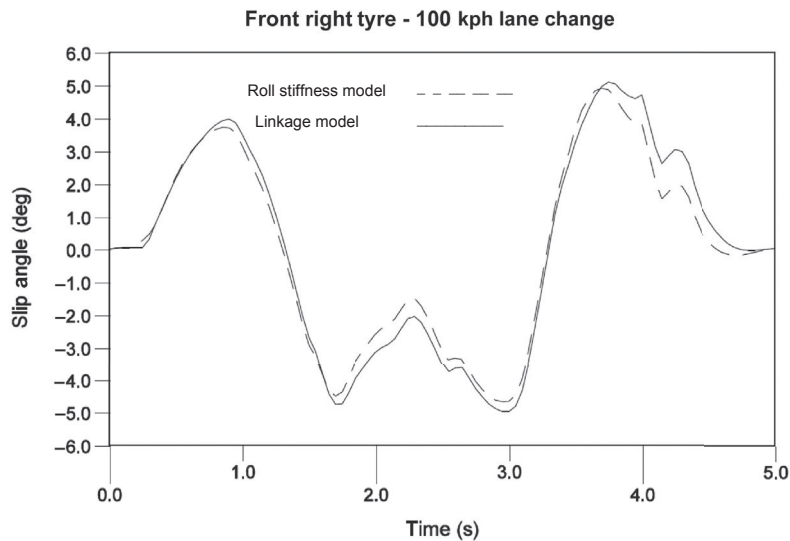


**FIGURE 6.65**

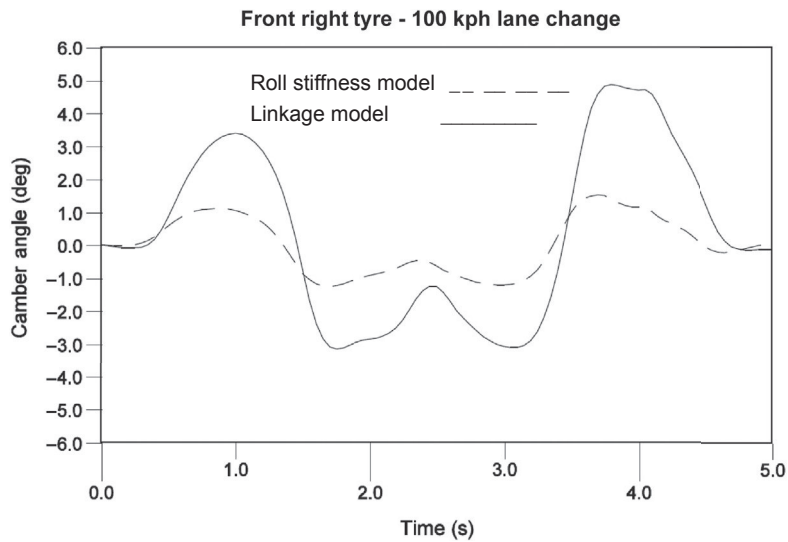
Vertical tyre force comparison — linkage and roll stiffness models.



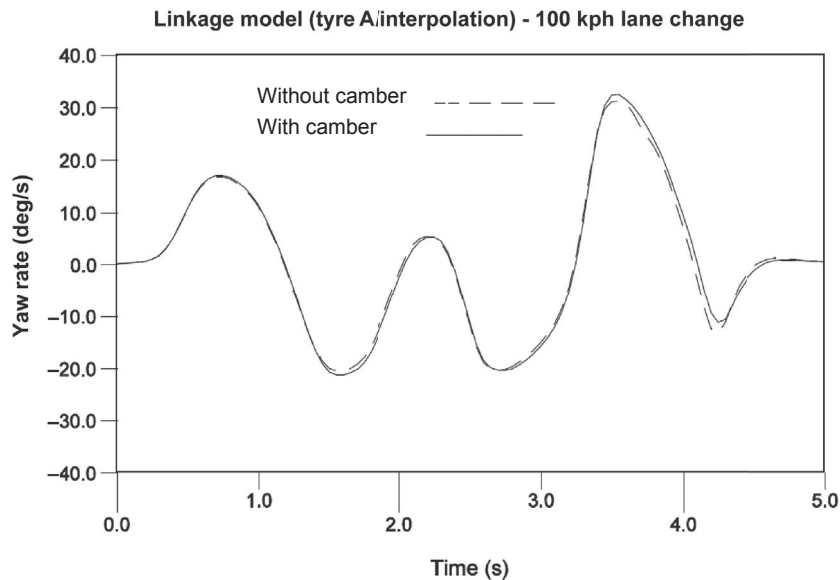
**FIGURE 6.66**  
Vertical tyre force comparison – linkage and roll stiffness models.



**FIGURE 6.67**  
Slip angle comparison – linkage and roll stiffness models.

**FIGURE 6.68**

Camber angle comparison — linkage and roll stiffness models.

**FIGURE 6.69**

Yaw rate comparison — interpolation tyre model.

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it can be seen that the prediction of yaw rate, for example, is not sensitive for this vehicle and this manoeuvre to the modelling of camber thrust.

To conclude this case study it is possible to consider an alternative modelling and simulation environment for the prediction of the full vehicle dynamics. As discussed earlier the incorporation of microprocessor control systems in a vehicle may involve the use of a simulation method that involves:

1. the use of MBS software where the user must invest in the modelling of the control systems.
2. the use of software such as MATLAB/Simulink where the user must invest in the implementation of a vehicle model or,
3. a co-simulation involving parallel operation of the MBS and control simulation software.

In this example the author (Wenzel et al., 2003)<sup>2</sup> has chosen the second of the above options and a vehicle model (Figure 6.70) is developed from first principles and implemented in Simulink. The model developed here is based on the same data used for this case study with 3 DOF: the longitudinal direction  $x$ , the lateral direction  $y$  and the yaw around the vertical axis  $z$ .

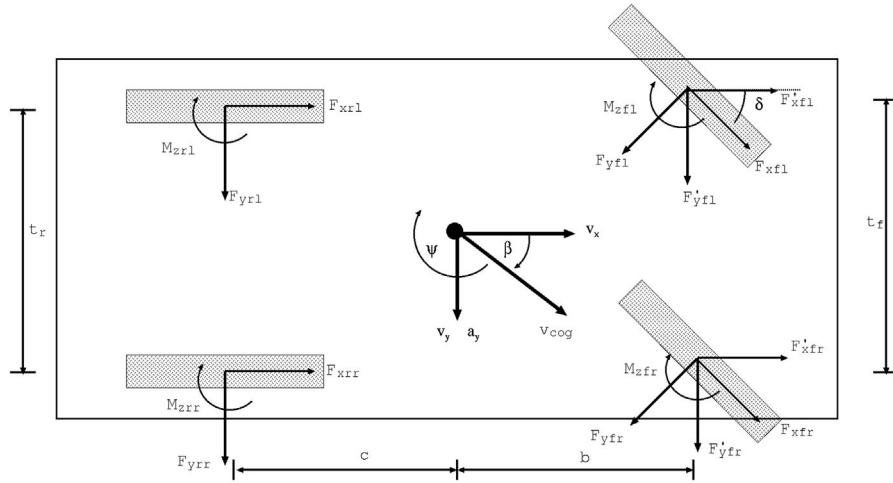
The vehicle parameters used in the following model include:

$v_x$	Longitudinal velocity (m/s)
$v_y$	Lateral velocity (m/s)
$v_{cog}$	Centre of gravity velocity (m/s)
$a_x$	Longitudinal acceleration (m/s <sup>2</sup> )
$a_y$	Lateral acceleration (m/s <sup>2</sup> )
$\Gamma$	Torque around $z$ -axis (Nm)
$\delta$	Steer angle (rad)
$\beta$	Side slip angle (rad)
$\alpha_{ij}$	Wheel slip angles (rad)
$\dot{\psi}$	Yaw rate (rad/s)
$F_{zij}$	Vertical forces on each wheel (N)
$l_j$	Position: $i = \text{front}(f)/\text{rear}(r)$ , $j = \text{left}(l)/\text{right}(r)$

Note that steer angle  $\delta$  and the velocity of the vehicle's centre of gravity  $v_{cog}$  are specified as model inputs.

The relationship between the dynamic vehicle parameters can be formulated as differential equations. Most of these can be found in the standard literature. Using

<sup>2</sup>Wenzel et al., 2003, describes preliminary work undertaken in a collaborative research project with Jaguar Cars Ltd, Coventry, UK and funded by the Control Theory and Applications Centre, Coventry University, Coventry, UK. It forms the PhD programme for Thomas A Wenzel.



**FIGURE 6.70**

Three degree of freedom vehicle model (Wenzel et al., 2003).

formulae by Wong (2001) and Will and Žak (1997) the following differential equations for acceleration, torque and yaw rate can be derived:

$$\dot{v}_x = \frac{1}{m}(F_{xfl}\cos\delta - F_{yfl}\sin\delta + F_{xfr}\cos\delta - F_{yfr}\sin\delta + F_{xrl} + F_{xrr}) + v_y\dot{\psi} \quad (6.57)$$

$$\dot{v}_y = \frac{1}{m}(F_{yfl}\cos\delta + F_{xfl}\sin\delta + F_{yfr}\cos\delta + F_{xfr}\sin\delta + F_{yrl} + F_{yrr}) - v_x\dot{\psi} \quad (6.58)$$

$$\Gamma = \frac{t_f}{2}F_{xfl}' - \frac{t_f}{2}F_{xfr}' + \frac{t_r}{2}F_{xrl} - \frac{t_r}{2}F_{xrr} + bF_{yfl}' + bF_{yfr}' - cF_{yrl} - cF_{yrr} + M_{zfl} + M_{zfr} + M_{zrl} + M_{zrr} \quad (6.59)$$

$$\ddot{\psi} = \frac{\Gamma}{J_z} \quad (6.60)$$

where the additional parameters are defined as:

$F_{xij}$	$F_{xij}$ longitudinal forces on tyre $ij$ (N)
$F_{yij}$	$F_{yij}$ lateral forces on tyre $ij$ (N)
$F'_{xij}$	$F'_{xij}$ longitudinal forces on tyre $ij$ in the vehicle's coordinate system (N)
$F'_{yij}$	$F'_{yij}$ lateral forces on tyre $ij$ in the vehicle's coordinate system (N)
$M_{zij}$	$M_{zij}$ self-aligning moment on tyre $ij$ (Nm)
$m$	$m$ mass of vehicle (kg)
$J_z$	$J_z$ moment of inertia around vertical axis ( $\text{Nm}^2$ )
$t_f, t_r$	$t_f, t_r$ front and rear track width (m)
$b, c$	$b, c$ position of centre of gravity between wheels (m)

Other important states are the wheel slip angles  $\alpha_{ij}$  and the body slip angle  $\beta$ , defined as follows:

$$\alpha_{fl/r} = \delta - \arctan\left(\frac{v_y + b\dot{\psi}}{v_x \pm \frac{1}{2}t_f\dot{\psi}}\right) \quad (6.61)$$

$$\alpha_{rl/r} = \arctan\left(\frac{-v_y + c\dot{\psi}}{v_x \pm \frac{1}{2}t_r\dot{\psi}}\right) \quad (6.62)$$

$$\beta = \arctan\left(\frac{v_y}{v_x}\right) \quad (6.63)$$

In this model roll and pitch of the vehicle are neglected but weight transfer is included to determine the vertical load at each wheel as defined by Milliken and Milliken (1998):

$$F_{zfl/r} = \left(\frac{1}{2}mg \pm m \frac{a_y h}{t}\right) \frac{c}{\ell} - m a_x \frac{h}{\ell} \quad (6.64)$$

$$F_{zrl/r} = \left(\frac{1}{2}mg \pm m \frac{a_y h}{t}\right) \frac{b}{\ell} + m a_x \frac{h}{\ell} \quad (6.65)$$

The additional parameters are the height  $h$  of the vehicle's centre of gravity, the wheelbase,  $\lambda$  and the gravitational acceleration,  $g$ .

In [Eqns \(6.66\) and \(6.67\)](#) it has to be considered that  $a_x \neq \dot{v}_x$  and  $a_y \neq \dot{v}_y$ . The yaw motion of the vehicle has to be taken into account (Wong, 2001) giving:

$$a_x = \dot{v}_x - v_y \dot{\psi} \quad (6.66)$$

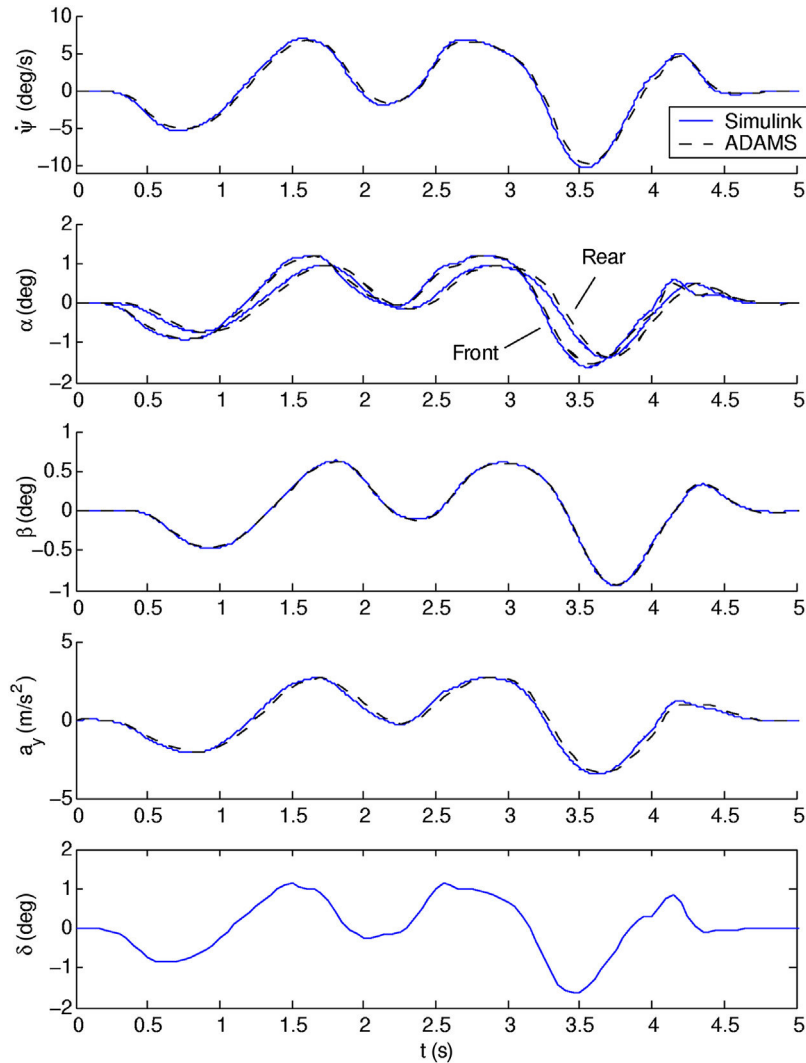
$$a_y = \dot{v}_y + v_x \dot{\psi} \quad (6.67)$$

In this work the author (Wenzel et al., 2003) has simulated a range of vehicle manoeuvres using both the Magic Formula and Fiala tyre models described in Chapter 5. The example shown here is for the lane change manoeuvre used in this case study with a reduced steer input applied at the wheels as shown in the bottom of [Figure 6.71](#).

Also shown in [Figure 6.63](#) are the results from the Simulink model and a simulation run with the MSC ADAMS linkage model. For this manoeuvre and vehicle data set the Simulink and MSC ADAMS models can be seen to produce similar results.

In completing this case study there are some conclusions that can be drawn. For vehicle handling simulations it has been shown here that simple models such as the equivalent roll stiffness model can provide good levels of accuracy. It is known however, that roll centres will 'migrate' as the vehicle rolls, particularly as the

vehicle approaches limit conditions. The plots for case study 1 in Chapter 4 show the vertical movement of the roll centre along the centre line of the vehicle as the suspension moves between bump and rebound. On the complete vehicle, the geometrically constructed roll centre will also move laterally off the centre line as the vehicle rolls, but the exact significance of this is questionable, as shown in Figure 4.13.



**FIGURE 6.71**

Comparison of simulink and MSC ADAMS predictions of vehicle response.

Using an MBS approach to develop a simple model may also throw up some surprises for the unsuspecting analyst. The equivalent roll stiffness model, for example, does not include heave and pitch DOF relative to the front and rear axles. During the simulation, however, DOF exist for the body to heave and pitch relative to the ground inertial frame. These DOF must still be solved and in this case are damped only by the inclusion of the tyre model. In the 3 DOF model these motions are ignored and solution is only performed on the DOF that have been modelled. While the main theme in this book is to demonstrate the use of MBS analysis, the Matlab/Simulink model is useful here in providing the basis for additional modelling and simulation of the modern control systems involved in enhancing the stability and dynamics of the vehicle. The effort invested in this modelling approach also provides educational benefits reinforcing fundamental vehicle dynamics theory.

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## 6.16 Summary

Many different possibilities exist for modelling the behaviour of the vehicle driver. That none has reached prominence suggests that none is correct in every occasion. In general, the road car vehicle dynamics task is about delivering faithful behaviour during accident evasion manoeuvres — where most drivers rarely venture. Positioning the vehicle in the linear region is relatively trivial and need not exercise most organisations unduly, but delivering a good response, maintaining yaw damping and keeping the demands on the driver low are of prime importance in the nonlinear accident evasion regime. For this reason, controllers that take time to ‘learn’ the behaviour of the vehicle are inappropriate — road drivers do not get second attempts. For road vehicles, the closed loop controller based on front axle lateral acceleration gives good results and helps the analyst understand whether or not the vehicle is actually ‘better’ in the sense of giving an average driver the ability to complete a manoeuvre.

In motorsport applications, however, drivers are skilled and practised and so controllers with some feed-forward capability (to reflect ‘learnt’ responses), plus closed loop control of body slip angle are appropriate to reflect the high skill level of the driver. Whether or not advanced gain scheduling models, such as the MRAS or Self Tuning Regulator, are in use depends very much on whether or not data exist to support the verification of such a model. The authors’ preference is that ‘it is better to be simple and wrong than complicated and wrong’ — in other words, all other things being equal, the simplest model is the most useful since its shortcomings are more easily understood and judgements based on the results may be tempered accordingly. With elaborate schemes, particularly self-tuning ones, there is a strong desire to believe the complexity is in and of itself a guarantee of success. In truth if a relatively simple and robust model cannot be made to give useful results it is more likely to show a lack of clarity in forming the question than a justification for further complexity.



# Simulation Output and Interpretation

# 7

*Education is not the learning of many facts but the training of the mind to think.*

**Albert Einstein**

## 7.1 Introduction

Vehicle handling simulations are intended to recreate the manoeuvres and tests that vehicle engineers carry out using prototype vehicles on the test track or proving ground. Some are defined by the International Standards Organization, which outlines recommended tests in order to validate the handling performance of a new vehicle. A list of these tests is given in Appendix D.

Standards also exist for categories of vehicles other than passenger cars and manufacturers in those markets will use them as appropriate. Manufacturers will probably go beyond these minimum procedures in most market segments. The goal of excellence in handling performance is driven not by the need to meet fixed legislation but rather the increasing demands of a competitive marketplace.

The attractions of simulation are summarised in Chapter 1 as

- improved comprehension and ranking of design variables,
- rapid experimentation with design configurations and
- genuine optimisation of numerical response variables.

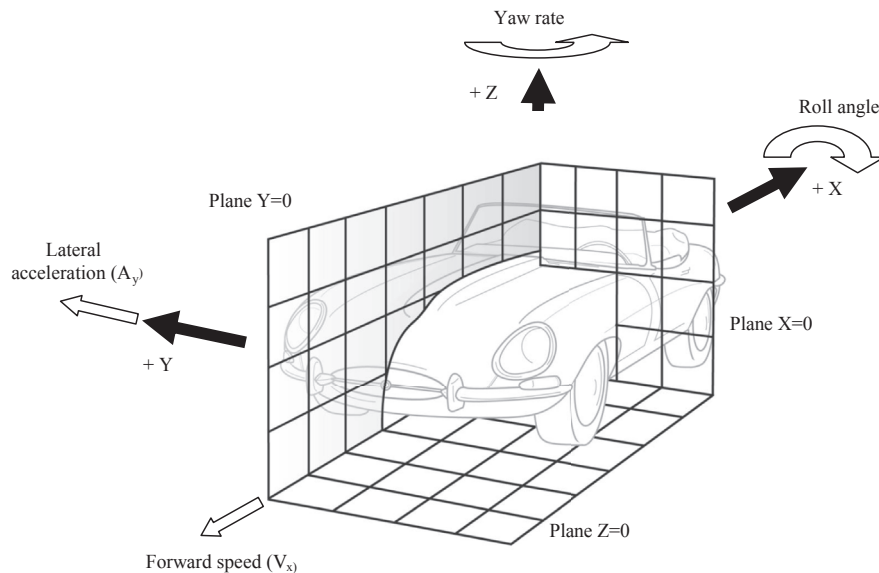
Also apparent is the ability to consider the behaviour of a vehicle before it exists physically. The cost for prototype vehicles can easily be £250,000 at the time of writing. Prudent use of simulation can ensure that only worthwhile prototype designs are turned into hardware. In many ways, well-considered simulation gives an equivalent to the use of a comprehensively instrumented vehicle in order to understand the sensitivity of the behaviour to the design variables available.

The use of instrumented vehicles to investigate handling performance can be traced back to the work of Segal in the early 1950s, which as mentioned in Chapter 1 was the subject of one of the seminal 'IME Papers' (Segel, 1956). Testing was carried out using a 1953 Buick Super four-door sedan (saloon), to investigate steady state behaviour with a fixed steering input at various speeds and also transient

response to sudden pulse inputs at the steering wheel. The instrumentation used at that time allowed the measurement of the following:

1. Left front wheel steer
2. Right front wheel steer
3. Steering wheel rotation
4. Lateral acceleration
5. Roll angle
6. Pitch angle
7. Yaw rate
8. Roll rate
9. Forward velocity.

Some of these responses are shown in [Figure 7.1](#). The trajectory (path) of the vehicle can also be recorded. With simulation this is straightforward but in the past it has been difficult to measure on the test track, testers resorting to measuring a trail of dye left by the vehicle on the test track surface. Modern satellite-based instrumentation has improved on this somewhat, with current systems able to record the trajectory of the front and the rear of the vehicle separately and concurrently at suitably high refresh rates. For each handling manoeuvre, or simulation, it is necessary for vehicle engineers to decide which responses are to be measured during the testing process.



**FIGURE 7.1**

Typical lateral responses measured in one of several possible vehicle coordinate frames.

All test and analytical activities are directed at understanding and improving the overall dynamic behaviour of the vehicle. As discussed in Chapter 1, the difficulty with the vehicle dynamics field is not the complexity of the effects in play but the level of interaction between them. A further difficulty is the level of change in behaviour required for a vehicle to be usefully ‘improved’. Typically, strong impressions of change are made with only modest variations of physical measures. When the difficulty of repeatable testing and the variation of impression with individual testers are both thrown in, the vehicle dynamics process lacks ‘capability’ – in the sense of quality control – compared to the task it is set. To explain this further, consider the following example.

## 7.2 Case study 9 – variation in measured data

Several attempts have been made to define single number measures for vehicle dynamic performance. A popular measure in the US is limiting lateral acceleration – frequently referred to as ‘grip’. Table 7.1 shows data that might be recorded during a steady state test, using a stopwatch to time a lap of a marked 200 m diameter circle.

If a honest error estimation is made in the recorded data then the results with the stopwatch are probably accurate to  $\pm 0.1$  s. Thus the raw data would appear as shown in Figure 7.2, with error bars on the figure as shown. This is a typical set of data for such a test; prolonging the test further might well degrade the tyres on the vehicle and lead to a ‘skewed’ result favouring the first test configuration with fresher tyres. It could be argued that tests should be repeated on fresh tyres for each different configuration to be examined but while academically sound, this argument neglects the commercial and temporal pressures placed on vehicle testing, particularly in intermediate configurations before matters are finalised and particularly if the tyres themselves are prototype items.

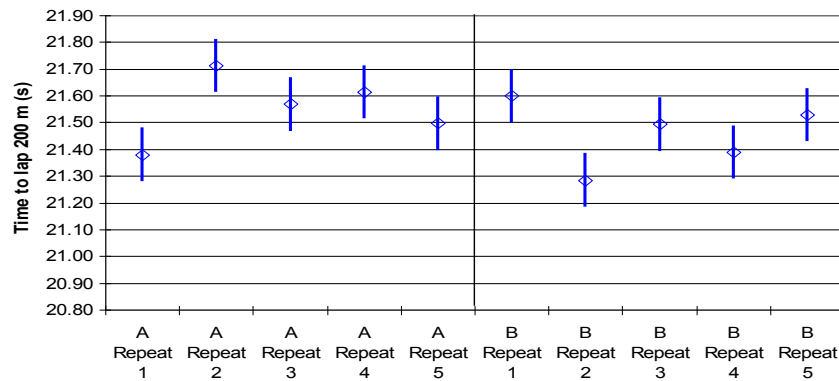
If these measurements are manipulated into lateral acceleration figures using the simple relationship

$$A_y = \frac{\left(\frac{200\pi}{t}\right)^2}{100} \quad (7.1)$$

then treated as the results that might be obtained from a production process and manipulated accordingly, the results in Table 7.2 emerge.

**Table 7.1** Time to Complete Lap of a Marked 200 m Diameter Circle (s)

	Test 1	Test 2	Test 3	Test 4	Test 5
Configuration A	21.38	21.71	21.57	21.61	21.50
Configuration B	21.60	21.49	21.29	21.39	21.53



**FIGURE 7.2**  
Raw Data from steady state test with error bars.

Table 7.2 Statistical Summary of Test Data		
	Mean	Population Standard Deviation
Configuration A	0.866 g	0.010 g
Configuration B	0.874 g	0.010 g
Difference A – B	0.008 g	

It is clear that the extrapolation to a population from only five samples is somewhat poor practice. However, it is equally clear that the difference between the two configurations is significantly less than the spread of the distribution of the tests. In statistical process control, a process is regarded as ‘capable’ if six times the standard deviation (so-called ‘ $6\sigma$ ’) is less than 75% of the tolerance on the measured attribute. For the measurement processes above, the ‘capability’ is no less than  $6 \times 0.01 \times 4/3 = 0.08 \text{ g}$  – 10 times the difference actually observed in the test. In other words, differences of less than 0.08 g cannot be controlled reliably using such a measurement process. However, such a difference is a significant one between otherwise similar vehicles, representing something around 10% of the total lateral acceleration available.

While the data set shown in the example is fictitious, the statistical character of the measurements is entirely typical, even under well-controlled conditions. When a vehicle is nearly optimised, this problem is typical. The resolution of the process in use – the test facilities and so on – is comparable to or greater than the control required to optimise the vehicle further. For this reason, in both motorsport and production vehicle engineering, a great deal of the final optimisation is based on subjective judgements of a few well-chosen individuals.

## 7.3 A vehicle dynamics overview

At this point, it is appropriate to develop some basic notions about vehicle dynamics. These definitions will be used later to suggest an interpretation of the subjective/objective relationship; however, it should be clear that by its very nature the absolute quantification of subjective qualities is impossible.

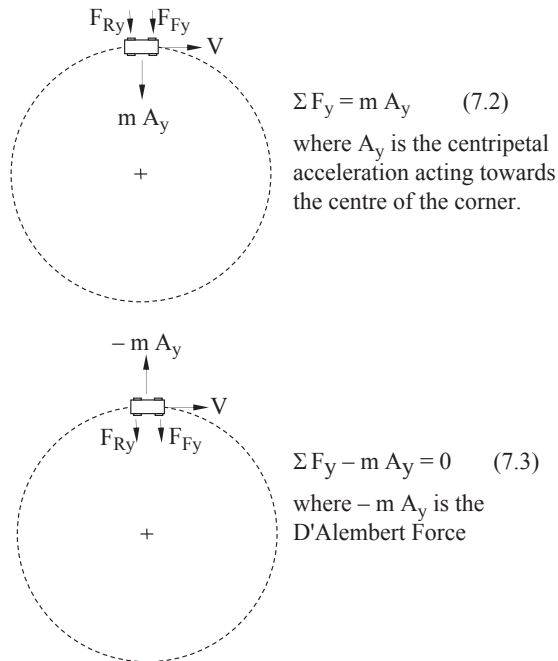
The driver has two primary concerns in controlling the vehicle. These are speed and path. Speed is controlled with engine power and braking systems. The use of separate controls for acceleration and deceleration is logical when the systems are separate but may become less so if vehicle architecture changes significantly; for example, some system with a ‘motor in each wheel’ that both accelerates and brakes might have a single pedal for control, of the type prototyped by Nomix AB in Sweden. Variation of speed is governed by vehicle mass and tractive/brake power availability at all but the lowest speed, and is easily understood.

### 7.3.1 Travel on a curved path

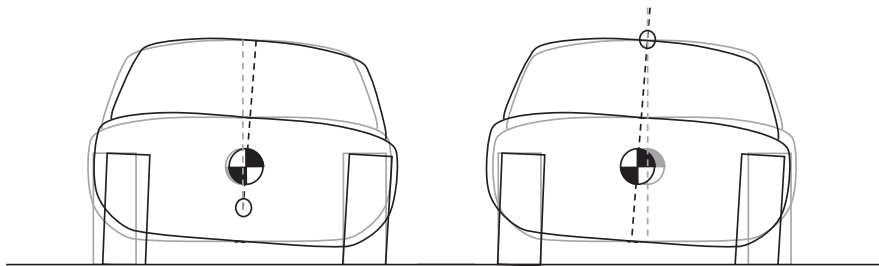
The adjustment of path curvature at a given speed is altogether more interesting. In a passenger car the driver has a handwheel, viewed by the authors as a ‘yaw rate’ demand — a demand for rotational velocity of the vehicle when viewed from above. The combination of a yaw rate and a forward velocity vector, which rotates with the vehicle, gives rise to a curved path. The curved path of the vehicle requires some lateral acceleration. The tyres on a car exert a force towards the centre of a turn and the body mass is accelerated by those forces centripetally — in a curved path. Thus the sum of the lateral forces that the tyres exert on the car is the centripetal force that produces the centripetal acceleration ([Figure 7.3](#)).

Note that the authors do not favour the use of the equivalent inertial (‘D’Alembert’) force since it can be misleading; it gives the impression that the analysis of the cornering vehicle is a static equilibrium problem, which it most certainly is not. The idea of an analogous static equilibrium condition is not in itself problematic but inappropriate ‘static’ thinking quickly becomes torturous and unwieldy.

For example, a common obsession is to attempt to find a ‘centre of rotation in roll’. This is usually performed with some sort of ‘point of zero lateral velocity’ logic but the reality is that this ‘zero velocity’ point is with respect to some arbitrary and ill-defined reference frame. [Figure 7.4](#) shows the same vehicle represented in two equally arbitrary reference frames; the first is anchored at the outboard wheel contact patch and the second at the inboard wheel contact patch. Given that most independent suspensions are not symmetric, the ‘lateral displacement’ for a given roll angle is entirely dependent on the choice of reference frame. Thus the idea of some ‘centre of instantaneous motion’ is difficult to pin down and even more difficult to ascribe any meaning to. If the reference frame is anchored somewhere on the vehicle body then that point becomes a point of zero translation, and the concept is again seen to be of no value. Nevertheless, within the authors’ experience there has been a great deal of effort to track down ‘centres of motion’ using elaborate static equilibrium analogies, although it has not notably added value to the vehicles involved.

**FIGURE 7.3**

Representation of inertial force during cornering.

**FIGURE 7.4**

The difficulty with arbitrary reference frames applied to 'pseudo static' cornering as suggested by the use of D'Alembert forces.

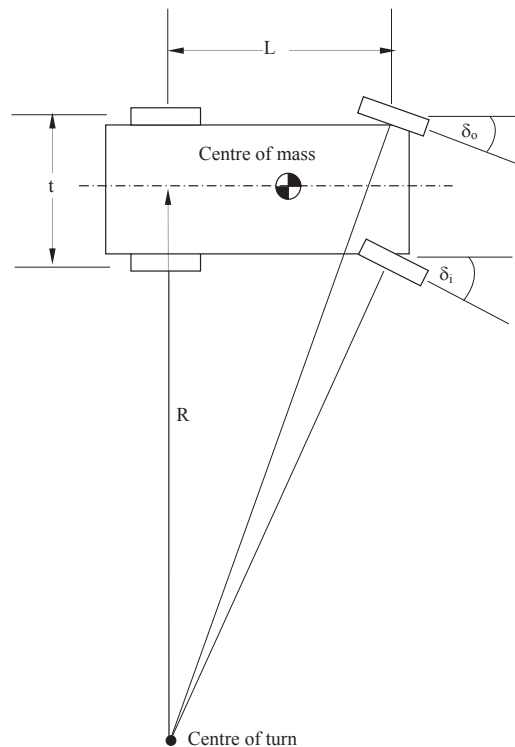
### 7.3.2 The classical treatment

The classical assessment of the behaviour of a vehicle was developed by the early work of vehicle dynamicists such as Olley, Milliken and Segel and has been documented in several textbooks dealing with the subject. In the following sections the classical treatment will be summarised followed by a consideration based on the transient dynamics of the driven vehicle.

The classical assessment of the behaviour of a vehicle is based on either testing or simulating steady state cornering. 'Steady state' means the vehicle states are unchanging with time — the car is 'settled' in a corner at a constant speed, on a constant radius and so on. Note that steady state is not the same as 'static'. Two traditional evaluation methods exist. The first involves driving the vehicle around a constant radius circle at a range of constant speeds that correspond to a range of increments in lateral acceleration. The second method involves driving the vehicle at a constant speed but with a progressive increase in handwheel angle that reduces the radius of turn with time and consequently increases the lateral acceleration. Which method is in use in a particular organisation is likely to be governed by the test facilities available and previous practice rather than on the merits or otherwise of a one or other test method. In particular, the first method is very practicable to perform and so it will be used for the basis of the following discussion.

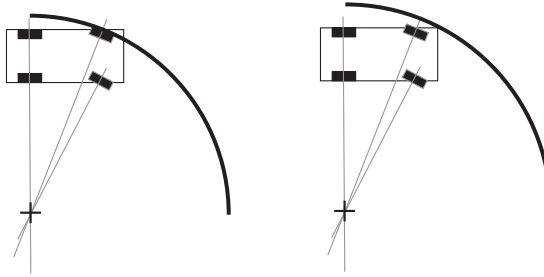
### 7.3.2.1 Low speed behaviour

The starting point for the consideration of steady state cornering behaviour is an assessment of the vehicle cornering at low speed ([Figure 7.5](#)).



**FIGURE 7.5**

Cornering at low speed.

**FIGURE 7.6**

Different definitions of turning circle.

The minimum radius available in normal driving is the turning circle, determined by turning the steering to its maximum (mechanically limited) extent and following the geometry illustrated in [Figure 7.5](#). Turning circles are usually quoted ‘between kerbs’ and ‘between walls’, as illustrated in [Figure 7.6](#).

At higher speeds the limit to path curvature is not the amount the wheels can be turned but rather the maximum centripetal force that can be generated. This is governed by the limiting coefficient of friction ( $\mu$ ) between tyres and road, typically around 0.9. Simplistically, if friction is independent of area (the Newtonian model), the theoretical maximum centripetal acceleration in units of  $g$  is identical to the friction coefficient,  $\mu$

$$\frac{A_y}{g} = \frac{F_y}{mg} = \frac{mg\mu}{mg} = \mu \quad (7.4)$$

This is for vehicles without additional force pressing them into the road – i.e. not ‘winged’ racing cars. Therefore, the maximum yaw rate possible (in degrees per second) is shown in [Figure 7.7](#) for a typical vehicle.

The yaw rate from ‘geometry’,  $\omega_{\text{geom}}$ , as shown in [Figure 7.7](#) is the maximum possible steering and varies with speed according to the simple relationship

$$\omega_{\text{geom}} = \frac{V (\delta_i + \delta_o) / 2 \left( \frac{180}{\pi} \right)}{L} \quad (7.5)$$

where  $V$  is the forward velocity in metres per second,  $\delta_i$ ,  $\delta_o$  are the steer angles in radians as shown in [Figure 7.5](#) and  $L$  is the wheelbase in metres.

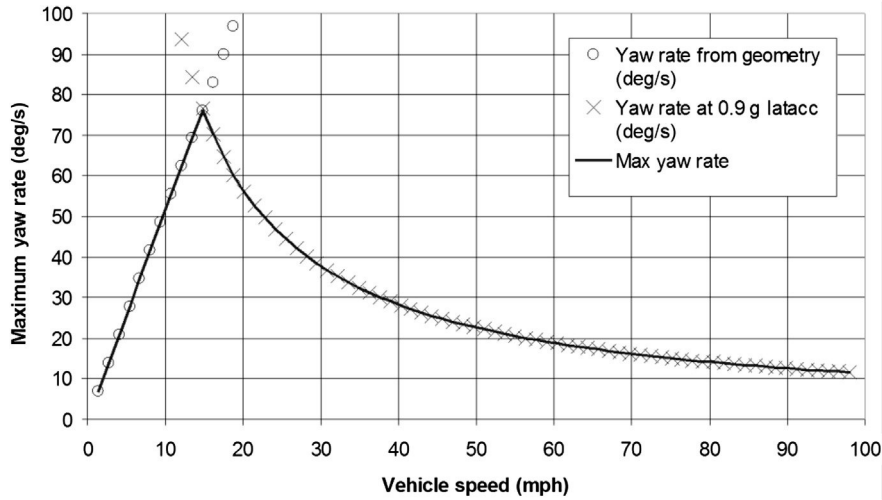
Yaw rate from limiting friction,  $\omega_{\text{friction}}$ , is equally simple.

$$\omega_{\text{friction}} = \left( \frac{180}{\pi} \right) \frac{\mu g}{V} \quad (7.6)$$

So it can be seen that above a limiting speed, given by

$$V_{\text{lowlimit}} = \sqrt{\frac{\mu g L}{\left( \frac{\delta_i + \delta_o}{2} \right) \left( \frac{\pi}{180} \right)}} \quad (7.7)$$



**FIGURE 7.7**

Maximum possible yaw rate for a typical vehicle.

it is the surface grip that determines the limiting yaw rate and hence path curvature. For typical passenger cars, the region in which geometry dominates steering behaviour is small – up to about 15 mph. This is the speed that may be counted as ‘low’ and in which Figure 7.5 is a reasonable description of the behaviour of the vehicle. For vehicles such as taxis and heavy goods vehicles, the low speed region is of more importance simply because these vehicles make more low speed, minimum radius manoeuvres. However, they rarely perform these manoeuvres at speeds exceeding walking pace and so the yaw rates remain low.

The steer angles required to avoid scrubbing at the inner and outer road wheels,  $\delta_o$  and  $\delta_i$ , are

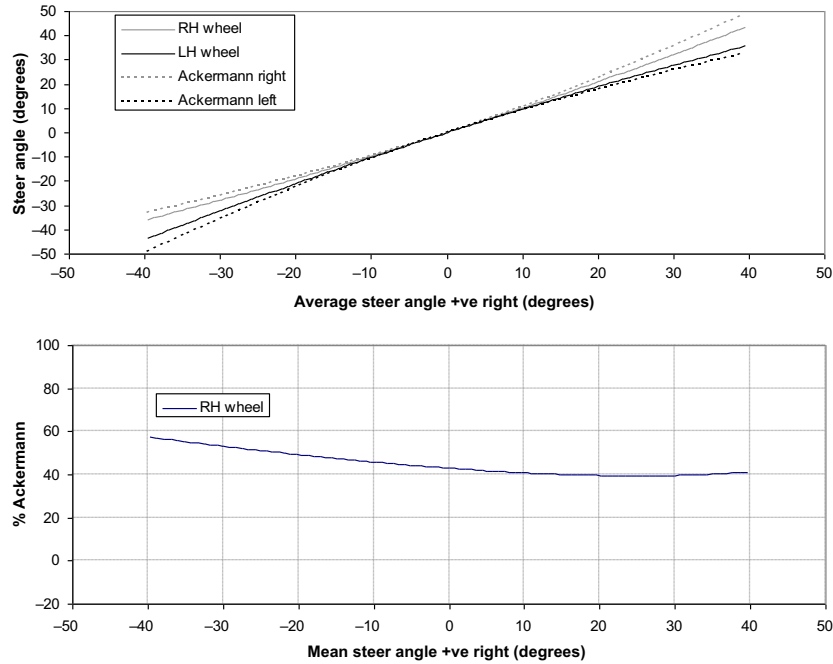
$$\delta_o = \frac{L}{(R + 0.5t)} \left( \frac{180}{\pi} \right) \quad (7.8)$$

$$\delta_i = \frac{L}{(R - 0.5t)} \left( \frac{180}{\pi} \right) \quad (7.9)$$

The angle of a notional ‘average’ wheel on the vehicle centreline,  $\delta$ , is the ‘Ackerman’ angle – the approximate form is in common usage

$$\delta = \frac{LR}{(R - 0.25t^2)} \left( \frac{180}{\pi} \right) \approx \frac{L}{R} \left( \frac{180}{\pi} \right) \quad (7.10)$$

In 1817, Rudolph Ackermann patented geometry similar to this as an improvement over a steered axle as was common on horse-drawn vehicles. That the geometry we today call ‘Ackermann’ was in fact a modification proposed by the Frenchman

**FIGURE 7.8**

Steering Geometry in Comparison with Ackermann/Jeantaud Geometry.

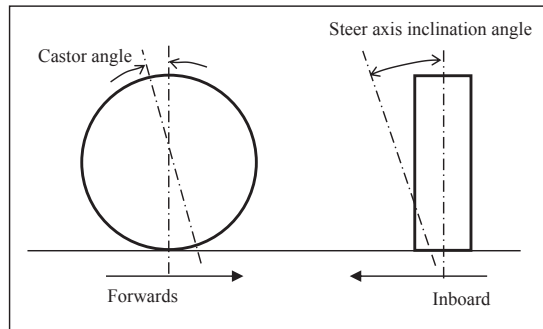
Charles Jeantaud in 1878 has been lost in the mists of time. Some measure of how accurately the steering geometry corresponds to the Ackermann/Jeantaud geometry is often quoted although rarely defined. The authors use a description as given in [Figure 7.8](#), which in turn uses the simplified form of [Eqn \(7.10\)](#).

The Ackermann/Jeantaud angles are calculated as given in [Eqns \(7.8\) and \(7.9\)](#). These are compared with the angles actually achieved by the front wheels. A fraction of the compensation is expressed for the outer wheel using  $\delta_{\text{mean}}$  for the mean of the actual angles:

$$\text{Ackermann Fraction} = \frac{\delta_{o\_actual} - \delta_{\text{mean}}}{\frac{L}{\left[ \left( \frac{180}{\pi} \right) \left( \frac{L}{\delta_{\text{mean}}} \right) + 0.5t \right]} \left( \frac{180}{\pi} \right) - \delta_{\text{mean}}} \quad (7.11)$$

This is the quantity graphed in [Figure 7.8](#) as ‘% Ackermann’. Road cars are successfully engineered with widely varying ‘% Ackerman’ values — from 20 to 80%.

Ackermann effects are noticeable at parking speeds. Typically, there is some inclination of the vehicle’s steering axis when viewed from the front and the side of the vehicle. These inclinations are known as castor and steer axis inclination or

**FIGURE 7.9**

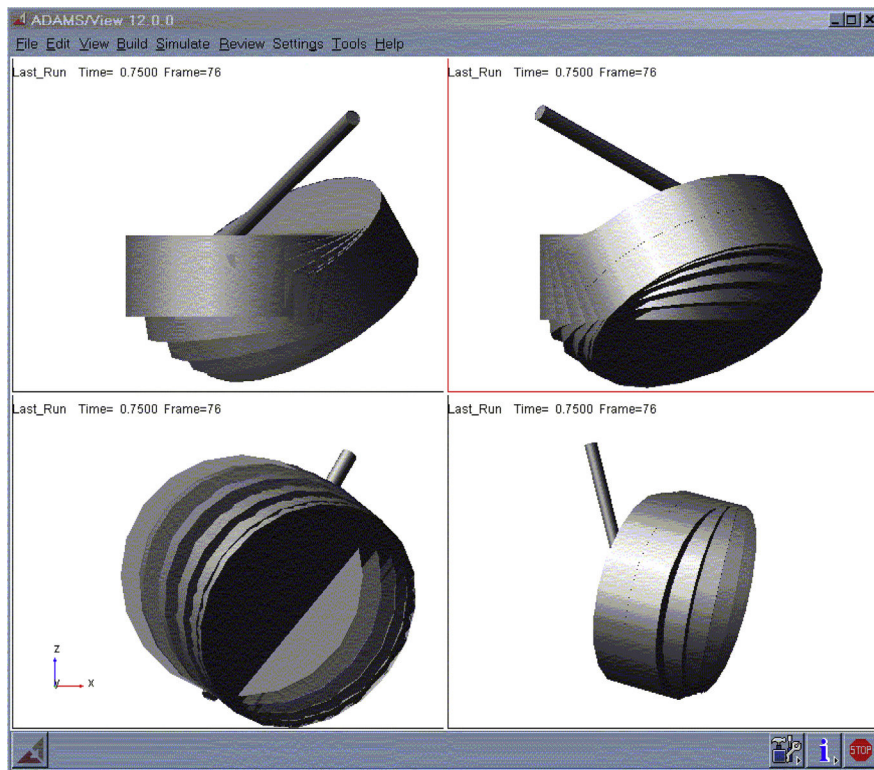
Steer Angle Geometry Definitions.

kingpin inclination, respectively, and are shown in Figure 7.9. Together with the offsets at ground level between the geometric centre of the wheel, these inclinations have the effect of moving the inboard wheel down and the outboard wheel up. Figure 7.10 shows the inboard wheel being moved down as it goes onto back lock. Given the constraint of the ground, this has the effect of imposing a roll moment on the vehicle that is reacted by the front and rear suspensions in series (Figure 7.11). The loading of the vehicle wheels becomes asymmetric.

This asymmetry is of the order of 10 N in a typical family saloon and is of little consequence. However, in a more stiffly sprung vehicle if the steering geometry is not constructed with care, the asymmetry can be more than 100 N. If this loading is applied in conjunction with significantly low Ackermann fractions, the result is that the inner wheel is emphasised over the outer wheel.

This emphasis has the consequence of producing an effective side force from the tyre because it is operating at a comparatively large slip angle (Figure 7.12). Normally, side forces from both tyres work in opposition if the wheels are effectively toed in when the Ackermann fraction is less than unity. However, if the forces are not in balance (because of the weight imbalance between inboard and outboard tyres) the additional side force on the inboard tyre has the effect of reducing the steer aligning torque when the vehicle is in motion. If the Ackermann fraction is particularly low, the vehicle will have a ‘wind on to lock’ behaviour at car park speeds beyond certain steer angles. Vehicles with stiff suspension, high steer axis angles and wide tyres are more prone to this effect.

There exists some confusion over the significance of Ackermann geometry at speed. For ride and handling work, the significance is sometimes overstated. Considering Figure 7.5, a turn at 50 mph (22 m/s) road speed at 0.4 g lateral acceleration may be calculated as producing a yaw rate of  $10.2^\circ/\text{s}$ . For a 2.7 m wheelbase vehicle, this requires a mean steer angle of  $1.26^\circ$ . The radius of turn is 123 m and so the Jeantaud modification gives  $1.25018^\circ$  on the inner wheel and  $1.26552^\circ$  on the outer wheel — an included angle of  $0.015^\circ$ . For a typical cornering stiffness of  $1500 \text{ N}^\circ$ , this gives a lateral force variation of 23 N between 0% Ackermann and



**FIGURE 7.10**

Steering geometry effects on wheel vertical position: fully constrained body and suspension, model runs from straight ahead to full back lock. Plan view (top left) rear view (top right), left view (bottom left) and three-quarter view (bottom right) of front left wheel with steering axis indicated by cylindrical graphic. Note the steering geometry used is atypical for emphasis.

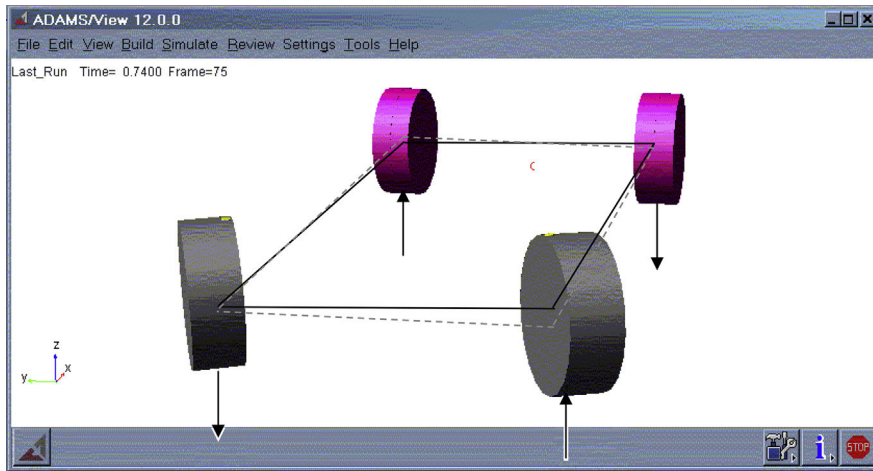
100% Ackermann. The lateral forces to achieve 0.4 g at 50 mph are over 5900 N for a typical 1500 kg vehicle, so the Ackermann effect amounts for lateral forces of some 0.4% of the total — a small modifier on the vehicle as a whole.

### 7.3.2.2 Higher speed linear region behaviour

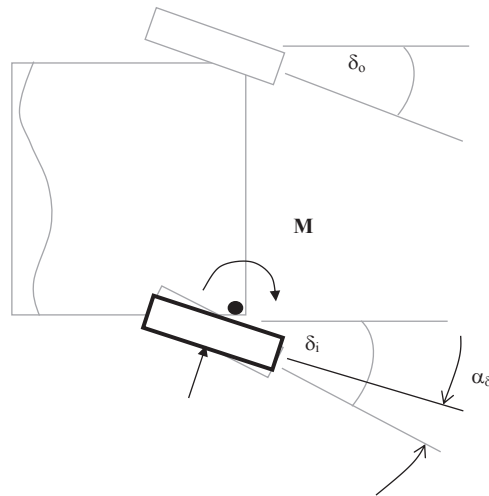
As speeds rise above the low limit calculated in Eqn (7.7), the behaviour of vehicles becomes less instinctive. As illustrated in Figure 1.4, it is generally true that the curvature of path *cannot* readily be estimated using a geometric approximation like Figure 7.5.

For a vehicle manoeuvring in the ground plane, it can be presumed the vehicle is broadly symmetric — note there are some exceptions to this, such as motorcycle combinations, but these are so uncommon as to be obvious.

Ground vehicles tend to be broadly flat on their lower surface and tend to have the majority of their weight distributed around a narrow height band; regardless of

**FIGURE 7.11**

Platform motion during steering at low speed (black) in comparison with the static platform position (grey dashed). Changes in wheel weight are indicated; front left wheel has an increased reaction force.

**FIGURE 7.12**

Effect of inside wheel loading in producing an excess side force due to slip angle  $\alpha_\delta$  when the vehicle is in motion.

the shape of the superstructure, most of the mass is in a 'stripe' between the wheel hubs and the top of the wheels.

Thus it may be presumed as a simplification that the local x-axis is parallel with the path of the vehicle when running in a straight line. In combination with the

symmetry, this gives a rather intuitive inertial axis set for the vehicle. Real vehicles may deviate from these presumptions but they are suitable for overview modelling.

It is often true for ground vehicles that they are longer than they are wide. It is also often true of passenger vehicles that they are wider than they are tall (though modern vehicle design is making this less obvious than it was). Thus it is coincidentally true that the principal inertias of the vehicle are least for the presumed x-axis and most for the presumed z-axis as discussed in Chapter 6 where a convenient axis set for a vehicle model is shown in Figure 6.3. It is conventional (though by no means universally so) to number the principal inertias 1, 2 and 3 from the most to the least and thus  $I_1 \approx I_{xx}$ ,  $I_2 \approx I_{yy}$  and  $I_3 \approx I_{zz}$ .

It is convenient with real bodies (i.e. not particles) to formulate the equations of motion in a body-centred manner to avoid recalculating inertia properties. A detailed treatment is provided in Chapter 2 but for convenience the full set of these equations – known as Euler equations and widely reproduced – is also given in Eqns (7.12)–(7.17) below:

$$\sum F_x = m(\dot{v}_x - \omega_z v_y + \omega_y v_z) \quad (7.12)$$

$$\sum F_y = m(\dot{v}_y - \omega_x v_z + \omega_z v_x) \quad (7.13)$$

$$\sum F_z = m(\dot{v}_z - \omega_y v_x + \omega_x v_y) \quad (7.14)$$

$$\sum M_x = I_1 \dot{\omega}_x - (I_2 - I_3) \omega_y \omega_z \quad (7.15)$$

$$\sum M_y = I_2 \dot{\omega}_y - (I_3 - I_1) \omega_z \omega_x \quad (7.16)$$

$$\sum M_z = I_3 \dot{\omega}_z - (I_1 - I_2) \omega_x \omega_y \quad (7.17)$$

The axes x, y and z move with the body. The principal inertia directions are presumed to be coincident with the body-centred axis definition as previously described.

Lateral motions occur with a frequency around 10 times higher than longitudinal motions, and thus the variation in forward velocity can be neglected in an initial formulation. The surface can be presumed smooth for initial evaluations and so heave, pitch and roll motions can also be neglected. This leaves only lateral velocity,  $v_y$ , and yaw velocity,  $\omega_z$ , as degrees of freedom (DOF):

$$\sum F_y = m(\dot{v}_y + \omega_z v_x) \quad (7.18)$$

$$\sum M_z = I_3 \dot{\omega}_z \quad (7.19)$$

Despite the large simplification, this representation of the vehicle — sometimes referred to as a 2 DOF model — can give good insight into overall vehicle behaviour.

Lateral forces and yaw moments are generated mainly by the tyres of the vehicle. Aerodynamic forces will be neglected for the initial formulation.

The tyres have been discussed in detail in Chapter 5. For the 2 DOF model they may initially be presumed to generate forces with slip angle — camber is neglected — and may be imagined as a single effective tyre on the centreline of the vehicle at each axle location. Equations 7.18 and 7.19 describe a single track vehicle but it should be noted that the dynamics of real single track vehicles (bicycles and motorcycles) are substantially different to the dynamics of this system. Although the term ‘bicycle model’ for the single track 2 DOF model is widespread, it is nevertheless very misleading.

Considering Figure 7.13, if the vehicle is viewed in plan, the slip angle at each axle from a lateral velocity  $v_y$  is given by

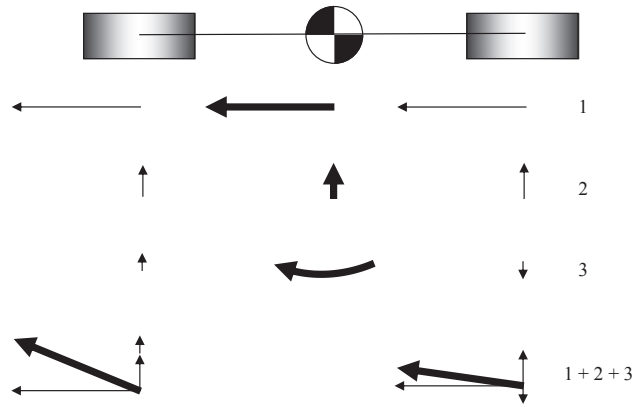
$$\alpha_f(v_y) = \alpha_r(v_y) = \tan^{-1}\left(\frac{v_y}{v_x}\right) \quad (7.20)$$

Similarly, the slip angle at each axle from a yaw velocity  $\omega_z$  is given by

$$\alpha_f(\omega_z) = \tan^{-1}\left(\frac{\omega_z \cdot a}{v_x}\right) \quad (7.21)$$

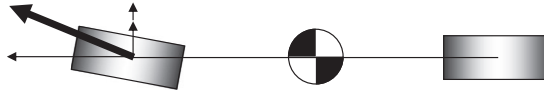
$$\alpha_r(\omega_z) = \tan^{-1}\left(\frac{-\omega_z \cdot b}{v_x}\right) \quad (7.22)$$

For small angles, less than around  $15^\circ$ , the value of the tangent function is approximately the same as the angle in radians.



**FIGURE 7.13**

A graphical representation of the summation of longitudinal, lateral and yaw velocities at the Centre of Mass as seen at the axles.

**FIGURE 7.14**

Steer angle requires algebraic subtraction.

The aggregate behaviour of the tyres in the presence of both velocities may be presumed to be the linear superposition of the two, since the DOF are mathematically orthogonal [Figure 7.13](#).

The steer angle on the front axle can be seen to need algebraic subtraction with the sign convention chosen (the SAE axis system), in which the steer angle has the same sign as the yaw rate ([Figure 7.14](#)).

Assembling all the terms it can be presumed that:

$$\alpha_f = \left( \frac{v_y + \omega_z \cdot a}{v_x} \right) - \delta \quad (7.23)$$

$$\alpha_r = \left( \frac{v_y - \omega_z \cdot b}{v_x} \right) \quad (7.24)$$

The behaviour of real tyres can be observed as broadly linear up to about half the available friction budget and can be expressed by a single ‘cornering stiffness’ coefficient,  $C_\alpha$ .

To linearise the model, a constant value will be used initially for the cornering stiffness coefficient. Therefore the lateral equation of motion can now be rendered thus:

$$C_{af} \cdot \alpha_f + C_{ar} \cdot \alpha_r = m(\omega_z v_x + \dot{v}_y) \quad (7.25)$$

Substituting the previous expressions for the slip angles gives:

$$C_{af} \cdot \left( \frac{v_y + \omega_z \cdot a}{v_x} - \delta \right) + C_{ar} \cdot \left( \frac{v_y - \omega_z \cdot b}{v_x} \right) = m(\omega_z v_x + \dot{v}_y) \quad (7.26)$$

Rearranging the components to group them by system states:

$$\left( \frac{C_{af} + C_{ar}}{v_x} \right) \cdot v_y + \left( \frac{C_{af} a - C_{ar} b}{v_x} \right) \cdot \omega_z - (C_{af}) \cdot \delta = m(\omega_z v_x + \dot{v}_y) \quad (7.27)$$

The terms within parentheses are terms in the Jacobian for the system and may be equivalently rendered as partial derivatives:

$$\left( \frac{\partial F_y}{\partial v_y} \right) v_y + \left( \frac{\partial F_y}{\partial \omega_z} \right) \omega_z + \left( \frac{\partial F_y}{\partial \delta} \right) \delta = m(\omega_z v_x + \dot{v}_y) \quad (7.28)$$



A single character symbolism for the individual Jacobian terms is common in aircraft dynamics where this approach was applied for the seminal IME papers of 1956 and will be familiar to any reader of the definitive 'Race Car Vehicle Dynamics' (Milliken and Milliken, 1995); it simplifies typesetting a great deal. In an affectionate nod to that source, the same symbols are used here:

$$(Y_{vy})v_y + (Y_\omega)\omega_z + (Y_\delta)\delta = m(\omega_z v_x + \dot{v}_y) \quad (7.29)$$

The yaw equation of motion can be similarly treated:

$$a \cdot C_{af} \cdot \alpha_f - b \cdot C_{ar} \cdot \alpha_r = I_3 \dot{\omega}_z \quad (7.30)$$

$$a \cdot C_{af} \cdot \left( \frac{v_y + \omega_z \cdot a}{v_x} - \delta \right) - b \cdot C_{ar} \cdot \left( \frac{v_y - \omega_z \cdot b}{v_x} \right) = I_3 \dot{\omega}_z \quad (7.31)$$

$$\left( \frac{a \cdot C_{af} - b \cdot C_{ar}}{v_x} \right) \cdot v_y + \left( \frac{a^2 \cdot C_{af} + b^2 \cdot C_{ar}}{v_x} \right) \cdot \omega_z + (-a \cdot C_{af}) \cdot \delta = I_3 \dot{\omega}_z \quad (7.32)$$

$$\left( \frac{\partial M_z}{\partial v_y} \right) v_y + \left( \frac{\partial M_z}{\partial \omega_z} \right) \omega_z + \left( \frac{\partial M_z}{\partial \delta} \right) \delta = I_3 \dot{\omega}_z \quad (7.33)$$

$$(N_{vy})v_y + (N_\omega)\omega_z + (N_\delta)\delta = I_3 \dot{\omega}_z \quad (7.34)$$

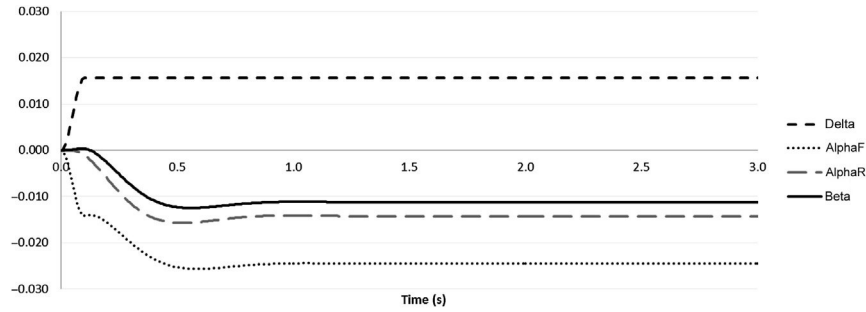
Given initial conditions and a time history for steering input, these equations are suitable for direct integration to find the solution with time. Numerical methods, as previously discussed, are eminently suitable for this. Some practitioners rearrange the equations further but these equations will satisfactorily illustrate the behaviour of the vehicle for the purpose at hand.

If a steer input is added early in the solution over a period of 0.1 s and held constant, and when typical values are used for vehicle parameters, certain features of the solution emerge, which can be seen in [Figure 7.15](#).

The immediately obvious one is that the slip angles on the tyres are negative compared to the steering angle. This arises because the lateral velocity is generally negative when the yaw rate is positive and vice versa. Note that the same sign for both tyre slip angles in the steady state implies the presence of a lateral velocity at the body, which may be thought of as a 'body slip angle' and is often denoted by  $\beta$  – the vehicle crabs sideways as it negotiates the turn (see [Figure 7.16](#)).

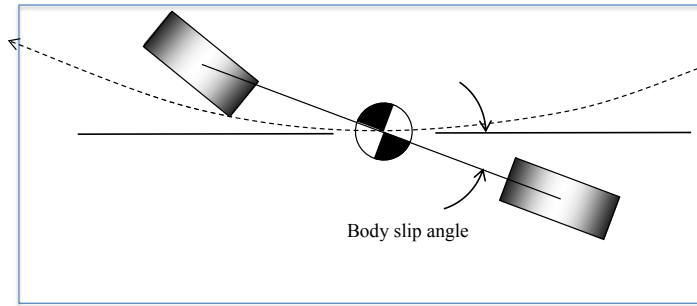
$$\beta = \tan^{-1} \left( \frac{v_y}{v_x} \right) \quad (7.35)$$

Note that in [Figure 7.15](#) the body slip angle is less than 0.02 rad, or just over 1°; the angle is not normally visible but is nevertheless always present.



**FIGURE 7.15**

Response of the two degree-of-freedom model to a sharp steer input.



**FIGURE 7.16**

Body slip angle.

Examination of the equations suggests that a steady state exists in which all time derivatives are zero, and the solutions do indeed display such a form, reaching constant values after a second or so. This may be thought of as a settled condition; some students believe that a ‘dynamic steady state’ is an oxymoron but this is not so; the steady state implies that all the phase relationships have become constant, as distinct from nothing changing — which would be better described with the word ‘equilibrium’.

Various texts reproduce this problem in various forms, which results from [Eqns \(7.36\) and \(7.37\)](#) with the derivative terms  $\dot{v}_y$  and  $\dot{\omega}_z$  set to zero. The result is a pair of simultaneous equations with two unknowns:

$$(Y_v)v_y + (Y_\omega - mv_x)\omega_z = -(Y_\delta)\delta \quad (7.36)$$

$$(N_v)v_y + (N_\omega)\omega_z = -(N_\delta)\delta \quad (7.37)$$

The steady state solutions for  $v_y$  and  $\omega_z$  can be calculated directly without recourse to time-domain integration. Rearranging [Eqn \(7.36\)](#) gives:

$$\omega_z = \frac{-Y_\delta\delta - Y_vv_y}{Y_\omega - mv_x} \quad (7.38)$$

Substituting  $\omega_z$  from Eqn (7.38) into Eqn (7.37) and rearranging yields:

$$N_v v_y + N_\omega \left( \frac{-Y_\delta \delta - Y_v v_y}{Y_\omega - m v_x} \right) = -N_\delta \delta \quad (7.39)$$

$$N_v v_y - \left( \frac{N_\omega Y_\delta \delta}{Y_\omega - m v_x} \right) - \left( \frac{N_\omega Y_v v_y}{Y_\omega - m v_x} \right) = -N_\delta \delta \quad (7.40)$$

$$v_y \left( N_v - \frac{N_\omega Y_v v_y}{Y_\omega - m v_x} \right) = -N_\delta \delta + \frac{N_\omega Y_\delta \delta}{Y_\omega - m v_x} \quad (7.41)$$

$$v_y = \frac{\left( -N_\delta \delta + \frac{N_\omega Y_\delta \delta}{Y_\omega - m v_x} \right)}{\left( N_v - \frac{N_\omega Y_v v_y}{Y_\omega - m v_x} \right)} \quad (7.42)$$

Equations (7.42) and (7.38) can be used in that order to calculate the steady state responses. When considering responses, it is helpful to consider the notion of a ‘gain’ in the control system sense, which is the ratio of response to input. Presuming the small angle approximation once more, the yaw rate gain (YRG) can be expressed as:

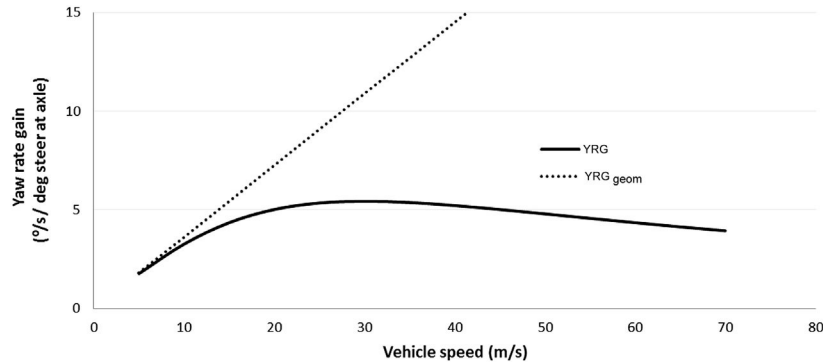
$$\text{YRG}(v_x) = \frac{\omega_z}{\delta} \quad (7.43)$$

YRG for a purely geometric vehicle ( $\text{YRG}_{\text{geom}}$ ) could be expressed as

$$\text{YRG}_{\text{geom}} = \frac{\omega}{\delta_{\text{mean}}} = \frac{V}{L} \quad (7.44)$$

This may be recognised as describing the low speed region of Figure 7.17. The geometric yaw rate is achieved by some notional vehicle that may be thought of as running on ‘blade’ wheels on indestructible ice — no sideslip of the wheels is possible. It is important for vehicle dynamicists to remember that the overwhelming majority of the driving population genuinely believe this is the way in which their vehicles function. For many typical drivers, the belief exists that the tyres are little miniature ‘rails’ that the vehicle carries around with itself to ‘lay tracks’ as it goes along.

For the linear 2 DOF model, the calculated YRG characteristic is substantially different from  $\text{YRG}_{\text{geom}}$ . When the characteristic is computed for a range of speeds and compared with the geometric response, a character very different emerges, as shown in Figure 7.17.

**FIGURE 7.17**

A comparison between geometric and 2 DoF linear steady state gains. YRG, yaw rate gain; YRG<sub>geom</sub>, geometric yaw rate gain.

This character arises from the yaw moment developed by the vehicle attempting to straighten itself out as it travels in a curved path and also the ‘weathercock’ influence of the vehicle, again trying to straighten itself out as it crabs somewhat in order to have a slip angle on the rear tyres.

The exact components of the numerator and denominator when calculating the response of the vehicle defy intuitive understanding, but the character in the above graph is typical. Note that the actual YRG is hugely reduced from YRG<sub>geom</sub> at high speeds.

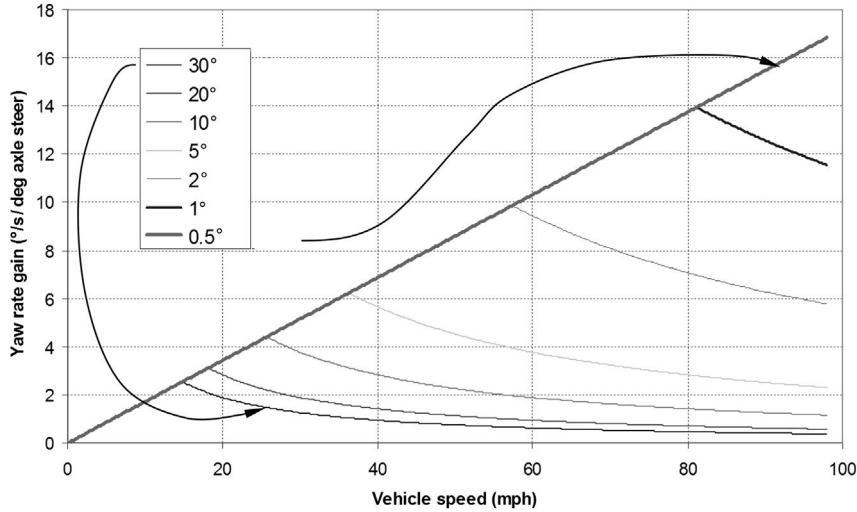
When YRG is less than YRG<sub>geom</sub>, this is the ‘understeer’ condition as defined in SAE J670. However, in a real vehicle and with a real operator, there is no sense of the operator being able to understand the actual geometric angle of the front wheels, only whether or not the car is travelling on the intended path. This is true even for open wheel cars where the wheels are directly visible from the driver’s seat. Thus the significance of the ‘technical understeer’ is minimal to the driver; there is a ratio between the angles at the handwheel and the road wheels, and a vehicle at a given speed with more ‘technical understeer’ but a more direct ratio will be indistinguishable from one with less ‘technical understeer’ and a less direct ratio.

This all points to the technical definition as being irrelevant from the driver’s perspective, since the driver will learn YRG of the vehicle adaptively as the vehicle changes speed. Even skilled development drivers do not report understeer when driving in the linear region — less than about half the available friction budget.

It is important to note that this technical definition of understeer has no connection to journalistic and development drivers’ description of it, which is confined exclusively to the nonlinear region and is discussed next.

### 7.3.2.3 Nonlinear region

The higher speed region, even with a simplistic Newtonian friction model, can be seen to have a nonlinear YRG characteristic even with a completely geometric



**FIGURE 7.18**

Yaw rate gains for an idealised vehicle with Newtonian friction.

vehicle description (Figure 7.18) since it is possible to turn the front wheels to an angle that corresponds to a yaw rate greater than that which the vehicle can achieve. The similarity between Figures 7.7 and 7.18 should be apparent to the reader.

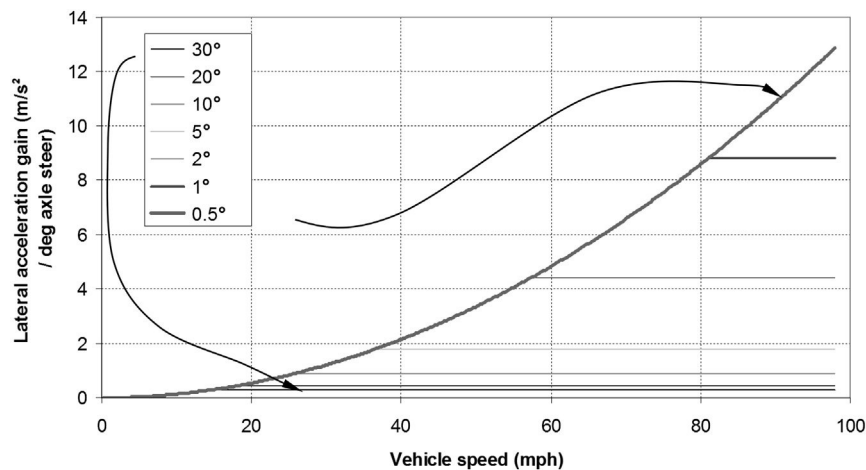
As previously noted, when a vehicle yaws less than expected, the term ‘understeer’ is used — the response of the vehicle is less than (‘under’) what might have been expected. When a vehicle yaws more than expected, the term ‘oversteer’ is used — the response of the vehicle exceeds (is ‘over’) what might have been expected. So far only mechanisms for generating understeer have been discussed. Remaining with the Newtonian friction model to describe the behaviour of the tyres, one further fundamental point is worth establishing. For a vehicle travelling in a circular path, forward speed,  $V$ , yaw rate,  $\omega$  and lateral (centripetal) acceleration,  $A_y$ , are related with

$$A_y = \omega V \quad (7.45)$$

Using the preceding relationship for  $YRG_{\text{geom}}$ , a geometric lateral acceleration gain ( $AyG_{\text{geom}}$ ) can be deduced for the geometric vehicle in the region where

$$AyG_{\text{geom}} = \frac{A_y}{\delta_{\text{mean}}} = \frac{\omega_{\text{geom}} V}{\delta_{\text{mean}}} = V^2 L \quad (7.46)$$

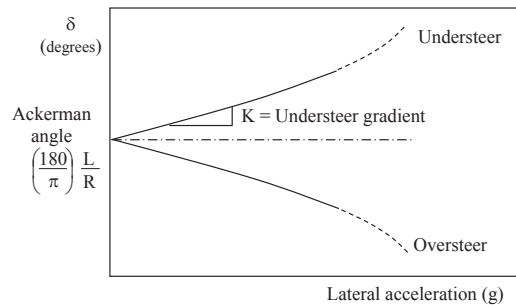
At high speeds it can be seen that axle steer inputs of much in excess of half a degree cause the available friction to saturate with the geometric formulation. Although it can be seen from Figure 7.17 that the vehicle has substantially reduced gains at speed — and so it might be expected that something of the order 2–3° of steering input might cause the available friction to saturate — the level of steer

**FIGURE 7.19**

Lateral acceleration gains for an idealised vehicle with Newtonian friction.

required to saturate is still very low compared to parking manoeuvres. To retain proportional control, the driver must keep inputs below this level. For this reason, quite high reduction ratios are generally used in steering gears to give a reasonable level of input sensitivity at the handwheel. For a European passenger car, a reduction ratio of around 16–18:1 is typical (16° of handwheel giving 1° of axle steer), meaning that at the highest speeds shown in Figure 7.18 and 7.19 handwheel inputs of 10–20° are enough to saturate the vehicle with respect to the available friction in a high-grip environment. For vehicles travelling faster — for example on unrestricted autobahns or competition vehicles — it can be seen that the overall steering ratio is of importance in order not to have the vehicle overly sensitive to driver inputs. There is, however, a trend among vehicle manufacturers to fit numerically lower steering ratios over time — compare a 1966 Ford Cortina at 23.5:1 with its current cousin the Focus at 17.5:1 — to promote a perception of agility. This fashion will require the adoption of more adventurous variable steering ratios (for example, as promoted by Bishop Technologies or as implemented by BMW and ZF in the 2003 five series ‘Active Front Steer’ system) in order to retain sensitivity at high speeds. In low-grip environments, the handwheel inputs needed to saturate the vehicle at speed are tiny, scaling down with coefficient of friction,  $\mu$ .

The range of lateral acceleration gains, from 0.6 m/s<sup>2</sup>/° at 20 mph to 12.9 m/s<sup>2</sup>/° at 100 mph, is quite a wide range to ask the driver to accommodate. When high-speed road systems lack curves of any kind there is no feedback information to allow the driver to adapt and so extremely straight designs of roads are unhelpful in this respect. Thus it is often true that normal drivers get into difficulty when faced with an emergency evasive manoeuvre at high speed. Some component of this is simply because the saturation (in terms of grip) at a comparatively low yaw rate

**FIGURE 7.20**

Determination of understeer gradient.

(Gillespie, 1992.)

of the car surprises the driver. Typically, road infrastructures are constructed so as not to need high lateral accelerations at high speeds. For normal drivers the difficulty is largely masked in day-to-day driving, which contributes to some complacency. In general, vehicles are tuned to have reduced YRG (and hence AyG) at higher speeds to compensate for this problem. The level of this tuning differs widely between different markets (Figure 7.20).

Although the tyres are not linear up until the point of saturation (as discussed at length in Chapter 5), the behaviour of the vehicle is not substantially different to that implied in the preceding discussion, it is merely a slightly distorted and smoothed version of the behaviour implied. To map the transition from linearity to nonlinearity, the constant radius turn test procedure (ISO 4138) may be summarised as:

- Start at slow speed, find Ackerman angle.
- Increment speed in steps to produce increments in lateral acceleration of typically 0.1 g.
- Corner in steady state at each speed and measure steering inputs.
- Produce a graph similar to that shown in Figure 7.12.

Considering the diagram, two regions are apparent. In the ‘understeer’ region, more steer angle is necessary compared to the Ackermann angle to hold the chosen radius. This may not seem intuitive unless the view is taken that the vehicle steers less than is expected (‘under’ the Ackermann response) and more steer angle is needed to compensate for it. Similarly, the ‘oversteer’ region needs less steer angle compared to the Ackermann angle. If the oversteer is large, the steer might need to become negative to trim the vehicle in the steady state. For many, oversteer is marked by the use of steer in the opposite direction to the corner – so-called ‘opposite lock’. However, the strict definition only requires that less steer than the Ackermann angle is applied – the transition to opposite lock merely marks a further degree of oversteer but there is nothing especially significant about the sign change. If the steer angle does not vary with lateral acceleration the vehicle is said to be ‘neutral steering’.

At low lateral acceleration the road wheel angle,  $\delta$ , can be expressed using Eqn (7.47).

$$\delta = \left( \frac{180}{\pi} \right) \frac{L}{R} + K A_y \quad (7.47)$$

where

- $\delta$  = road wheel angle (degree)
- $K$  = understeer gradient (degree/g)
- $A_y$  = lateral acceleration (g)
- $L$  = wheelbase (m)
- $R$  = radius (m).

Note that the use of understeer gradient in degrees per g can be expressed at either the axle or the handwheel if appropriate regard is taken of the steering reduction ratio. For vehicle dynamicists it is easy to declare that the only measure of consequence is the axle steer; however this is to ignore the subjective importance of handwheel angle to the operator of the vehicle. Note also that this parameter  $K$  is not to be confused with the more common ‘stability factor’  $K$  as developed by Milliken and Segel and used later in this chapter.

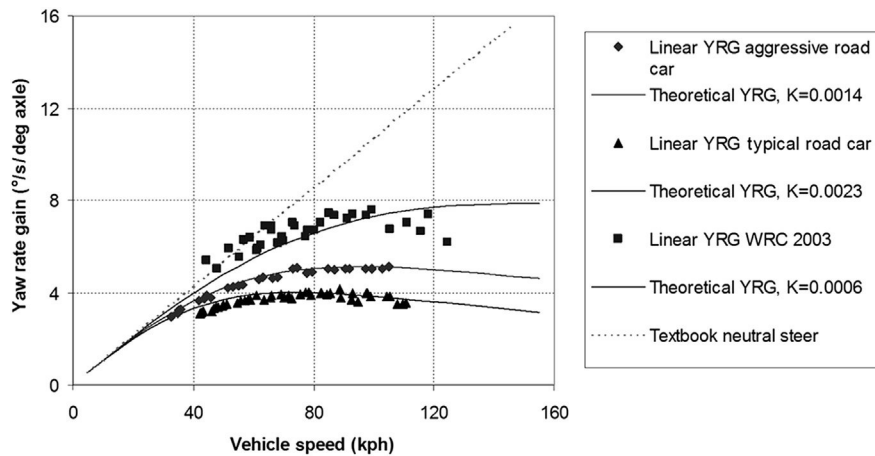
Olley makes an important distinction between what he calls the primary effects on the car affecting the tyre slip angles and secondary effects affecting handwheel angles and body attitudes, which are acutely sensed by the operator (Milliken and Milliken, 2001). Perhaps the biggest source of difficulty between practical and theoretical vehicle dynamicists is that the large modifiers of the primary vehicle dynamics are generally fixed by the time the practical camp get their hands on a vehicle and so are not considered by them; the secondary modifiers, used to great effect in delivering the required subjective behaviour of the vehicle for its marketplace, are frequently overlooked by the theoretical camp as being ‘small modifiers’ despite being important to the emotional reaction of the driver to the vehicle. For this entirely prosaic reason it is common that members of each fraternity understand little of what goes on in the other.

### 7.3.3 The subjective/objective problem

As suggested in the opening chapter, there are two strongly divided camps in the vehicle dynamics field – practitioners and theoreticians. One of the reasons for the difficulties between the two camps is the use of common vocabulary with different meanings. On the theoretical side, for example, there are clear definitions of such basic terms as ‘understeer’ and ‘oversteer’. Subjectively, these terms are used to describe quite different behaviours.

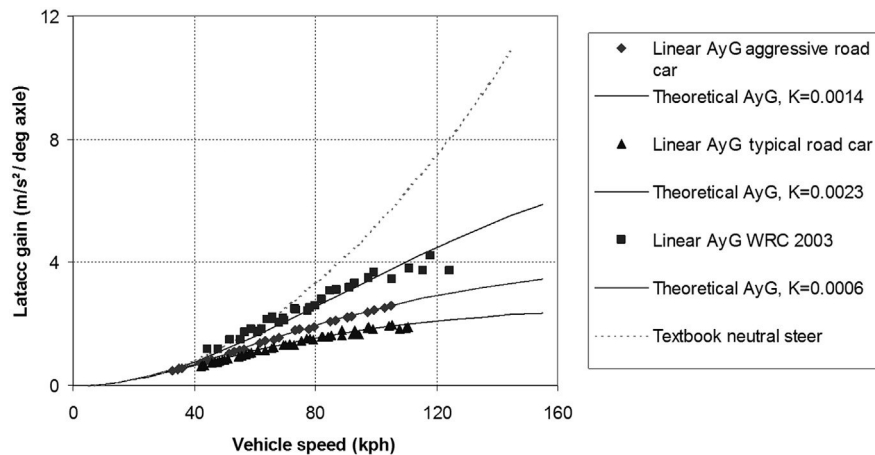
Understeer and oversteer have been defined in Section 7.3.2.3 as yaw rates under and over what might be expected, respectively. The subjective/objective problem has a great deal to do with *the nature of the expectation*.



**FIGURE 7.21**

Yaw rate gain characteristics for three different vehicles. The stability factor,  $K$ , is defined later in the text. YRG, yaw rate gain.

Objectively, what is expected is the 'idealised' or 'geometric' yaw rate (Figure 7.19). At the lowest vehicle speeds this corresponds very closely with the performance of the actual vehicle, as noted earlier. As vehicle speeds rise, the previously described effects further reduce YRG even when the tyres are not saturated. Figures 7.21 and 7.22 compare two road cars with a nonwinged competition car.

**FIGURE 7.22**

Lateral acceleration gain characteristics for three different vehicles. AyG, Lateral acceleration gain.

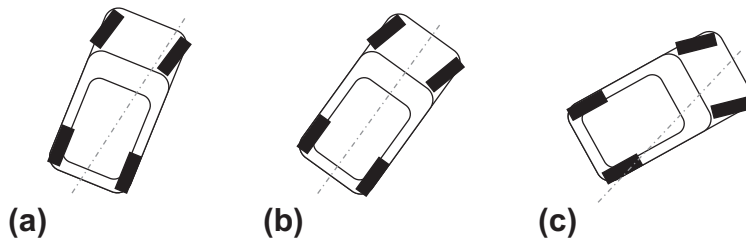
Objectively, understeer is when the actual yaw rate is less than the idealised yaw rate and is often expressed as the ratio of the two

$$US = \frac{\Omega_{\text{geom}}}{\Omega} \quad (7.48)$$

If US is greater than unity, the vehicle is understeering and if US is less than unity the vehicle is oversteering. A vehicle that produces the geometric yaw rate is described as ‘neutral steering’ and is often regarded as something of a holy grail. However, for all the reasons described in [Section 7.3.2.3](#) it is rarely engineered in vehicles. In [Figure 7.21](#), all three cars understeer in their linear regions. The range of lateral acceleration gains is greatly reduced for the understeering road car in [Figure 7.22](#), requiring less skilful adaptation to vehicle behaviour as speed varies compared with a textbook neutral steer vehicle. For the competition car the lateral acceleration gain can be seen to be substantially linear with speed.

Subjectively, drivers instinctively learn these base characteristics of vehicles very quickly. Only the most inexperienced novice drivers have difficulty with the steering ratios being higher or lower than expected. Within a few hours of driving experience, errors in steering (as distinct from positional errors caused by the size of the vehicle) are almost entirely absent. When the vehicle is driven at low lateral accelerations (i.e. when there is significant grip in reserve) the vehicle behaves in a substantially linear fashion — i.e. more steering gives more yaw rate in a proportional manner. Under these conditions, vehicles are rarely evaluated as ‘understeering’ although the vast majority of road cars do in fact understeer at highway speeds. These gains (both yaw and lateral) become the ‘datum’ condition, which is often rendered as ‘neutral’ when described subjectively. Differences between vehicles, in particular with YRG, are likely to be ascribed to differences in steering ratio and not differences in fundamental vehicle behaviour.

As lateral accelerations rise towards some significant fraction of the available coefficient of friction (typically above half), the vehicle behaviour becomes nonlinear due to the behaviour of the tyres. Depending on suspension geometry, tyre characteristics, elasto-kinematic behaviour of the suspension and suspension calibration, the yaw rate characteristics in the nonlinear region can vary in either direction — either up or down — from the linear characteristics. Of particular interest is the relationship between AyG and YRG in the nonlinear region. [Figure 7.18](#) graphically suggests three possibilities for the departure from linearity for a vehicle. The situation imagined is that a vehicle has been driven steadily close to the subjective linearity limit and then the vehicle speed is increased in order to increase the lateral acceleration. The increase in vehicle speed is gradual and so details such as driveline layout are not relevant because the drive torque is low. This situation might occur, for example, on a long, constant radius, downhill motorway interchange ‘cloverleaf’.

**FIGURE 7.23**

Possibilities for departure from linearity.

The three scenarios illustrated in [Figure 7.23](#) can be summarised as:

- a.** reduced AyG and further reduced YRG
- b.** reduced AyG and YRG in proportion to each other
- c.** reduced YRG and further reduced AyG.

In scenario (a), YRG is reduced further than lateral acceleration gain. In order to accommodate the changes in both lateral acceleration and yaw rate, the radius of the path must increase and so the vehicle has a period of adjustment to a new, wider line in the curve. Most drivers notice this and instinctively reduce vehicle speed to restore the desired path over the ground. It is described subjectively as an ‘understeer departure’ or ‘pushing’ or perhaps in the USA as ‘plowing’ (ploughing). If uncompensated, it leads to a vehicle departing the course (road, track, etc.) in an attitude that is basically forwards. This is by far the most common behaviour for road vehicles. It is desirable since, if the vehicle does leave the road, it is least likely to roll over and will correctly present the engineered crash structure between the occupants and any obstacles encountered. For sporty drivers the sensation of the vehicle ‘turning out’ of the corner as it departs from linearity can become tiresome.

In scenario (b), lateral acceleration gain and YRG change in some connected manner and the vehicle will maintain course although it might need some modification to steering input. Subjectively this vehicle will be described as ‘neutral’ although objectively it might well be understeering. Excess speed for a curve will lead to the vehicle running wide but with no sense of ‘turning out of the curve’. Such a vehicle generally feels benign although the progressive departure can mean it is unnoticed by inattentive drivers. Enthusiastic drivers will not be so frustrated by this behaviour.

In scenario (c), AyG reduces more than YRG. This leads to an ‘over rotation’ of the vehicle when viewed in plan. Depending on the severity of the mismatch, the change may lead to a spin out of the curve. From inside the vehicle there is a pronounced sense of the rear end of the vehicle departing first but objectively the vehicle may not actually oversteer in the classical sense — it may simply move ‘towards neutrality’. This is the nature of rear-wheel-drive vehicles when driven to departure using the throttle. Subjectively, there is a pronounced sense of ‘oversteer’ —

sometimes described as ‘loose’ in the USA. Vehicles that preserve YRG as they lose linearity are widely regarded as fun to drive and sporty.

A further difficulty between theoretical and practical dynamicists is that the former group often considers the vehicle on the basis of ‘fixed control’ and ‘free control’ where the latter almost always uses ‘driver input to complete a set task’. With fixed and free control, the inputs are consistent and the response of the vehicle is used to evaluate it. With driver input, the vehicle response is substantially constant and the vehicle is evaluated on the basis of the required changes in driver input to complete a task. More and more, so-called ‘black lakes’ — large flat areas of high grip surface — are being added to vehicle testing facilities to allow the evaluation of fixed and free control manoeuvres for experimental correlation purposes. Also gaining in popularity are theoretical ‘driver models’. These range from simple path-followers to sophisticated multiloop, multipass adaptive controllers. At present there are many such models and none has gained precedence, suggesting perhaps that none is ideal for the task at hand — understanding and improving vehicle behaviour. Methods for modelling driver behaviour are discussed in Chapter 6. A very real problem with such models is that there is a fine line between evaluating the quality of the driver model and evaluating the change on the vehicle. Did the vehicle performance improve because the modification suited the driver model? Does that behaviour reflect a typical driver in an emergency situation? These criticisms are not unique to modelling and can also be levelled at highly skilled development drivers. Indeed, within motorsport circles this particular difficulty is widely recognised. There are drivers who can drive a given setup in the fastest way that it can be driven but who cannot articulate how the vehicle could be faster. Such drivers are an asset on race days but less so during development testing. For this reason, ‘test’ or ‘development’ drivers are frequently employed who have a different set of skills to event drivers.

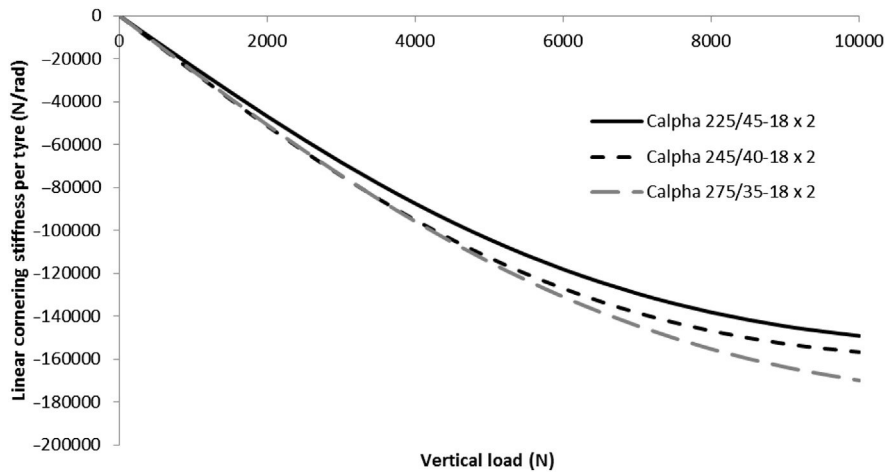
In summary then, the subjective evaluation of a vehicle depends largely on the nature of its departure from linearity while objective evaluation is an absolute positioning against a geometric datum. From inside the vehicle it is generally difficult to distinguish between a vehicle that is operating at a large body slip angle and one that is truly oversteering. In any case, to control a large body slip angle it is frequently necessary to reduce or reverse steering input (‘opposite lock’), changing the measured YRG substantially towards oversteer.

### 7.3.4 Mechanisms for generating understeer and oversteer

At this stage we must return to the tyre characteristics. Neglecting camber angle, version three of the Magic Formula (Table 5.4) gives cornering stiffness, BCD, by the following expression:

$$BCD = a_3 \left( \sin \left( 2 \cdot \tan^{-1} \left( \frac{F_z}{a_4} \right) \right) \right) \quad (7.49)$$

where  $F_z$  is the vertical load on the tyre and  $a_3$ ,  $a_4$  are empirical coefficients.

**FIGURE 7.24**

Influence of vertical load on cornering stiffness — published data.

Using data for three known tyre sizes from the same manufacturer, [Figure 7.24](#) can be plotted.

It can be seen that there is a behaviour that is broadly related to tyre width. Adding data for two additional tyres and plotting both  $a_3$  and  $a_4$  against tyre width, an empirical approximation can be made ([Figure 7.25](#)).

Using the coefficients for the linear fit, which can be seen to be representative but not particularly high quality, the curves in [Figure 7.26](#) can be plotted for notional tyres of different widths.

While clearly not an accurate representation of any particular tyre, these curves represent ‘plausible’ tyres of the type used on modern vehicles and display the kind of load sensitivity needed to manipulate vehicle behaviour to an extent that is of broadly similar magnitude to that experienced in developing vehicles. In the absence of detailed data, these estimates provide direction; it is suspected they overrepresent the fall-off in performance for smaller tyres and thus underpredict vehicle performance — a conservative error.

Considering one particular tyre, [Figure 7.27](#) shows a typical plot of tyre lateral force with tyre load at a given slip angle. The total lateral force produced at either end of the vehicle is the average of the inner and outer lateral tyre forces. From the figure it can be seen that  $\Delta F_y$  represents a theoretical loss in tyre force resulting from the averaging and the nonlinearity of the tyre. Tyres with a high load will not produce as much lateral force (in proportion to tyre load) compared with other tyres on the vehicle.

More weight transfer at either end therefore reduces the total lateral force produced by the tyres and causes that end to drift out of the turn. At the front this will produce understeer and at the rear this will produce oversteer. It should be noted that this behaviour is true for all slip angles of interest; at low slip angles, the cornering

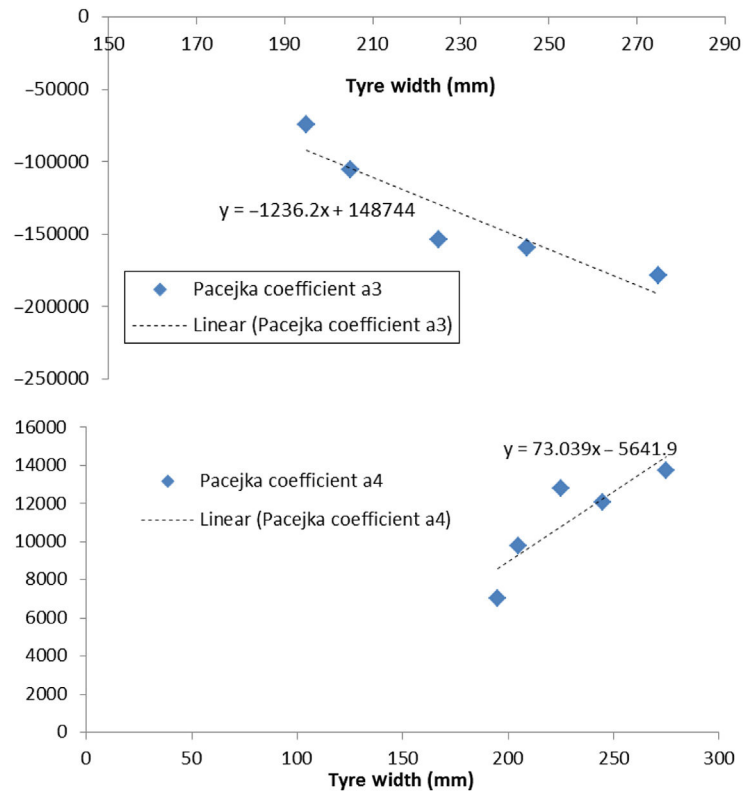


FIGURE 7.25

Fitting a linear regression line to the published data with respect to tyre width.

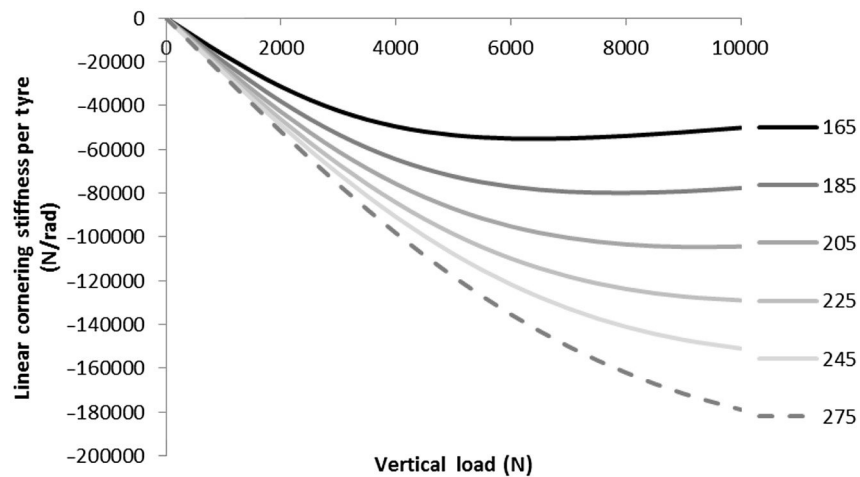
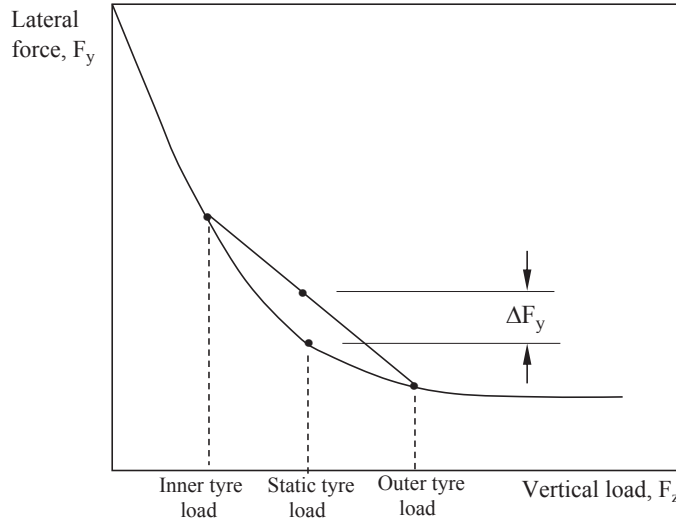


FIGURE 7.26

Reconstructed tyre cornering stiffness versus load characteristics using the fitted equations.

**FIGURE 7.27**

Loss of cornering force due to nonlinear tyre behaviour.

stiffness of the tyre is reduced nonlinearly with increasing load, promoting a larger slip angle at an axle with a greater roll moment. At saturated slip angles, the peak lateral force is reduced as a proportion of the vertical load, producing a lower coefficient of friction for an axle with a greater roll moment. Thus mechanisms that adjust sublimit understeer and oversteer also adjust departure plough and spin behaviour.

Returning to the 2 DOF model, a common criticism levelled at it is the lack of representation of the effects of load transfer on tyre behaviour. This is easily incorporated into the model by presuming static roll behaviour — i.e. roll moment is proportional to lateral acceleration — and calculating weight transfer on the basis of mass centre height and track width. The frequency of oscillation for the yaw/sideslip mode is something under 0.5 Hz generally. The authors' experience of the roll mode of vibration is that it is well damped and around 2 Hz. Thus there are about two octaves separating two quite well damped modes of vibration. This means the static representation of roll is slightly inaccurate but not unacceptably so for the purpose at hand.

With the tyre characteristics modelled, the lateral acceleration and mass centre height are used together to produce a roll moment:

$$(\dot{v}_y + \omega_z v_x) h = M_x \quad (7.50)$$

This roll moment is distributed arbitrarily between front and rear axles by a fraction,  $R$ , sometimes known as the front roll moment distribution (FRMD) and which in turn produces a change in wheel loads via the wheel track distance,  $t$ :

$$F_{z\text{ fr}} = \frac{1}{2} mg \left( \frac{b}{a+b} \right) - \frac{RM_x}{t} \quad (7.51)$$

$$F_{zn} = \frac{1}{2} mg \left( \frac{b}{a+b} \right) + \frac{RM_x}{t} \quad (7.52)$$

$$F_{zrr} = \frac{1}{2} mg \left( \frac{a}{a+b} \right) - \frac{(1-R) M_x}{t} \quad (7.53)$$

$$F_{zrl} = \frac{1}{2} mg \left( \frac{a}{a+b} \right) + \frac{(1-R) M_x}{t} \quad (7.54)$$

The static roll approximation allows for the addition of roll moment distribution to exercise the load sensitivity of the tyres and produce more useful results. In combination with the current slip angles at the tyres, this allows the resulting side forces on the vehicle to be calculated, allowing the acceleration state to be calculated for the next integration step. Alternatively, the steady state gains can be calculated as was performed previously.

With the extra static roll DOF, the character of the 2 DOF solution is maintained but the ability to tune the behaviour of the car with roll moment distribution is captured. This model — which perhaps might be called a two-and-a-half DOF ('2½ DOF') model — is all that is needed to examine the behaviour of vehicles in an overview sense for normal driving.

Note that in principle such a model could be further extended to calculate the nonlinear behaviour of the tyres and its validity extended to the entire handling envelope. While the 2½ DOF model is readily soluble in a generic tool such as MS-Excel, as the calculations become more complex it is observed that MS-Excel gets into difficulties and becomes less and less responsive. For this reason a nonlinear model is best implemented in another environment, such as Octave or Matlab/Simulink.

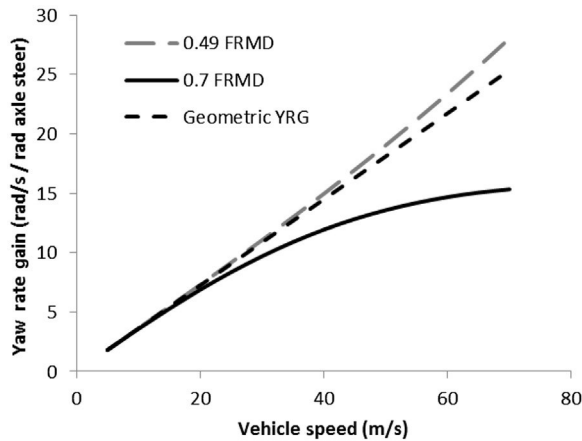
A brief consideration of varying vehicle configurations is warranted with the 2½ DOF model. It is a matter of record that the underlying facts have led to the development of, broadly speaking, three distinct types of vehicles, each of which have their own foibles.

#### 7.3.4.1 50/50 (BMW, Mercedes, Jaguar, MX-5)

The vehicle layout in which the mass is carried midway between the wheels has little or no technical understeer (i.e. a high YRG) and also a large amount of yaw/sideslip damping. That the original layout chosen for horseless carriages also happens to be technically meritorious is simply a matter of serendipity. However, it remains quite robust when subject to development and for this reason it has been the layout of choice for BMW, Mercedes, Jaguar, etc. for sometime.

Practical disadvantages include poor winter mobility when combined with a rear-wheel-drive powertrain and reduced interior package due to the intrusion of transmission and differential into cabin and luggage space. The high YRG generally leads to the addition of technical understeer through roll moment distribution, as shown in [Figure 7.28](#).



**FIGURE 7.28**

Yaw Rate Gain manipulation of a 50/50 vehicle with roll moment distribution. FRMD, front roll moment distribution; YRG, yaw rate gain.

For the baseline 50/50 vehicle, the solution does not converge at less than 49% front roll moment distribution (FRMD). The system responds strongly, becoming unbounded and progressing into the nonlinear region, at which time the model is no longer valid — but a disturbing lack of convergence may be expected in a real vehicle as the local slope of the tyre characteristics reduces and the vehicle slides ‘over the edge’.

The addition of a strongly front-biased roll moment distribution aids traction performance by reducing the burden on the differential in a rear-wheel-drive powertrain and naturally balances the tendency of the powertrain to add YRG as the lateral capability of the rear tyres reduces with tractive effort.

50/50 cars display a distinctive subjective ‘calmness’ to their behaviour that some observers describe as competence and others as dullness.

Note also that recent high-power versions of such vehicles tend to display a tyre stagger that is also largely for marketing reasons; homologation documents often support the fitment of the same-size tyres all round if the owner prefers; the author’s (Harty) BMW 540 was homologated with an ‘M5 style’ tyre stagger but was run with 225 tyres all round, also homologated.

#### 7.3.4.2 Tail heavy (VW Beetle, Porsche 911)

Low-powered vehicles like the original 1938 Beetle attained a top speed of around 75 mph. When mass is carried rearward in the vehicle, the equations of motion display technical oversteer, in which  $YRG$  exceeds  $YRG_{geom}$ . There exists a speed beyond which the car cannot restore itself to a straight ahead condition without intervention from the driver — the yaw damping becomes negative. Early VW Beetles did not exceed this speed and therefore simply harvested the extra traction afforded by the weight over the driven wheels to retain mobility and improve accelerative performance.

When speeds for this platform came up with post-war performance tuning and evolution into the Porsche 911 family, the solution to the problem was empirically determined as the use of larger tyres on the rear of the vehicle than the front. The process of empirical determination took some time and for a period, such cars displayed somewhat difficult behaviour as poignantly illustrated by James Dean's nickname for his Porsche Speedster: 'Little Bastard'. Not until the group of papers surrounding Segel's published by the IMechE (1956) did the underlying system properties become clear.

The axle weights modify the effective tyre cornering stiffness through the vertical load sensitivity mechanism. To recover stable behaviour, the roll moment distribution needs to be pushed even further forward. Extreme roll moment distributions are difficult to achieve in practice, and so it is more likely that a tyre stagger will be adopted. A 'two-size' stagger (using the previously modelled tyre properties), gives acceptable behaviour for a 45% front weight distribution, with a 'four-size' stagger being helpful for a 40% front weight distribution. In both cases, the vehicle tolerates a range of roll stiffness distributions from 50% to 70% to deliver any preferred YRG characteristic. Without the tyre stagger, the vehicle is extraordinarily sensitive to roll moment distribution which would mean, even if it were possible to achieve, that the vehicle would be unduly sensitive to build quality and setup adjustments, rendering it not robust in service, as illustrated in Figure 7.29 where small amounts of roll moment distribution turn the vehicle from stable to unstable.

The high level of roll moment carried on a relatively light axle gives rise to the distinctive wheel-in-the-air cornering stance of Porsche 911s and such like, as shown in Figure 7.30.

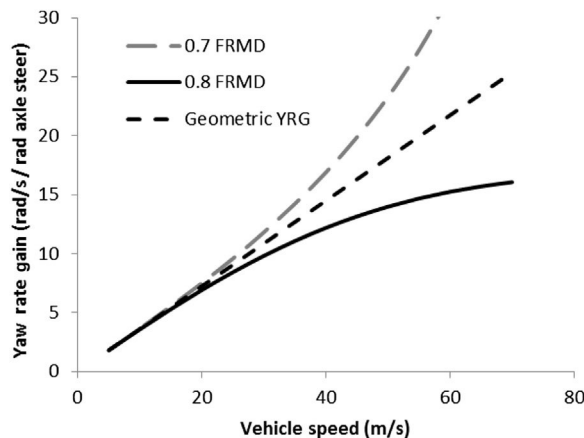


FIGURE 7.29

Tail-heavy vehicles with equal sized tyres are unduly sensitive to roll moment distribution. FRMD, front roll moment distribution; YRG, yaw rate gain.

**FIGURE 7.30**

High front roll moment distribution gives a distinctive cornering stance.

(Picture courtesy of [Pistonheads.com](http://Pistonheads.com).)

Note that current 911s display rather more tyre stagger than is necessary for the minimum stability criterion, this being driven to some extent by marketing expectations and to some extent by the requirement for acceptable tyre life with increased power outputs.

#### **7.3.4.3 Nose heavy (*Issigonis Mini, VW Polo, Ford Focus*)**

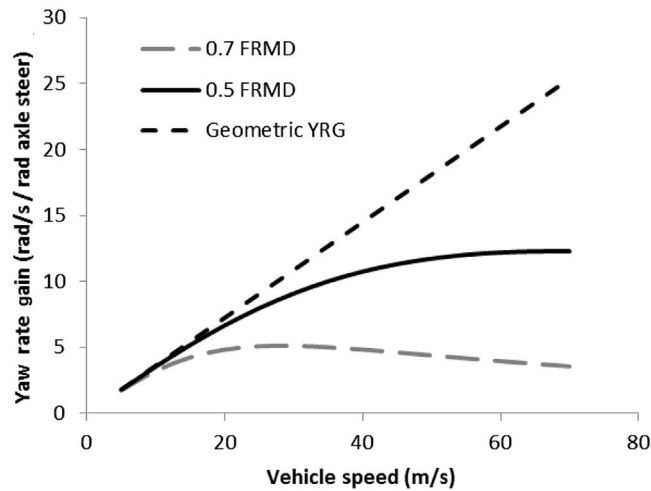
Issigonis recognised that the primary purpose of a car was accommodating passengers and proposed turning the driveline through 90° and pushing it forward in the car. In the equations of motion, this results in a large amount of technical understeer but can lead to very low levels of yaw/sideslip damping. It can be seen from typical eigen-solutions that damping is highest at low speed and falls off in a fairly smooth way with increasing speed. Thus the most lightly damped condition is at maximum speed.

While there is no abrupt safe-to-unsafe boundary for yaw/sideslip damping, most organisations will not release a vehicle with a damping ratio of less than about 0.6–0.7. One solution to this would be to fit larger tyres at the front of the vehicle, but since the package is already congested widthwise this is not a preferred solution. Economies of scale with the same tyre all round add to the pressures to find other solutions to replacing the lost yaw damping.

These solutions include roll steer to induce yaw moments out of the turn, compliant steer to the same end and a less front-biased roll moment distribution. Modifying the roll moment distribution to 50% front improves the damping slightly and lifts the control gains from their very low values. The development of nose-heavy vehicles is something of a chase for yaw damping, depending on the required top speed of the vehicle and its mass and inertia characteristics. Again it must be emphasised that this is damping in the yaw/sideslip mode and not suspension damping (Figure 7.31).

The reduced damping and increased yaw/sideslip frequency gives nose-heavy cars a distinctive subjective ‘liveliness’ to their behaviour that some observers describe as eager and others as nervous.

The requirement to manipulate vehicle characteristics with suspension characteristics means that, unlike the 50/50 platform, nose-heavy cars require a large amount of subjective development to avoid awkward transient sensations. They are also

**FIGURE 7.31**

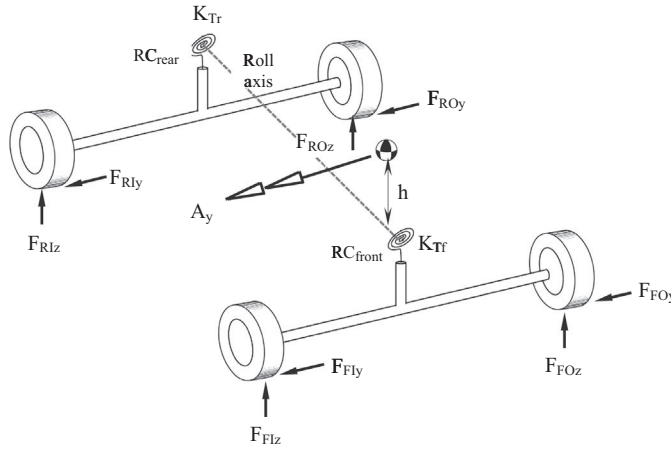
Influence of roll moment distribution on control gains — nose-heavy vehicle. FRMD, front roll moment distribution; YRG, yaw rate gain.

heavily reliant on the detail of the suspension systems to deliver good behaviour; the last 30 years has seen an explosion of complexity in suspensions, with particular emphasis on the rear suspension.

It can be seen that each of the three categories of vehicle can be tuned for acceptable behaviour without excessive difficulty, and desirable behaviour — stability and adequate yaw damping — looks broadly similar for all the vehicle categories. A fully featured vehicle model must be manipulated in its entirety to represent different vehicles and may not respond well to simple editing of headline wheelbase, mass and inertia properties without accompanying detailed suspension changes. The 2½ DOF linear formulation described here has the advantage of being somewhat simpler and does not obscure the underlying behaviour of the vehicle with such details; the sense that emerges from using such a formulation remains valid despite the simplicity of the model.

It may be readily imagined that most passenger cars cross between two of the identified types with loading condition, and for light commercial vehicles they may transition from nose to tail heavy — that is to say crossing between three of the identified types. While this would seem to be cause for some consternation, it is the author's experience that dynamic considerations are applied to a vehicle with two passengers and an expectation is applied that the vehicle will not be subject to emergencies when fully laden — see for example National Highway Traffic Safety Administration (NHTSA) (2001). Many organisations define a 'performance test weight' that is somewhat less than gross vehicle weight. It would seem that, generally, drivers of laden vehicles are considerably more circumspect.

To understand in more detail the vehicle parameters that have an influence on oversteer, understeer and departure behaviour, we can use the roll stiffness model

**FIGURE 7.32**

Free body diagram roll stiffness model during cornering.

described earlier to consider a series of free bodies and the force moment balance on each during steady state cornering. Figure 7.32 shows a version of the roll stiffness model while travelling in a curved path, with a lateral acceleration,  $A_y$ .

Consider next the components of force and moment acting on the vehicle body in isolation.

Using the roll stiffness model as the basis for the analysis we are treating the body as a single rigid axis with forces and moments transmitted from the front and rear suspensions (axles) at points representing the front and rear roll centres as shown in Figure 7.33. Consider the forces and moments acting on the vehicle body rigid roll axis. Note that we are ignoring the inclination of the roll axis. A roll moment ( $m A_y \cdot h$ ) acts about the axis and is resisted in the model by the moments  $M_{FRC}$  and  $M_{RRC}$  resulting from the front and rear roll stiffnesses,  $K_{Tf}$  and  $K_{Tr}$ .

$$F_{FRCy} + F_{RRCy} - m A_y = 0 \quad (7.55)$$

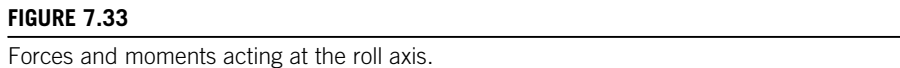
$$M_{FRC} + M_{RRC} - m A_y \cdot h = 0 \quad (7.56)$$

The roll moment causes weight transfer to the inner and outer wheels (Figure 7.34). Taking moments for each of the front and rear axles shown gives:

$$\Delta F_{FzM} = \frac{M_{FRC}}{t_f} = m a_y \cdot h \left( \frac{K_{Tf}}{K_{Tf} + K_{Tr}} \right) \frac{1}{t_f} \quad (7.57)$$

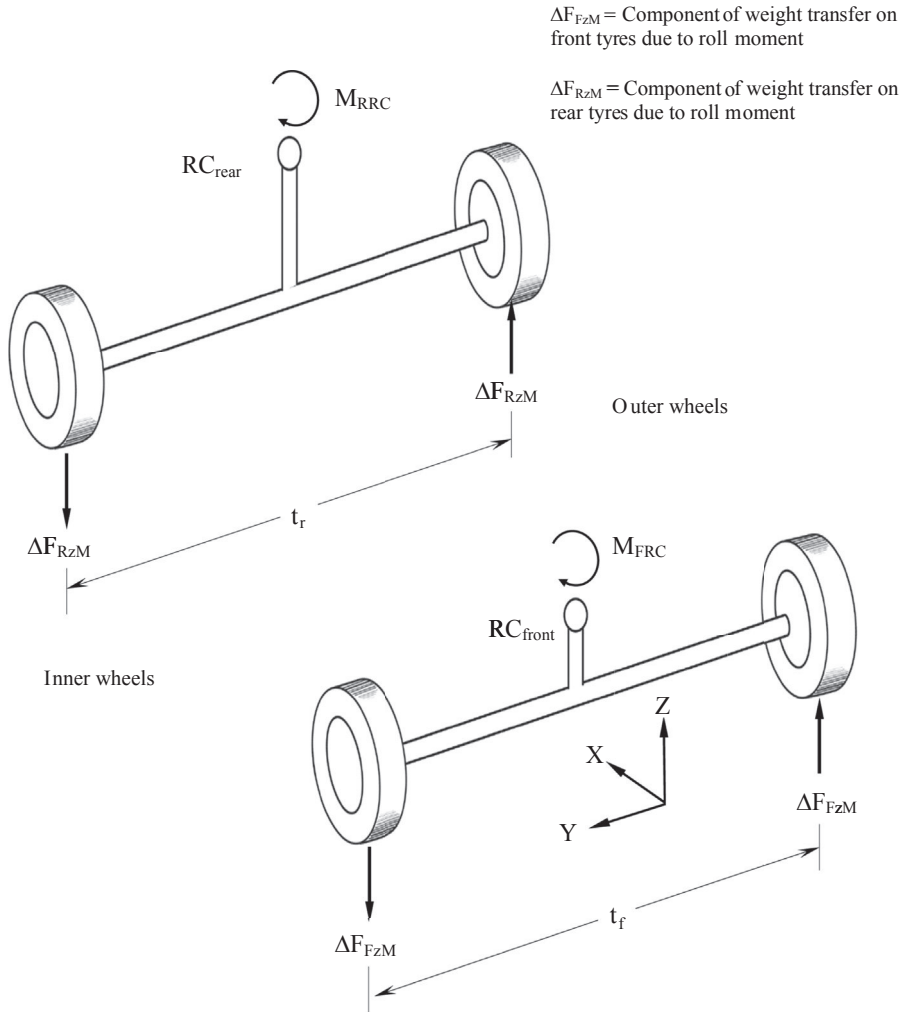
$$\Delta F_{RzM} = \frac{M_{RRC}}{t_r} = m a_y \cdot h \left( \frac{K_{Tr}}{K_{Tf} + K_{Tr}} \right) \frac{1}{t_r} \quad (7.58)$$

It can be seen from Eqns (7.58) and (7.59) that if the front roll stiffness,  $K_{Tf}$  is greater than the rear roll stiffness,  $K_{Tr}$  there will be more weight transfer at the front



This gives:

$$F_{RRCy} = m a_y \left( \frac{a}{a+b} \right) \quad (7.60)$$



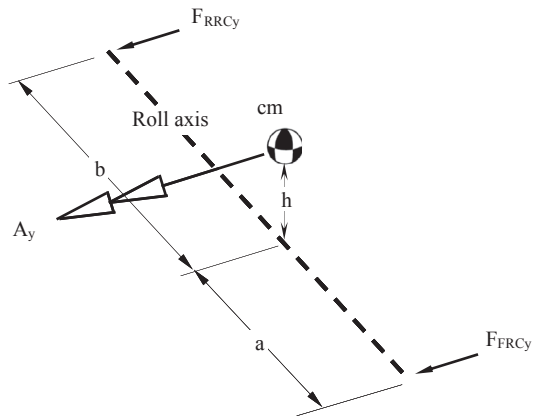
**FIGURE 7.34**

Components of weight transfer due to roll moment.

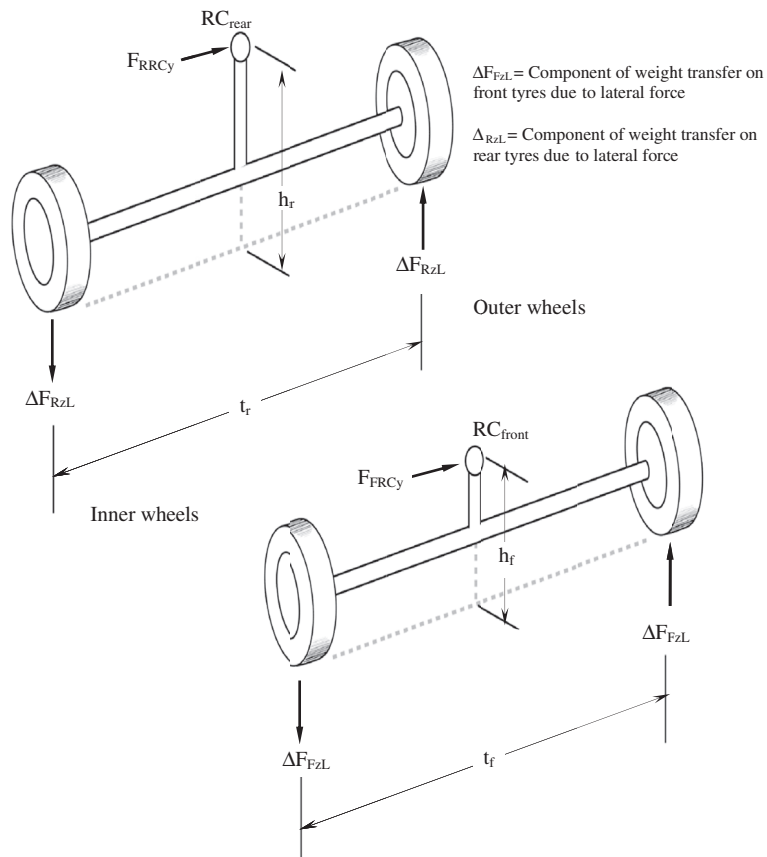
From Eqns (7.60) and (7.61) we can see that moving the body centre of mass forward would increase the force, and hence weight transfer, reacted through the front roll centre (and vice versa). We can now proceed to find the additional components,  $\Delta F_{FzL}$  and  $\Delta F_{RzL}$ , of weight transfer due to the lateral forces transmitted through the roll centres.

Taking moments again for each of the front and rear axles shown in Figure 7.36 gives:

$$\Delta F_{FzL} = F_{FRCy} \left( \frac{h_f}{t_f} \right) = m a_y \cdot h \left( \frac{b}{a+b} \right) \left( \frac{h_f}{t_f} \right) \quad (7.61)$$


**FIGURE 7.35**

Forces acting on the body roll axis.


**FIGURE 7.36**

Components of weight transfer due to lateral force.



$$\Delta F_{RzL} = F_{RRCy} \left( \frac{h_r}{t_r} \right) = m a_y \cdot h \left( \frac{a}{a+b} \right) \left( \frac{h_r}{t_r} \right) \quad (7.62)$$

It can be seen from Eqns (7.62) and (7.61) that if the front roll centre height,  $h_f$ , is increased there will be more weight transfer at the front (and vice versa).

We can now find the resulting load shown in Figure 7.37 acting on each tyre by adding or subtracting the components of weight transfer to the front and rear static tyre loads ( $F_{FSz}$  and  $F_{RSz}$ ).

This gives:

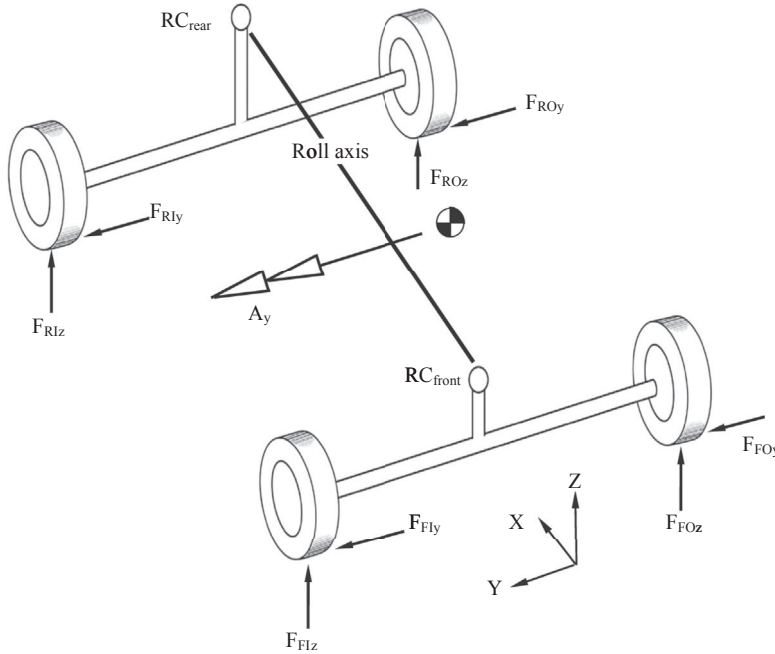
$$F_{FIZ} = F_{FSz} - \Delta F_{FzM} - \Delta F_{FzL} \quad (7.63)$$

$$F_{FOz} = F_{FSz} + \Delta F_{FzM} + \Delta F_{FzL} \quad (7.64)$$

$$F_{RIz} = F_{RSz} - \Delta F_{RzM} - \Delta F_{RzL} \quad (7.65)$$

$$F_{ROz} = F_{RSz} + \Delta F_{RzM} + \Delta F_{RzL} \quad (7.66)$$

Although substantially simplified, the preceding analysis helps with understanding the essential mechanisms in play when a vehicle is cornering.



**FIGURE 7.37**

Resulting forces acting at the inner and outer tyres.

## 7.4 Transient effects

Vehicle dynamics would be a very simple field if the preceding text described it entirely. The full transient equations developed earlier can be considered again, considering the vehicle travelling in a straight line with no steering input. It is reasonable to presume solutions to the differential equations of the form  $Ae^{\lambda t}$ , which can represent many different behaviours depending on the value of  $\lambda$ .

$\lambda$  is the ‘eigenvalue’ of the equation describing the system. Systems with multiple DOF have multiple eigenvalues. For a system with DOF  $x_i$ , the general form of the equations of motion is given by the so-called ‘Jacobian’ notation, a matrix of partial derivatives:

$$\begin{bmatrix} \frac{\partial}{\partial x_1}(\dot{x}_1) & \frac{\partial}{\partial x_2}(\dot{x}_1) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_1) \\ \frac{\partial}{\partial x_1}(\dot{x}_2) & \frac{\partial}{\partial x_2}(\dot{x}_2) & & \frac{\partial}{\partial x_n}(\dot{x}_2) \\ \dots & & & \dots \\ \frac{\partial}{\partial x_1}(\dot{x}_n) & \frac{\partial}{\partial x_2}(\dot{x}_n) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \quad (7.67)$$

We have presumed a solution of the form

$$x(t) = Ae^{\lambda t} \quad (7.68)$$

$$\dot{x}(t) = \lambda Ae^{\lambda t} = \lambda x(t) \quad (7.69)$$

When derivatives are substituted into the equations, it can be seen that

$$\begin{bmatrix} \frac{\partial}{\partial x_1}(\dot{x}_1) & \frac{\partial}{\partial x_2}(\dot{x}_1) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_1) \\ \frac{\partial}{\partial x_1}(\dot{x}_2) & \frac{\partial}{\partial x_2}(\dot{x}_2) & & \frac{\partial}{\partial x_n}(\dot{x}_2) \\ \dots & & & \dots \\ \frac{\partial}{\partial x_1}(\dot{x}_n) & \frac{\partial}{\partial x_2}(\dot{x}_n) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad (7.70)$$

This leads directly to the eigenvalue problem in familiar form as the determinant of the Jacobian:

$$\begin{vmatrix} \frac{\partial}{\partial x_1}(\dot{x}_1) - \lambda & \frac{\partial}{\partial x_2}(\dot{x}_1) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_1) \\ \frac{\partial}{\partial x_1}(\dot{x}_2) & \frac{\partial}{\partial x_2}(\dot{x}_2) - \lambda & & \frac{\partial}{\partial x_n}(\dot{x}_2) \\ \dots & & & \dots \\ \frac{\partial}{\partial x_1}(\dot{x}_n) & \frac{\partial}{\partial x_2}(\dot{x}_n) & \dots & \frac{\partial}{\partial x_n}(\dot{x}_n) - \lambda \end{vmatrix} = 0 \quad (7.71)$$

For the 2 DOF linear vehicle system modelled, the solution to the eigenvalue problem is given by

$$\begin{vmatrix} \frac{C_{af} + C_{ar}}{mv_{x'}} - \lambda & \frac{C_{af}a - C_{ar}b}{mv_{x'}} - v_{x'} \\ \frac{a \cdot C_{af} - b \cdot C_{ar}}{I_{zz}v_{x'}} & \frac{a^2 \cdot C_{af} + b^2 \cdot C_{ar}}{I_{zz}v_{x'}} - \lambda \end{vmatrix} = 0 \quad (7.72)$$

Writing the terms in a more convenient form...

$$a_{11} = \frac{C_{af} + C_{ar}}{mv_x} \quad (7.73)$$

$$a_{21} = \frac{C_{af}a - C_{ar}b}{mv_x} - v_x \quad (7.74)$$

$$a_{12} = \frac{a \cdot C_{af} - b \cdot C_{ar}}{I_{zz}v_x} \quad (7.75)$$

$$a_{22} = \frac{a^2 \cdot C_{af} + b^2 \cdot C_{ar}}{I_{zz}v_x} \quad (7.76)$$

...allows the solution of the eigenvalue problem with the rather trivial quadratic equation in  $\lambda$ .

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21} \cdot a_{12} = 0 \quad (7.77)$$

$$\lambda^2 + (-a_{11} - a_{22})\lambda + (a_{11} \cdot a_{22} - a_{21} \cdot a_{12}) = 0 \quad (7.78)$$

$$A = 1 \quad (7.79)$$

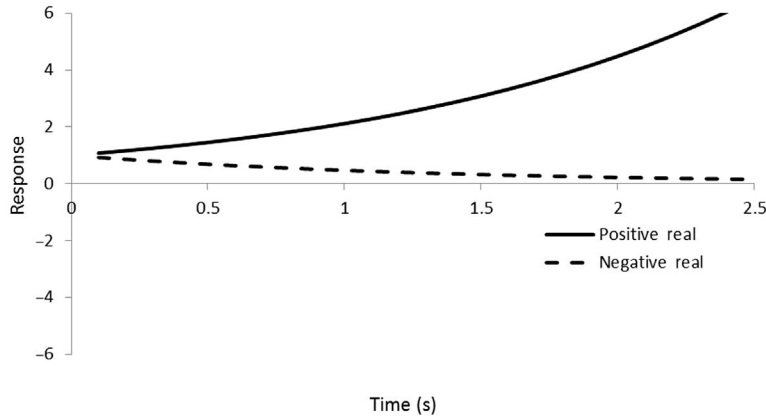
$$B = (-a_{11} - a_{22}) \quad (7.80)$$

$$C = (a_{11} \cdot a_{22} - a_{21} \cdot a_{12}) \quad (7.81)$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad (7.82)$$

It can be seen that there are two types of outcome, the generation of two real roots or the generation of a pair of complex conjugate roots, depending on the magnitude of the terms B and C.

If  $\lambda$  is real and positive then the response of the system is unbounded and grows with time. If  $\lambda$  is real and negative then the response of the system is bounded and converges to a constant value. Both of these outcomes are illustrated in [Figure 7.38](#), which is a repeat of [Figure 3.62](#).

**FIGURE 7.38**

The influence of the sign of the real part of the roots of the characteristic equation.

For such a response, the time constant of the response is given by the magnitude of the eigenvalue; a numerically large eigenvalue suggests a very rapid response, whether positive or negative.

If  $\lambda$  is imaginary then the response of the system is oscillatory with the frequency given by the eigenvalue since  $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$  which is oscillatory in time. If  $\lambda$  is complex and the real part of  $\lambda$  is positive the response is oscillatory and unbounded. If  $\lambda$  is complex and the real part of  $\lambda$  is negative the response is oscillatory and convergent. These outcomes are illustrated in Figure 7.39, which is a repeat of Figure 3.63.

For oscillatory responses the period of oscillation is given by the imaginary part of the eigenvalue. Note that the undamped natural frequency is given by the Pythagorean sum of the real and imaginary components, but is of little practical interest.

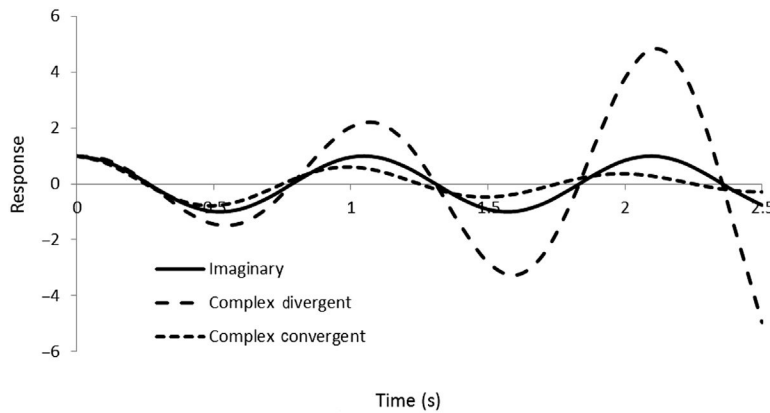
For an oscillatory response, the damping ratio,  $\zeta$ , is given by

$$\zeta = \cos \left( \tan^{-1} \left( \frac{\text{imag}(\lambda)}{\text{real}(\lambda)} \right) \right) \quad (7.83)$$

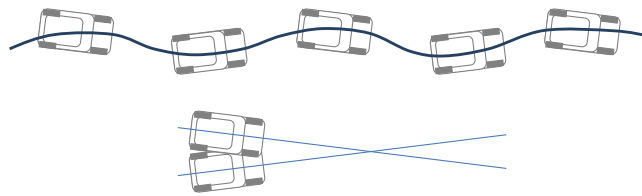
The eigenvalues for the vehicle system are of particular interest since they change with speed. It can be shown that if the ratio of cornering stiffness front to rear is less than the mass ratio front to rear then the vehicle will always be stable (Milliken and Milliken, 1998).

$$\frac{C_{af}}{C_{ar}} < \frac{b}{a} \Rightarrow \text{stable} \quad (7.84)$$

If this is not true then there will be some speed above which the eigenvalues become positive real, implying an inability to reject disturbances. It is observed in the literature that vehicles that are always stable display reduced damping in the yaw/sideslip mode of vibration — see for example Milliken and Milliken (1998).

**FIGURE 7.39**

The influence of imaginary components of the roots of the characteristic equation.

**FIGURE 7.40**

Yaw/Sideslip mode of vibration for a running car.

The eigenvalue represents a resonant behaviour of the vehicle in the ground plane, often described as a ‘yaw/sideslip’ mode, illustrated at the top of [Figure 7.40](#). If it is imagined that the car could be shadowed with a helicopter travelling at exactly the same speed, when in resonance the motion of the car as observed from directly above in the helicopter would resemble a pendulum swinging about some ‘apparent pendulous centre’ — a point ahead of the car when viewed from a platform travelling at the same speed as the vehicle, illustrated at the bottom of [Figure 7.40](#). The figure is an illustrative view and does not capture the phase difference between yaw and sideslip present in the real vehicle motion.

Like a pendulum, a system resonance exists despite the fact that there is no stiffness — a fact that is not always intuitively grasped at first reading. Unlike a pendulum, even the very simple representation of the car has some damping in the mode of vibration; this arises due to the yaw/sideslip coupling terms in the equations of motion. Note that the yaw/sideslip damping ratio has nothing to do with the suspension dampers.

The idea of the body slip angle has already been introduced in [Section 7.3.2.2](#). The need for a body slip angle gives rise to an additional yaw rate, rotating the body to the correct slip angle, for a brief period near the start of any manoeuvre.

There is also a corresponding yaw rate reduction at the end of a manoeuvre required to bring the vehicle back to its trimmed, straight ahead state. These variations in yaw rate are acutely remarked by even the least skilled driver and greatly influence the emotional reaction of the driver to the car — although the driver might not consciously recognise it.

A vehicle that rotates slowly to give the required body slip angle feels sluggish and unresponsive; a vehicle that overshoots and oscillates feels lively and poorly controlled. Subjectively, these two states might be described as ‘transient understeer’ and ‘transient oversteer’, respectively — although the latter might not objectively be oversteer, it is certainly a greater yaw rate than might be expected. Objectively, the quantity of interest is the rate of change of body slip angle, known as  $\dot{\beta}$  or ‘beta-dot’. The bulk yaw rate of the vehicle is made up of two components: the first is the yaw rate associated with a curved path — it might be thought of as the yaw rate that would be experienced by a stone on a string being swung around. Milliken and others refer to this as the ‘no-slip yaw rate’ — the yaw rate predicted without any body slip angle. The second is the beta-dot component. By inspection, it can be seen that:

$$\dot{\beta} = \frac{A_y}{V} - \omega \quad (7.85)$$

Thus in general, beta-dot is available as a simple combination of vehicle states, particularly during multibody simulation work. For real vehicles, noise on accelerometer data and the difficulty of knowing the genuine forward speed of the vehicle under conditions of longitudinal tyre slip mean that beta-dot is difficult to discern in real time, although it is amenable to offline processing.

With changes in suspension configuration, elasto-kinematic calibration and damper behaviour, beta-dot can be modified significantly. The overall behaviour of the vehicle may well only be slightly modified and so to concentrate merely on bulk yaw rate would be to underestimate the significance of the change from the driver’s perspective.

In the steady state, the body slip angle is constant and so it is of little consequence except when something changes. In practice, real vehicles spend very little time in the steady state and so the short-lived events — described as ‘transients’ — are very important to the driver’s reaction to the vehicle because of the acute perception of beta-dot. This perception is not only in terms of absolute levels but also in terms of delays between driver requests and vehicle response. When the handwheel is in motion, the driver is implicitly requesting a change in body slip angle, since body slip angle is linked to lateral acceleration. If the change in body slip angle occurs in a way that is substantially connected to the change in steering, the driver feels reassured. If however the change in body slip angle occurs with perhaps a significant delay or perhaps with a characteristic shape different to the steer rate, the driver forms an impression that the vehicle has ‘a mind of its own’. This leads to poor subjective ratings for handling confidence whatever the objective measures might

say. Thus the correlation between steer rate and beta-dot is a useful one in measuring improvements in vehicle behaviour using theoretical models and real vehicles alike.

Transients are also important to the objective behaviour of the whole vehicle. Under circumstances of a steering reversal, particularly at or near the grip limit, there is a substantial increase in yaw rate transiently. To understand this, imagine the vehicle travelling in a steady curve with a body slip angle of, say,  $5^\circ$ . Now imagine the steering is reversed to produce a lateral acceleration in the other direction with a corresponding slip angle of  $-5^\circ$ . If the adjustment of the body slip angle took place over a 1 second period, this would make a transient yaw rate of  $10^\circ$  per second. Considering Figure 7.6, it can be seen this is a substantial proportion of the total available yaw rate at speeds over about 50 mph. If the steady state lateral acceleration produced a yaw rate of, say,  $15^\circ/\text{s}$  (below the friction limit) then the additional  $10^\circ/\text{s}$  transient yaw rate would be more than the vehicle is capable of sustaining. For this reason alone, vehicles that converge (i.e. settle to a steady state solution) under normal conditions may well become unstable (spin) under conditions of steer reversal. Therefore any studies of transient vehicle dynamics must at some stage consider responses to steer reversals.

Vehicle mass and particularly inertia properties are a modifier to vehicle behaviour. There is a popular belief that the minimum mass moment of inertia in yaw is the best. However, this is not necessarily so. To understand this, consider a vehicle manoeuvring in the ground plane, which may be described by the classical 2 DOF formulation:

$$C_{af}\alpha_f a - C_{ar}\alpha_r b = \dot{\omega} I_{zz} \quad (7.86)$$

$$C_{af}\alpha_f + C_{ar}\alpha_r = m v \dot{\omega} + m \dot{v} \quad (7.87)$$

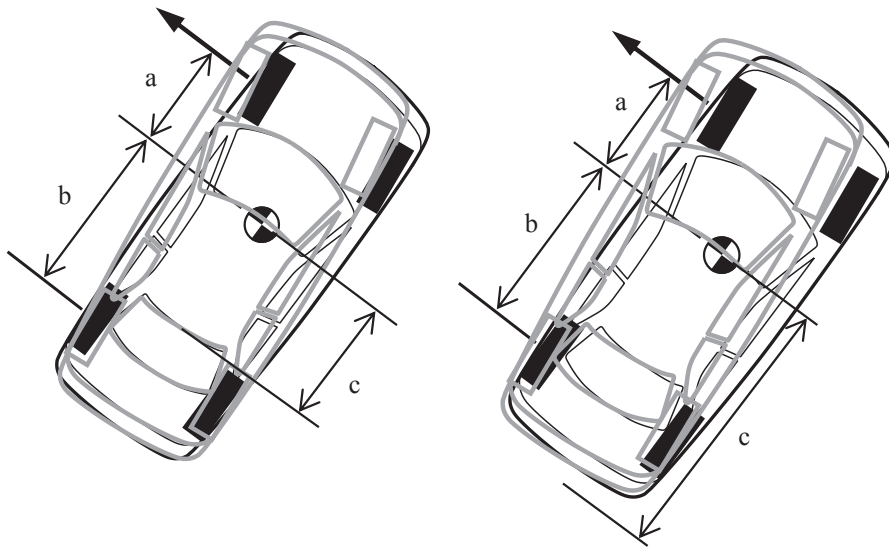
At the moment of turn-in, yaw rate,  $\omega$  and rear slip angle,  $\alpha_r$  are both zero. Assuming a constant acceleration solution for the differential equations (a reasonable assumption for the first few moments of a turn-in event or following a disturbance at the front axle), the response of the builds in the manner given in Eqns (7.90) and (7.91).

$$\omega = \frac{C_{af}\alpha_f a t}{I_{zz}} \quad (7.88)$$

$$v = \frac{C_{af}\alpha_f t}{m} \quad (7.89)$$

Combining these quantities gives an instant centre at a distance  $c$  behind the mass centre

$$c = \frac{v}{\omega} = \frac{\frac{C_{af}\alpha_f t}{m}}{\frac{C_{af}\alpha_f a t}{I_{zz}}} = \frac{I_{zz}}{m a} = \frac{m k^2}{m a} = \frac{k^2}{a} \quad (7.90)$$

**FIGURE 7.41**

Dynamic Index influences the centre of rotation of the vehicle in yaw.

Expressing the distance  $c$  as a fraction of the distance to the rear axle,  $b$ , leads to

$$\frac{c}{b} = \frac{k^2}{ab} \quad (7.91)$$

This quantity may be recognised as the ‘dynamic index’ (DI) defined by the SAE and used to good effect by Olley, albeit in pitch rather than yaw. Its importance for vehicle behaviour is that a DI less than unity results in an increase in lateral velocity at the rear axle, and hence an increase in slip angle at the rear tyres (Figure 7.41). If DI is greater than unity then lateral velocity — and hence slip angles — are reduced. In a situation where the vehicle is cornering at the critical slip angle, it can be seen that a DI less than unity will increase the rear slip angle and therefore reduce the cornering force available at the rear axle — promoting the tendency to spin. Low yaw inertia does indeed promote agility and speed of response, but excessively low inertia — as quantified by DI — makes the car difficult even for skilled drivers to manage in yaw. There is thus a trade-off between response and ease of control; for a broad spectrum of vehicles from the Mk 1 Lotus Elan to the BMW M5, this trade-off is a DI of just over 0.9. Motorsport vehicles differ significantly from road cars and the exact figures are jealously guarded. It can be seen with some reflection that a low-mass mid-engined car errs towards a DI that is ‘too low’ since the wheels have to be at a certain distance from each other to accommodate engine and passengers. The interested reader is invited to compare the mass and wheelbase of a Fiat X/1-9 and an MG-F.

In general, an important goal for vehicle dynamicists is to have the vehicle ‘look after itself’ from a body slip angle point of view. Most drivers have no conscious knowledge of body slip angle and beta-dot until they become large. The authors



estimate that a body slip rate of less than around  $3^\circ/\text{s}$  is probably the threshold between subconscious and conscious awareness of body slip rate, but this is necessarily extremely sensitive to context. This means that a body slip angle of  $5^\circ$  or more can develop more or less unnoticed by a typical driver. In general, body slip angles of greater than  $10^\circ$  are difficult for the majority of the driving population to recover control from, and so  $5^\circ$  represents something 'halfway to irrecoverable'. When the body slip rate and/or angle exceeds some threshold, drivers suddenly become aware that something is amiss and so they report that vehicles 'abruptly' skid because the event was well developed before they recognised it. For this reason, an unexpected skid during otherwise normal driving can be surprisingly traumatic and leave people with a large amount of anxiety since they felt 'ambushed' by the vehicle. Drivers become sensitised and acclimatised to variations in body slip rate and angle through familiarity and for this reason, skid pan training for normal drivers is an excellent idea.

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## 7.5 Steering feel as a subjective modifier

A further difficulty for theoretical dynamicists is the question of steering 'feel'. Subjectively, impressions of vehicle behaviour are gathered to a significant extent through the handwheel, whether consciously or subconsciously. A great deal of effort is concentrated in modern road cars on the manner in which torque is transmitted back to the driver up the steering column. Those skilled in the art have no difficulty distinguishing between steering issues and vehicle issues. However, a lack of clarity can lead to confusion if steer effects are not separated from vehicle effects. For example, two otherwise identical vehicles with different steering ratios will be judged quite differently by most drivers. Presuming the underlying behaviour of the vehicle is satisfactory, most drivers will rate a numerical reduction in steering ratio ('quicker' steering) as giving 'better' handling due to the increased YRG of the vehicle as seen by the driver from the handwheel. Yet the vehicles are identical and it would be possible to reproduce the behaviour of one vehicle by steering at a different rate in the other, to different final positions.

Steering feel is correctly given a great deal of importance in road car design since it is the primary means by which the customer comprehends the dynamics of the vehicle. Accurate modelling of steering feel is difficult and requires a great deal of data about friction in individual joints, plus also a good characterisation of the hydraulic or electrical power assistance used in the steering system. Nevertheless, work to understand the relative importance of individual contributions is possible with comparatively inaccurate models so long as good judgement is used and conclusions are correlated with work on real systems.

Changes in steering torque are a primary input for skilled drivers to detect vehicle behaviour. A common description of 'steering feel' involves discussion of tyre aligning torques. When driving normally, the tyres generate forces by distortions in the contact patch (see Chapter 5) that result in a moment attempting to return the tyre to a zero slip angle condition. If the steering system is well designed, it is delivered with very little corruption from vehicle weight and frictional effects directly to the hands of the

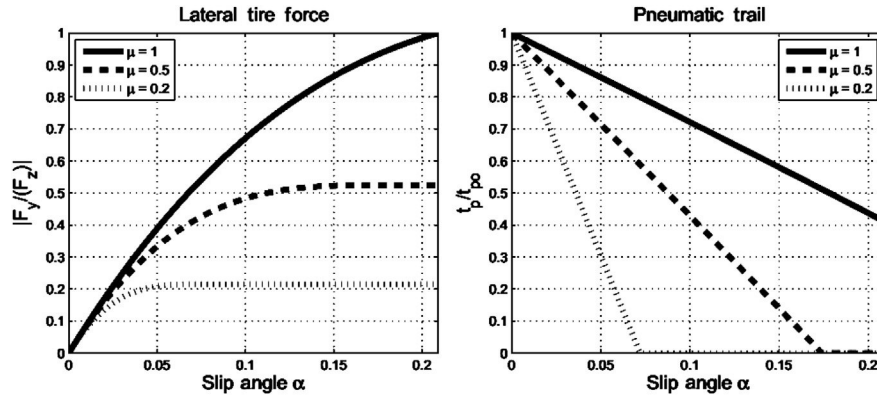


FIGURE 7.42

The use of inferred pneumatic trail for predicting surface friction characteristics even at very low slip angles.

(Hsu, 2006.)

driver. As discussed in Chapter 5, the relationship between side force and aligning moment implies a physical distance known as ‘pneumatic trail’ — the distance between the geometric centroid of the contact patch and the force centre observed in the contact patch. It is typically observed that pneumatic trail falls away in a fairly straightforward manner as the tyre delivers more and more force. It seems likely that skilled drivers develop an intuitive model in their heads that allows them to read this slope of falling pneumatic trail and thus simultaneously assess the available friction and the front tyre saturation state on a moment-by-moment basis. Hsu (2006) discusses this matter at some length and illustrates it very clearly in Figure 7.42.

The author’s (Harty) experience in World Rally Car steering development suggests another mechanism is in play. It is widely recognised in accident reconstruction communities that the skid marks left by a tyre when it is laterally saturated display distinctive patterns described as ‘striations’. The formation of such marks, which are a combination of the tyre slip angle, slip ratio, tread block design and rubber compound is described elsewhere in detail (Beauchamp et al., 2009) and will not be repeated here. The key item to observe is that the tyre is typically somewhat deformed during this process and there exists a distinctive pattern of vibration in the carcass of the tyre associated with successive deformation and release of individual tread blocks in the contact patch. In addition, the large lateral forces along with typical suspension geometries cause some energy to bleed into the wheel hop mode of vibration. Taken together, this means there is a distinctive vibration spectrum associated with the tyre operating in a saturated condition. A body of work by practitioners (Giacomin, 2013) suggests that humans are very capable of distinguishing vibration environments from one another even with drastically limited bandwidths and it seems likely that skilled drivers learn to associate the vibration environment presented by the steering system with the stress state of the front tyre.

In addition, if a vehicle starts to spin then the steering system informs the driver within around 0.1 s using the ‘castoring’ torque generated by operating the entire

vehicle at a large slip angle. This mechanism ensures minimum handwheel torque when the wheels are placed so as to recover the skid; in this way the steering system fairly directly signals the current body slip angle to the driver. Skilled drivers are extremely sensitive to these messages, which arrive ahead of the brain's processing of the results of its data from the inner ear and significantly ahead of messages decoded purely from the visual environment.

It can be seen that steering systems that mask any or all of these mechanisms might be less helpful than steering systems that do not. Therefore it is suggested that proposing metrics to capture these aspects of the steering system and examining design variables that influence them will lead to well-behaved steering systems even without extensive prototyping and subjective test activities.

One final aspect of steering feel is worth commenting on, since it is noted in the foregoing discussion that the usefulness of the feedback is predicated on a skilled operator taking time to learn the sensations available and their meaning with respect to the behaviour of the vehicle. When hydraulically assisted power steering was first introduced in the USA, it was clearly an effort-saver in premium (i.e. heavy) vehicles. In the 1970s and 1980s as it started to be applied to more performance-oriented cars in Europe and Japan, there was earnest discussion about whether or not a power-assisted steering system could ever feel 'as good' as a manual system, or whether they were somehow 'impure' and a lot of other such nonsense. It has been interesting to relive most of those discussions with the introduction of electric power-assisted steering (EPAS), which had fairly awful feedback in most of its early incarnations; it has been interesting to see that hydraulic power steering is now held up as the reference and EPAS is somehow viewed as 'not as good', although even at the time of writing the tide of opinion seems to be accepting electric power steering. The important point in noting this is that when assessing steering feedback metrics it is important to compare with some absolute standard of information available to a human about tyre and vehicle states (and their corruption by unintended factors) rather than to concentrate on replicating the characteristics of a currently prevalent system to which people are temporarily habituated.

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## 7.6 Roll as an objective and subjective modifier

So far, no mention has been made of body roll. There are two important effects of roll, they are objective and subjective. In Chapter 4, some discussion of the so-called 'roll centre' was put forward as well as the notion of suspension movement leading to adjustments of the wheel camber and toe angle as the body rolls.

Both of these effects lead to a modification of the way the tyre is presented to the road. These in turn lead to variations in the yaw moment on the car and therefore some real (objective) influence on the behaviour of the vehicle. These are mostly quite straightforward. For example, toe-out on bump on the front suspension will lead to a reduction in the slip angle of the tyre as the vehicle rolls out of the turn, reducing the yaw moment on the vehicle and hence reducing the steady state yaw rate if no adjustment is made by the driver. A slightly more subtle effect is associated with the typical inclination of the so-called roll axis. If the vehicle is imagined to rotate purely about its longitudinal axis (to roll in a pure sense) then the lower front

anti-roll geometry will lead to a greater lateral velocity of the front wheels in comparison with the rear. This modification of the lateral velocity will modify the tyre slip angles, with a correspondingly greater increase in front slip angle than rear. As a result, the tyres will produce a yaw moment out of the turn — against the yaw rate but phased with roll rate. Although the motion of the vehicle is not purely roll when entering a turn or reacting to a disturbance, this mechanism may be seen to be one that couples roll and yaw.

Modern road vehicles are comparatively taut in roll, with compliances of  $6^\circ/\text{lateral g}$  and less being commonplace on quite ordinary vehicles. This is in comparison with 20 years ago, when roll compliances of  $12^\circ/\text{lateral g}$  were quite normal. Circuit competition cars are typically at something under  $2^\circ/\text{lateral g}$ . Low levels of roll compliance mean that body roll is no longer the modifier to vehicle dynamics that it once was — its effect is now quite small in the overall scheme of things.

Subjectively, roll has an important effect. Upon entering a turn, the vehicle may be thought of as ‘relaxing out’ to a final body roll angle. This means the lateral response of the driver’s head, high in the vehicle, is reduced during the transient roll-out section of turn-in. Consideration of the expression for  $\beta\dot{\cdot}$  (Eqn 7.85), shows that subjectively this leads to a momentary overestimation of body slip rate by the driver and hence roll transients often degrade driver confidence by introducing a delay in perceived lateral acceleration. Note that for vehicles typically instrumented, that delay is not captured by an accelerometer mounted on the floor of the passenger compartment and some processing of roll rate is necessary to compute the lateral response of the vehicle at driver’s head height. This is also true of MBS models. Although the driver subconsciously compensates for motion of their head on the flexibility of their neck, they do not generally compensate for motion of the platform of the vehicle in the same manner.

The nature of the roll-out event is also important, with many in the vehicle dynamics community believing that roll acceleration profiles and roll jerk (the time derivative of roll acceleration) are important modifiers in the perception of roll as a modifier to platform dynamics. Makers of motion simulators understand and use this effect to ‘simulate’ angular events, applying an angular jerk and then providing a visual environment to suggest the roll rate is persisting. Although quite reproducible, these small events are difficult to capture with instrumentation on vehicles and so there exists a belief that these phenomena are not amenable to objective quantification. That this is currently true is a consequence of pragmatism rather than any underlying principle — it is very quick to have someone experienced and skilled in the art develop damper tuning for these subjective qualities, compared to instrumenting a vehicle and going through a research programme to define numerical goals at which to aim. In general, a fluid development of initial roll rate and a progressive deceleration to the final roll angle are recognised as necessary to reduce the perception of roll angle. Directional changes towards this end are certainly amenable to predictive analysis with MBS models, and simulation work using the final, released damper calibrations would go a long way towards improving the quality of work on the next vehicle programme. However, institutionally there is little time in modern engineering

organisations for such work since it does not immediately contribute to the task at hand. Historically it has been difficult to get good data to define vehicle dampers (primary modifiers for roll transients) although modern system identification techniques mean this is more possible than it once was.

There is some work that suggests that roll–pitch interaction is important for subjective evaluation of roll (Kawagoe et al., 1997) and this certainly seems plausible. Again, such behaviour is amenable to analysis with MBS modelling and allows directional selection of design alternatives if not final tuning on the real vehicle. Essentially, a pitch nose-upward that accompanies roll is often subjectively described as ‘the rear of the vehicle rolling more than the front’ — a statement that is quite mystifying to objective vehicle dynamicists but common currency among skilled development drivers. It is recognised as undesirable and a small but not excessive amount of nose-down pitch is preferred for road vehicles. The exact amount varies with market segment and changes over time as market tastes change. This is something of a challenge for vehicle dynamicists since often there is a desire to have the vehicle roll onto front bump stops before rear in order to guarantee limit understeer, and hence stability — unfortunately this promotes subjectively undesirable pitch nose-up with large roll transients. Development work with MBS models and real vehicles allows combinations of damper tuning and anti-roll geometry to overcome this difficulty. Once again, a fluid development of initial pitch rate and progressive deceleration to the final pitch angle are recognised as necessary to reduce the perception of pitch angle.

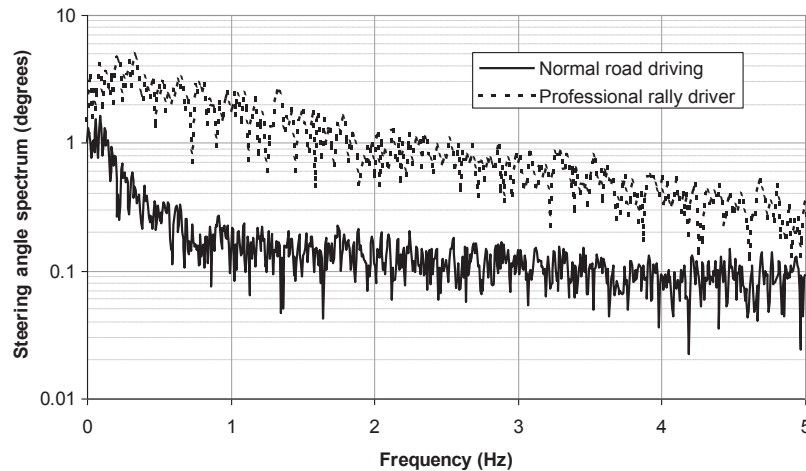
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## 7.7 Frequency response

As mentioned in [Section 7.4](#), a transient demand for yaw rate change (a nonzero steering rate) is also a transient demand for a change in body slip angle. That is to say there is a strong link between expectations of beta-dot and rate of handwheel motion. Delays between the two are also acutely remarked.

The seminal IME papers in 1957 showed the existence of a yaw/sideslip mode of vibration for the vehicle, which has been illustrated with the 2½ DOF model in [Section 7.3.4](#). The mode of vibration may be thought of as analogous to a pendulum but in the ground plane. It is this mode of vibration that rally drivers use when they ‘flick’ the car from one side to the other before a turn. Like any other vibrating system, the gains of the vehicle vary with frequency. Substantially below the resonant frequency, the behaviour of the vehicle is as already described. Around the resonant frequency, the gain is controlled by the level of damping present and above the resonant frequency the gain is controlled by the mass and inertia of the vehicle.

Also like any other vibrating system, a phase shift builds up between input and output as input frequencies approach resonance. At resonance, response is 90° behind input and beyond resonance, response is 180° behind inputs. For normal drivers this is particularly problematic. When driving normally, the significant frequency content of steer input is very low, typically below the primary ride frequencies at around 1 Hz.

**FIGURE 7.43**

A frequency-domain comparison between road and competition driving.

A professional rally driver, in contrast, makes rapid and high-frequency steering inputs. Figure 7.43 shows a pair of spectral estimates made from handwheel angle recorded over time. The spectral density measure of degrees per hertz is multiplied by the spectral resolution of the two estimates to allow meaningful overplotting of signals sampled at different rates. A single 4096 point buffer, tapered using the ‘Hanning’ (cosine) window function has been used for both signals. At the lowest frequency, the professional driver (Petter Solberg during the third special stage of the 2002 Argentina round of the World Rally Championship) is using around three times as much steering input as the road driver, the author (Harty) on a Saturday in Warwickshire. By 0.5 Hz, the normal road driver’s steer input is nearing the noise floor and is around one-eighth of its peak. In contrast, the professional driver’s input is more or less flat to 1 Hz and tails off slowly, extending all the way to 5 Hz before reaching the same levels as the normal road driver’s input at 0.5 Hz. At 0.5 Hz, the professional driver is using between eight and nine times the normal road driver’s steer input — and this on a rack ratio that is some 50% faster than the road car on which the data were gathered.

Thus the Milliken assertion (Milliken and Milliken, 1998) that for road use the vehicle can be treated as a series of connected quasi-static events is mostly true — this is the basis of the automotive statics analysis embodied in the MRA Moment Method software (Milliken and Milliken, 1998). However, during emergency manoeuvres, the amplitude and frequency content of driver’s steer inputs rise tremendously. Published data suggest that the highest steer rate sustained for 200 ms or more is likely to be around  $1000^\circ/\text{s}$  by the population as a whole (Forkenbrock and Elsasser, 1995). Data from the World Rally Championship series confirm this is the highest rate used by those drivers also; the authors’ logged data also achieve — but have never exceeded — these levels on occasion.

When such large and high-frequency inputs are used, the response of the vehicle is no longer controlled only by the steady state yaw characteristics but may also be amplified compared to the base level as well having a phase delay imposed as described previously. The dynamic amplification is a result of the yaw/sideslip resonance of the vehicle being excited to produce responses that may be substantially greater as well as delayed by some 150 ms compared to the linear result that the driver might be expecting.

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## 7.8 The problems imposed by ...

### 7.8.1 Circuit racing

Here the authors draw unashamedly on the material (Milliken and Milliken, 1998) presented in Chapter 1 of the Milliken and Milliken book 'Race Car Vehicle Dynamics'. Circuit racing is about using the acceleration vector of the vehicle to maximum effect at all times, be it in braking, accelerating, turning, or some combination thereof. Circuit racing is a highly rehearsed behaviour with 'braking points' and 'turn-in points' all being prescribed for a given type of vehicle. The driver's task is to apply control in a largely open-loop manner and to add minor trim inputs in a closed-loop manner. The 'Milliken Moment Method' uses, as one of several tools, a diagram that describes the manoeuvring envelope for the vehicle by plotting yaw moment against lateral force (or some variants thereof). Some examples are given of the diagrams for various different possible configurations. Notionally, the 'ideal' circuit car is one that is neutral steering and retains its YRG characteristic right to the limit of adhesion since this maximises the lateral acceleration possible. In truth, most circuit cars exhibit a small amount of 'push' or 'plough' right at the limit but to a first approximation, the yaw moment versus lateral force diagram looks a little like that shown in [Figure 7.44](#). The right-hand vertex aligns with the  $A_y$  axis, indicating a departure that is neither 'push' nor 'loose'.

### 7.8.2 Rallying

In contrast, rallying is largely unrehearsed, with line-of-sight driving plus prompts from co-drivers. These prompts take the form of memory-joggers: 'left 5, 150 over crest; caution, do not cut'. This is interpreted by the driver as 'left (turn, of severity) 5, (where 1 is gentle and 5 is aggressive based on likely speed before braking commences), 150 (metres) over crest; caution, do not cut (the inside of the corner because there is some hazard not obvious on entry)'. A great deal of skill and rapport is required; both driver and co-driver need to mould each other's expectations on the style of the co-driver's prompts. This information, though, is very incomplete compared to the circuit-racer's knowledge. Rally driving is all about control and adaptation. The driver's inputs have a very small open-loop content and very large and active closed-loop (trim) content. During a typical drift event lasting 2 s, there are five changes of steering angle direction (i.e. from left to right) and handwheel rates as defined in Bartlett (2000)

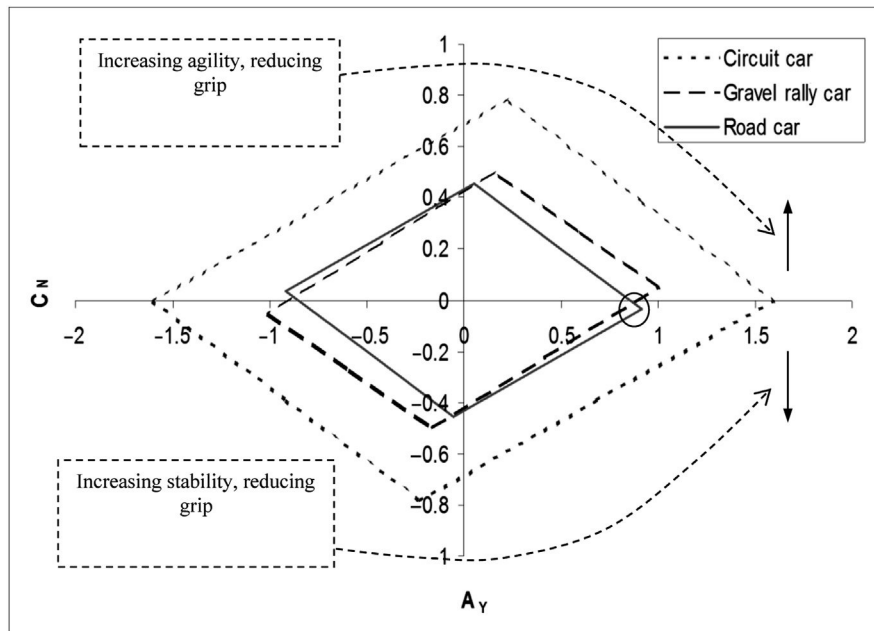


FIGURE 7.44

Force–moment (F–M) Diagrams for circuit racing, rally and road car performance – based on static weight distribution.

exceed  $1000^\circ/\text{s}$  regularly despite the use of ultralow (‘fast’) steering ratios. The yaw moment versus lateral force diagram is typically skewed much more towards control, which often produces stability concerns at the very highest speeds (Figure 7.44). The right-hand vertex is above the  $A_Y$  axis, indicating a departure that is ‘loose’ – that is to say the vehicle will spin unless actively trimmed by the driver.

### 7.8.3 Accident avoidance

Accident avoidance in road cars is quite different to both the previous scenarios. In terms of the percentage of miles driven it is statistically insignificant; if it were not for the fact that collisions have such serious consequences we would ignore it altogether. This is important because it means the driver is (in general) unpractised and unprepared. The driving task may be thought of as possessing two components – the *command* component, concerned with the speed and path of the vehicle and the *control* component, concerned with maintaining the heading of the vehicle in line with its path. That this split mimics the way the authors model the driver behaviour (Chapter 6) is no coincidence. Under normal circumstances the control part of the driving task is minimal. However, during aggressive manoeuvres it emerges to become an important component of the overall task because of the transient and



frequency-domain issues outlines earlier. Typically, the importance of command decisions is also high at these times, leading to either a loss of control or poor command decisions under circumstances where the vehicle could have been physically able to evade the emergent hazards, had the driver been ‘more able’. For normal driving tuition, only command driving is taught and control is presumed linear. Progression into the nonlinear region is regarded as a failure and so is inadequately addressed. In the UK, for example, a single ‘emergency stop’ manoeuvre is the only nonlinear event in DSA test. While notionally it is true that perfect command decisions could maintain the vehicle always in the linear region, most individuals are imperfect and will misjudge matters from time to time — misjudging either their own speed and position or that of others, requiring some emergency avoidance driving of one form or another.

There are two requirements for the vehicle under these circumstances. The first is that it must have sufficient cornering ability to be able to curve the vehicle path in such a manner as to avoid the hazards. The second is that it must be stable — that is to say the control demands on the user are minimised. A strongly desirable attribute is that the vehicle displays behaviour that does not distract the attention of the driver by thinking he ‘might’ need some control in the near future. In general this is linked with stability but not necessarily in a direct manner. Considering the Force—Moment (F—M) diagram in Figure 7.44, the right-hand vertex (circled) is below the  $A_y$  axis, indicating a departure that is ‘push’ — that is to say the vehicle loses YRG with increasing lateral acceleration, gaining stability. If stability were the only criterion, then it would be easy to simply ensure an early ‘push’ departure with the vehicle design. However, increasing expectations for cornering ability to avoid hazards means that excessive sacrifice of grip for stability (a traditional recipe for the North American market) is becoming less and less acceptable.

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## 7.9 The use of analytical models with a signal-to-noise approach

The attraction of using predictive methods is mentioned in Chapter 1 and reiterated in Section 7.1. One key benefit of analytical methods is that they are very repeatable — the same simulation run twice will generally yield the same results. There are specific exceptions to this, where ‘random’ numbers are included in analyses, but in general it holds as a premise. This removes an important obstacle in vehicle dynamics — the lack of repeatability of measured data as mentioned at the start of the chapter. This is certainly one form of ‘noise’ but not the form referred to in the title of the section. In general, drivers expect linearity from their vehicles. A departure from linearity may be regarded as ‘noise’ compared to the ‘signal’ that represents the driver’s inputs.

Most readers will be familiar with the idea of variation in output being some measure of repeatability. For example, if a machine is required to produce a piece of a certain nominal size then a better ‘quality’ machine will produce pieces closer

to that nominal size than a worse ‘quality’ machine. To quantify the level of match between machine and desired results, we use the notion, introduced in [Section 7.1](#), of ‘process capability’,  $C_p$

$$C_p = \frac{0.75 \Delta_d}{6 \sigma_d} \quad (7.92)$$

where  $\Delta_d$  is the allowable range for the attribute  $d$  and  $\sigma_d$  is the standard deviation of the attribute  $d$  as produced by the process. If  $C_p$  is greater than unity, the process is ‘capable’ and if not, it is not. This notion is used to assess production methods for intended tolerances and to focus attention on the least capable processes where resources are limited. As previously noted, the capability for the vehicle dynamics measurement process is questionable, at best – though analytical methods improve the capability of the process at the risk of introducing systematic inaccuracies through modelling errors and the like.

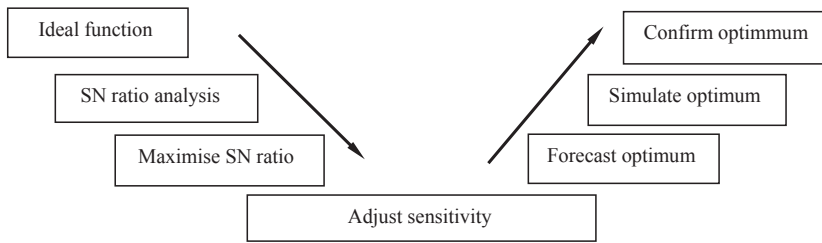
As part of a design process, the idea of a signal-to-noise (SN) ratio becomes much more useful. If we imagine that same manufacturing process, SN ratio is the ability of the process to produce any desired value of  $d$ . Thus, for some form of computer numerically-controlled (CNC) machine tool, some dimension  $d$  might be anywhere from 10 to 500 mm. A snapshot of the device’s ability to produce, say, 276 mm is less useful than an overall knowledge of the relationship between input (desired dimension) and output (dimension produced). There are three types of ‘lack of quality’ that concern us:

- *Linearity*: the proportionality of output to input. For example, if on small dimensions the stiffness of the workpiece reduced, the machine might systematically make smaller things than required but as the dimension increases this effect might become less significant.
- *Variability*: the consistency of output to input. If the machine is insufficiently stiff then random vibrations induced by the cutting action might lead to random variations in the size of the workpiece.
- *Sensitivity*: the scale of output to input. If some calibration error were present between sensors and tool positioning shafts, it could be that so-called ‘millimetres’ in one axis were different to ‘millimetres’ on another axis.

A fourth effect is the ability of our measuring technique to discern the output to the required resolution. If we ask for a 0.1 mm change in size of the product but can only measure to the nearest millimetre, we are unable to discern whether or not we have been successful. This, however, is a matter of good experimental or analytical technique rather than something innate in the process itself and so it is laid aside as a difficulty for the moment.

To capture all three types of ‘lack of quality’, a single measure is used – the SN ratio. This is simply defined as

$$SN = \eta = 10 \log_{10} \left( \frac{\kappa^2}{\sigma^2} \right) \quad (7.93)$$

**FIGURE 7.45**

The Robust Design Process. SN ratio, signal-to-noise ratio.

where  $\kappa$  is the sensitivity of the process (i.e. the amount the output changes for a unit input change) and  $\sigma$  is the standard deviation of the results from the nominal.

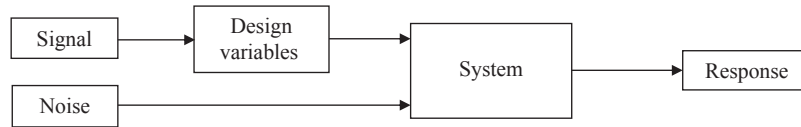
To successfully use the idea of SN ratio in vehicle dynamics design, a process is required that looks something like that shown in Figure 7.45. Notice the similarity between this and Figure 1.7 in Chapter 1. This type of process is variously and vaguely known as Robust Design, Parameter Design or the Taguchi Method.

The first and crucial step is to define the so-called ‘ideal function’. In vehicle dynamics it is very easy to become mired in talk of detailed effects of springs, dampers, bump rubbers, anti-roll bar bushings and so on and to dive into an intensive programme of investigation that quickly spirals out of control. It is only the experience of the empirical vehicle dynamicists that delivers vehicles that perform under these circumstances. Those same empirical vehicle dynamicists have become rightly suspicious of analytical methods since they can take a long time and run the risk of delivering little if not well directed. This need not be so. The authors suggest that the ideal functions should be generic enough to be applicable to any vehicle, without exception; only the tuning of sensitivities should need matching to the application (Figure 7.45).

If we consider the brake system as an example, it is simple to state an ideal function:

*to proportionally transfer the driver’s inputs from the pedal to the deceleration of the vehicle up to the available friction level.*

Brake systems have several unintended functions as implemented. Squeal, judder and so on are of prime concern to road vehicle manufacturers and the ability to use all the available friction is of prime concern in motorsport applications. The conditions under which this ‘ideal function’ holds are simple — all conditions under which the vehicle operates. This includes in reverse (when drum brake systems can be startlingly ineffective), in car parks, when cold, when hot, when wet, when new, when worn, without the engine running and so on. Individually, these circumstances lead to a great deal of elaborate testing. However, when using SN ratio methods they can be robustly combined into a relatively small number of tests that will deliver the best performance available from the existing combination of ingredients and design variables (Figure 7.46).

**FIGURE 7.46**

A Generic system from the Robust Design perspective.

‘Signal’ is that part of the input to the system, via the design variables (‘control factors’) under the engineer’s control that is intended to produce a response. ‘Noise’ is the collection of all other factors that influence the response. For the brake system, these are ambient temperature, brake temperature, vehicle loading condition, direction of travel, fluid condition and so on. Design variables available to the engineer will be such things as disc mass, friction material, calliper stiffness, seal material, master cylinder mounting stiffness, pipe stiffness and so on. Some knowledge of the most likely dominant variables is preferred since the number of variables can become unmanageable. Wherever possible, variables should be compounded together for initial studies. If identified as important, further work can be performed to isolate the effects of individual variables. For example, calliper stiffness, master cylinder mounting stiffness and pipe stiffness come together under ‘system stiffness’. As long as our tests incorporate some contribution from each of the compounded ‘sub-variables’, the conclusions will be valid. Similarly, noise inputs can be compounded together to give a ‘noise reducing performance the most’ and ‘noise increasing performance the most’. For example, the former condition might be very cold brakes, fully loaded, worn brake pads. The latter condition might be optimally warm brakes, driver only, new bedded-in brake pads.

Each variable could be varied individually while maintaining the others constant. While this is a valid method, it rapidly leads to a large number of experiments with only a small number of variables. For example, with eight variables the number of experiments required to test two levels of each variable is  $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$  experiments. The use of orthogonal arrays allows meaningful results to be produced from a reduced set of experiments. With only eight variables, results can be obtained from only 72 runs. Adhering to so-called ‘Taguchi’ principles and using dynamic SN ratios as described allows interactions between inputs to be ruled out, allowing eight variables to be handled in just 16 experimental runs. Modern MBS codes usually come with some form of experimental design built into them, although not necessarily the dynamic SN ratio calculations described.

To perform the experiment, several levels of ‘signal’ are set and the design configuration is set. Results are collected at each signal level, in each design configuration for both noise conditions. The nature of those results should be such that the comparison with the ideal function is a meaningful one. For the brake system, mean deceleration between two speeds is suggested as a suitable response variable. Any modern data logging or MBS analysis software can easily capture and process such data. Signal factors should be chosen to give a reasonable spread of results over the operating envelope. In the case of a brake system, three levels of

	A	B	AxB	C	AxC	BxC	DxE	D	AxD	BxD	CxE	CxD	BxE	AxE	E
1	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+
2	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-
3	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-
4	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+
5	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-
6	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+
7	-	+	-	+	-	+	-	-	+	-	+	-	+	-	+
8	+	+	+	+	+	+	+	-	-	-	-	-	-	-	-
9	-	-	+	-	+	+	-	+	-	-	+	-	+	+	-
10	+	-	-	-	-	+	+	+	+	-	-	-	-	+	+
11	-	+	-	-	+	-	+	-	+	-	-	-	+	-	+
12	+	+	+	-	-	-	-	+	+	+	+	-	-	-	-
13	-	-	+	+	-	-	+	+	-	-	+	+	-	-	+
14	+	-	-	+	+	-	-	+	+	-	-	+	+	-	-
15	-	+	-	+	-	+	-	+	-	+	-	+	-	+	-
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

**FIGURE 7.47**

An  $L_{16}^{(5-1)}$  two-level orthogonal array used for processing five variables in 16 runs. Columns AxB handle interactions between A and B, and so on.

deceleration might be suggested as 0.2, 0.5 and 0.8 g, with the pedal inputs selected to give results around these levels. Note that the signal factors (i.e. pedal inputs) should remain as consistent as possible. The actual array to be used is best selected from an existing library of such arrays; there is no particular need to derive one's own. Figure 7.47 shows a typical such array. Software tools external to the modelling or test environment, such as Minitab, have greatly simplified the experimental design process in recent years. Once the results have been generated — either by an experiment or by simulation — they are processed to calculate SN ratio and sensitivity for each of the conditions in the orthogonal array.

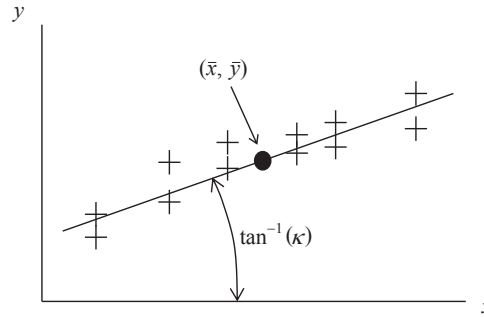
For vehicle dynamic behaviour, the most useful form for the SN ratio calculation is the so-called 'linear' form, which presumes that output is related to input in the following manner

$$y = \kappa(x - \bar{x}) + \bar{y} + e \quad (7.94)$$

where  $y$  is the response,  $x$  is the signal, and  $e$  is the error.  $\bar{y}$  and  $\bar{x}$  are the averages over the whole experimental data set (Figure 7.48).

If the system is nonlinear (as is typical), the ideal function is defined such that the linear form of this equation can still be applied. As a further example, if acceleration performance is being considered then the ideal function is 'constant power acceleration' such that

$$Pt = \frac{1}{2}mv^2 \quad (7.95)$$

**FIGURE 7.48**

Linear form of presumed response.

that is to say all of the power of the engine goes into kinetic energy in the vehicle. The ideal function may be expressed as:

$$x = V_0 + \sqrt{\frac{2P(t - t_0)}{m}}, y = V \quad (7.96)$$

If various speed intervals are used for testing then the constant power formulation allows the expected results, or signal values, to be set according to this formulation. Actual test values can be compared with them using the linear formulation to calculate SN ratios and sensitivity. Changes that improve the sensitivity will be reduced losses in the system (i.e. better efficiency) and changes that improve the SN ratio will be improved robustness. For example lower levels of driveline vibration reducing random losses, reduced aerodynamic drag maintaining constant power acceleration to a higher speed and improved area under the torque curve giving reduced sensitivity to gear ratios will all improve SN ratio.

So how then should this be applied to simulation models to improve the lateral dynamics of vehicles? The key to success lies totally in the definition of ideal functions. We have already discussed the brake system and powertrain. The handwheel remains the only driver input left. Its function is as a yaw rate demand. Previously, the behaviour of typical vehicles and drivers has been discussed. On the basis of this, a suggested ideal function for the handwheel is

*to deliver a yaw rate proportional to handwheel angle, related to speed by the stability factor for the vehicle in question, up to the limit of available friction.*

Figure 7.21 shows the linear YRG in relation to speed for some different vehicles. Thus, the ideal function for the handwheel could be expressed:

$$x = \frac{V\delta}{L(1 + KV^2)}, y = \omega \quad (7.97)$$

With  $n$  pieces of logged or predicted information, SN ratio,  $\eta$ , is calculated according to the following method

$$\eta = 10 \cdot \log_{10} \left( \frac{S_{\kappa} - V_e}{\left[ \sum_{i=1}^n (x_i - \bar{x})^2 \right] \cdot V_e} \right) \quad (7.98)$$

where the following intermediate calculations are defined:

$$\text{Error Variance} \quad V_e = \frac{S_e}{n-2} \quad (7.99)$$

$$\text{Error Variation} \quad S_e = S_T - S_{\kappa} \quad (7.100)$$

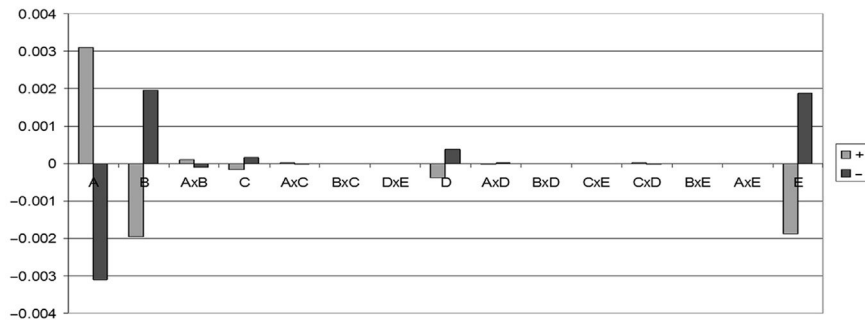
$$\text{Total Variation} \quad S_T = \sum_{i=1}^n y_i^2 - \frac{\left( \sum_{i=1}^n y_i \right)^2}{n} \quad (7.101)$$

$$\text{Variation due to Linear Effect} \quad S_{\kappa} = \frac{\left( \sum_{i=1}^n y_i (x_i - \bar{x}) \right)^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad (7.102)$$

In addition, the sensitivity of the data set (the ratio of input to output) is calculated thus

$$\kappa = \frac{\sum_{i=1}^n y_i (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad (7.103)$$

The similarity to the standard least squares method is apparent. The interested reader is urged to consult Wu & Wu (2000) for further, more detailed description and background to the method. In that text, the symbols used differ slightly. The symbols used here have been chosen to avoid a clash with the familiar vehicle dynamics symbols in use. To use the method, a typical set of predicted or logged values is taken, consisting of yaw rate,  $\omega(t)$ , handwheel angle,  $\delta(t)$  and forward speed  $V(t)$ . Values for stability factor,  $K$ , and wheelbase,  $L$ , are presumed known for the vehicle. Using the ideal function, ‘expected’ values for  $\omega$  are calculated and taken as the input data series,  $x$ . The ‘real’ – either logged or predicted – values of  $\omega$  are taken as the output  $y$  and the calculations for  $\eta$  and  $\kappa$  performed as described above.

**FIGURE 7.49**

A typical effects plot produced by processing the results from an orthogonal array.

Once SN ratios have been calculated for each 'state' in the orthogonal array, they are processed to produce an effects plot of the type shown in Figure 7.49. In this case, the array was an  $L_{16}^{(5-1)}$  two-level array for processing five design variables and their possible interactions. Variable A is shown to be dominant, with variables B and E also important. The signs of the effects are not all the same; variable A is 'less A gives more response' while B and E are 'less B/E gives less response'. There are no significant interactions between the variables. Effects plots are produced for  $\kappa$  and  $\eta$  separately. To calculate effects with the array shown in Figure 7.47, results from runs 1,3,5,7,9,11,13 and 15 are averaged to give the '-' result for variable A. The remaining columns give the '+' result for variable A. For variable B, runs 1,2,5,6,9,10,13 and 14 give the '-' result and so on. The combination of design variables is selected to maximise SN ratio to maximise robustness of the system. The following steps in the process are the adjustments to the sensitivity,  $\kappa$ , being made through those variables showing themselves as affecting  $\kappa$  but of less influence on  $\eta$ .

Such an approach to the ideal function for the steering and handling behaviour covers a great many different situations although for motorsport applications, in particular rallying, it may lack one important aspect.

Rally drivers use the nonlinearity of the car to their advantage. Recall the F-M diagram in Figure 7.44. This is derived on the basis of quasi-static calculations. However, professional rally drivers use the dynamic amplification of the yaw/sideslip resonance along with features such as ruts and road surface edges (Figure 7.50) to achieve yaw accelerations significantly greater than that suggested by the Milliken Moment Method, as shown by the comparison in Figure 7.51.

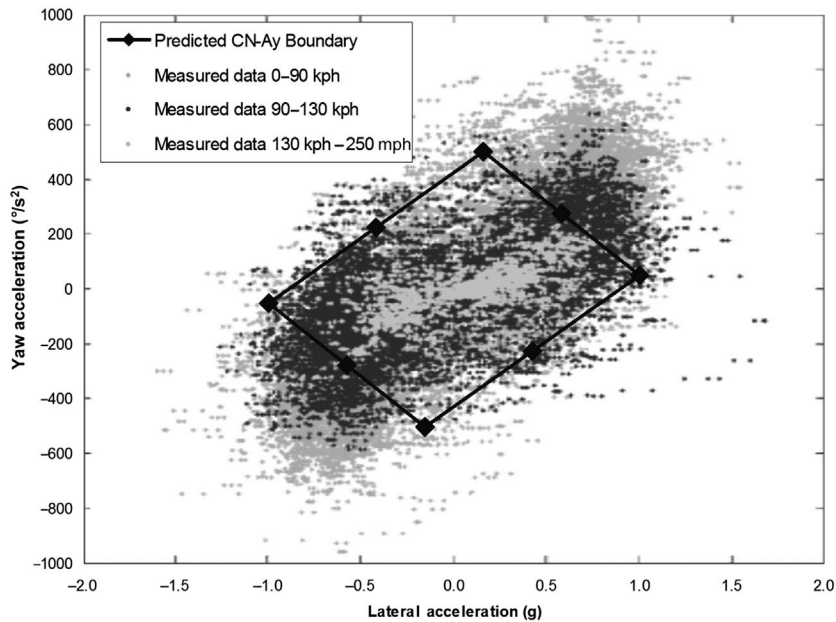
The lightest data in the centre represent the highest speeds achieved during the stage and fit well within the estimated boundaries of the F-M diagram. However, the two other shades show extremely large levels of yaw acceleration at high lateral acceleration in strong contrast to the general form of the Milliken diagram. This is a symptom of the use of the dynamic amplification in the yaw/sideslip mode as used



**FIGURE 7.50**

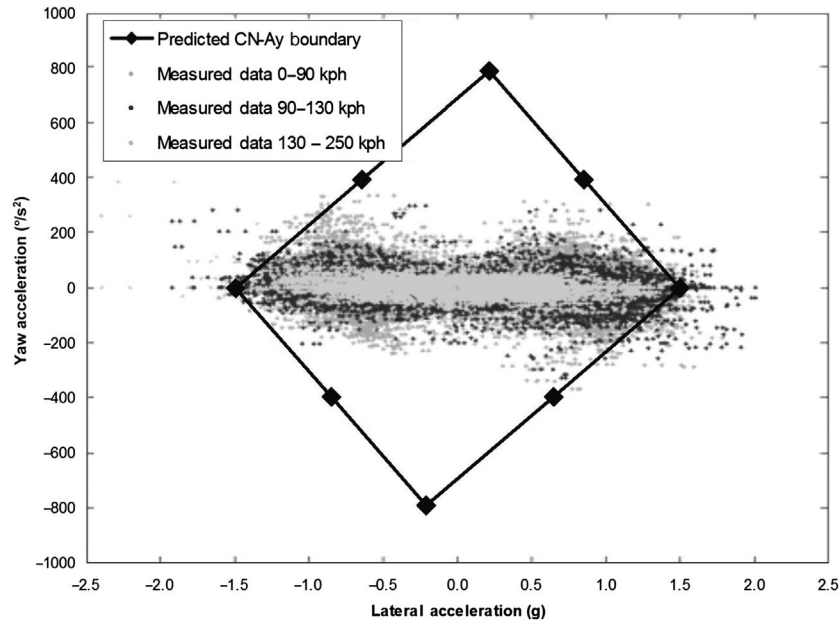
Dani Sordo hooks the front left wheel of his 2011 Mini Countryman John Cooper Works WRC off the road and ensures a large yaw moment will be available if he needs it.

(Image: McKlein.)

**FIGURE 7.51**

Yaw acceleration versus lateral acceleration for the Subaru WRC 2002, Petter Solberg, Argentina.

by the rally drivers. The use of an ideal function that relates steer angle to yaw rate will promote an increase in yaw damping, which is good for road cars but precludes the ‘flick’ style of driving preferred by rally drivers.

**FIGURE 7.52**

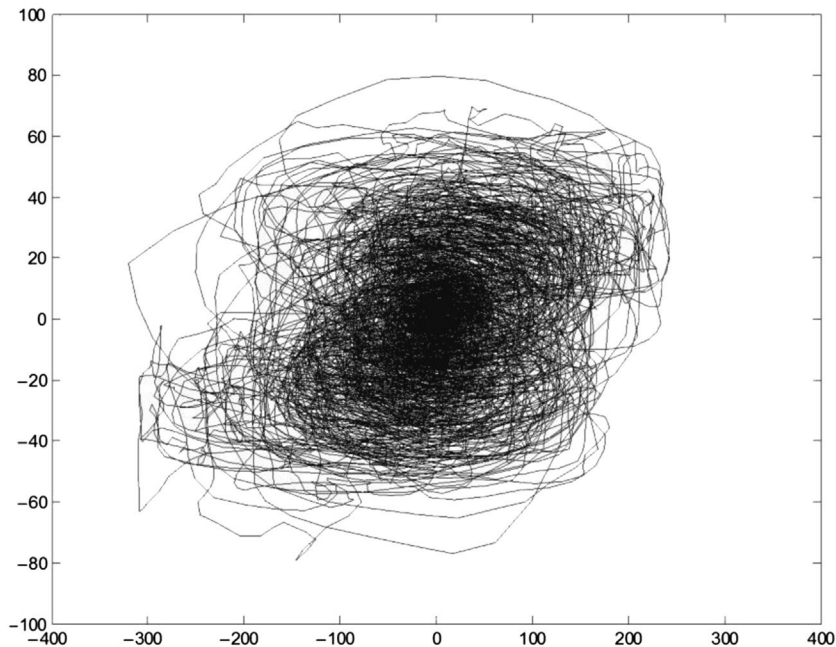
Yaw acceleration versus lateral acceleration for Subaru WRC 2002, Petter Solberg, Germany.

Figure 7.52, by contrast, shows the behaviour of the rally car on tarmac and suggests that the vehicle stays well within the quasi-static boundary predicted by the Milliken Moment Method.

Examination of the recorded data shows there is little ‘steady state’ about the rally stage, on loose surfaces and the character of the handwheel angle versus yaw rate trace for the entire stage has a distinctly ‘circular’ quality about it (Figure 7.53). This indicates that output (yaw) has a phase shift of around  $90^\circ$  – an indication that the system is indeed at resonance. In this instance it might be imagined the proposed ideal function is inadequate. However, substantial experience with vehicles for this application has suggested that pursuit of linearity under all possible circumstances delivers behaviour that is easily learnt and does not obstruct good performance. The behaviour of the Mini WRC, admired by a spectrum of drivers from ‘gentleman racers’ to world-class competitors and finishing only 8 s behind the event winner in its debut tarmac event in France, 2011, suggests that for all the ‘technical’ shortcomings of the pursuit of linearity, it does actually deliver well-behaved vehicles in a very focused manner and with limited engineering resources.

## 7.10 Some consequences of using SN ratio

Previously, vehicle dynamicists have sought to address a wide range of phenomena – unintended yaw rate change following lift-off in a turn, delays



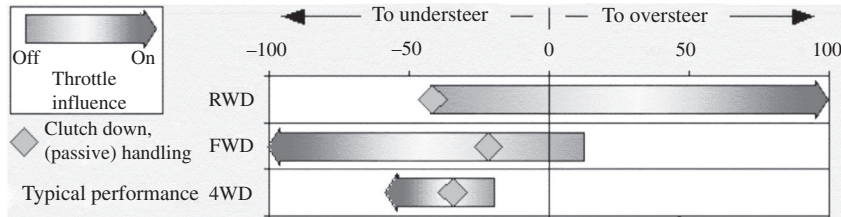
**FIGURE 7.53**

Yaw versus Steer Angle — note the circular quality of the data traces, indicating a  $90^\circ$  phase difference between the two quantities.

following turn-in, nonlinearity with increasing lateral acceleration and so on. It is eminently possible to attack each of these issues in turn, although it is also likely that solutions to some may become problematic for others.

The use of a single measure — SN ratio — and an ideal function allows experimentation to optimise the performance of the vehicle in its entirety. Questions of balancing one attribute against another no longer become subjective but can be expressed objectively, since an increase in yaw damping will increase SN ratio where it controls yaw overshoots in response to disturbances but reduce SN ratio where it blunts turn-in. This is not to say that the endeavour of optimising vehicle dynamics will become trivial; defining the correct ‘handling sign-off’ usage over which to record SN ratio remains difficult, instrumentation challenges remain, and the difficulties of accurately simulating vehicle and driver behaviour over a lengthy handling sign-off test are far from trivial.

Also nontrivial is the engineering of areas of vehicle behaviour where the departure from the ideal function might be regarded as ‘character’ rather than a non-optimum. An example of such behaviour is the oversteer departure — i.e. the loose behaviour at the limit — of rear-wheel-drive vehicles (Figure 7.54). If smart systems on board disallow such a departure, keeping the vehicle stubbornly linear

**FIGURE 7.54**

Vehicle behaviour on throttle — flaws or brand attributes? The illustration is informal, the quantities of understeer and oversteer being some undefined ‘Subjective %’.

until the end of the available friction, will it surprise the driver when the friction is exceeded? Will it bore the driver and tempt him or her to buy a different, less perfect brand next time? Or will the retention of YRG as AyG falls off give a uniformly better driving experience, less tainted by some of today’s compromises?

Certainly the use of SN ratio can potentially address many issues simultaneously:

- linearity of YRG with lateral acceleration
- delays in response degrading driver confidence
- roll/yaw interaction
- disturbance rejection
- throttle/steer interaction
  - FWD — minimising ‘push’ under power
  - RWD — reducing oversteer departure tendency under power
  - 4WD — minimising ‘push’ under power
- brake/steer interaction

Of some importance for the successful application of the method is the balancing of the proportion of different events in the overall SN ratio calculation. For instance, when comparing the influence of crosswind disturbance rejection with the influence of roll/yaw interaction, due account must be taken of how many ‘crosswind’ events there are for every ‘aggressive turn-in’ event and so on, lest one become unduly weighted in comparison with others. It is likely that the selection of the sign-off cycle will be what ultimately determines the dynamic brand attributes of the vehicle. This is not so different to the current situation where vehicles are tuned to match the preferences of key individuals at their preferred test facilities. These differences are a source of richness and diversity in vehicle design and are part of the larger commercial process of ‘differentiation’ to retain or increase market share.

# Active Systems

# 8

*We can do anything – we've been to the moon. The trick is in working out what we want.*

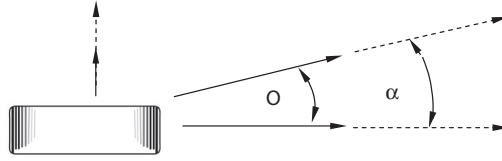
**Damian Harty, AVEC 10 Keynote Address**

## 8.1 Introduction

Modern passenger vehicles are developed to a very high level in terms of their dynamic behaviour. Indeed, there are many who believe that little more can be done to improve the performance of the road car with passive means. While it may be a little premature to make such a statement, there have been many steps forward that have been made since the start of the vehicle engineering industry:

- addition of suspension damping
- adoption of independent suspension
- adoption of hydraulic suspension damping
- adoption of hydraulic brakes
- progressive stiffening of body structure
- addition of isolating elastomers in suspension
- adoption of radial tyre construction
- decoupling of lateral and longitudinal loadpaths
- optimisation of suspension geometry
- optimisation of elastokinematic behaviour
- adoption of low profile radial tyre construction.

It is probably true that there is less to do than has been done. However, some fundamental difficulties remain with pneumatically tyred vehicles. In particular, the fall-off in yaw damping with speed is problematic. Since the fundamental nature of the tyres is to generate side forces with respect to slip angle, this leads to increased lateral velocities at increased vehicle speeds in order to respond to a given level of disturbance (Figure 8.1). This behaviour leads directly to Fonda's 'tyre-as-damper' analogy (Fonda, 1956). Since lane width in general does not increase with speed, this means the vehicle's ability to reject a given disturbance within the lane is reduced with increasing speed and hence driver workload increases. As well as the

**FIGURE 8.1**

An increase in forward speed requires an increase in lateral speed to respond to the same disturbance.

straightforward increase in lateral velocity, consideration of the stability derivatives for the classical vehicle dynamics equations given in Milliken and Milliken (1995), and derived again in Chapter 7, shows that four important derivatives have vehicle speed as a denominator:

$$Y_r = \left( \frac{a \cdot C_F - b \cdot C_r}{V_x} \right) \quad (8.1)$$

$$N_r = \left( \frac{a^2 \cdot C_F - b^2 \cdot C_r}{V_x} \right) \quad (8.2)$$

$$Y_{vY} = \frac{C_F + C_R}{V_x} \quad (8.3)$$

$$N_{vY} = \frac{a \cdot C_F - b \cdot C_r}{V_x} \quad (8.4)$$

The dimensions  $a$  and  $b$  are the longitudinal distances of the vehicle mass centre from the front and rear axles as shown previously in Figure 7.22. The first two derivatives are the side force and yaw moment with respect to yaw velocity. The remaining two are side force and yaw moment with respect to body slip angle, which may be expressed as a function of lateral and forward velocities and hence implicitly contains forward velocity in the denominator. It is thus a fundamental consequence of the tyre behaviour that the vehicle loses restoring force with speed. This is in marked contrast to aircraft (in the subsonic region, at least) where restoring forces increase with speed since aerodynamic forces rise with the square of speed.

Note the contrast between these statements and those at the start of Chapter 7, where the control gains for ground vehicles can be seen to increase significantly with speed. This presents vehicle engineers with some particular problems since control authority is increasing while system stability is reducing — a ‘closing gap’ scenario in which the safe behaviour of vehicles at Autobahn speeds remains difficult to engineer.

## 8.2 Active systems

This ‘closing gap’ means the industry as a whole is considering the use of active systems to complement the passive behaviour of the vehicle. Traditionally, ‘active’ has meant a system in which energy is added in some substantial amount. In modern usage, this distinction is becoming lost and so the term is being used for any system that performs something other than a passive mechanical reaction. Previously, systems that modified their behaviour without adding energy were referred to as ‘adaptive’. In general, systems that combine mechanical actuation with electronic control are referred to as ‘mechatronic’, although this term appears infrequently in industry literature.

To model any type of mechatronic system requires the introduction of sensors into the vehicle model and the implementation of the control law. Note that sensors in this context are different to software entities that may be also called ‘sensors’. One convenient way is to add state variables in a manner similar to that described for driver modelling. One important difference, however, is that the driver model is ‘continuous’ – that is to say the resulting outputs may be differentiated without discontinuity.

An aspect of many active systems is that they use ‘logic controller’ principles (see Chapter 6) and thus may branch between conditions in a very short space of time. If carelessly modelled, this may give convergence problems inside the vehicle model and so the use of some smoothing method is required. Note that these difficulties are present when implementing real-life systems and not confined to simulation. In both MSC ADAMS and Simpack, the STEP function described in Chapter 3 allows the transition between one level and another using a half sinusoidal form that is continuously differentiable.

### 8.2.1 Full authority active suspension and variable damping

Lotus performed a great deal of work in the 1980s looking at ‘active suspension’, a fast acting system that varied the vertical load on each corner in accordance with control laws attempting to preserve ride height, minimise wheel load variation and minimise body acceleration. This system was undoubtedly very effective, particularly on the attitude-sensitive Formula 1 car. However, expensive components and significant power consumption mean that full authority active systems, which have the authority to substantially raise and lower the vehicle if so commanded, remain uncommon. In the 10 years since the first edition of the book, to the authors’ knowledge only the Mercedes ‘Active Body Control’ system has remained available (Figure 8.2), despite the earnest attempts of marketing departments elsewhere to convince us that there is no difference between variable damping and full authority active suspension.

When compared to the original Lotus system, the Mercedes system is a down-specified version, working only with primary ride motions (up to around 5 Hz) rather than the higher frequency wheel-hop control achieved by the Lotus motorsport and SID research vehicles. In April 1990, Infiniti, the luxury division of Nissan,



**FIGURE 8.2**

Mercedes Active Body Control (left) and a passively suspended vehicle (right) in action at high lateral accelerations — around  $1^\circ$  of body roll (a combination of tyre compliance and some programmed roll compliance for ‘feel’) with the ABC car compared to  $4.5^\circ$  of body roll for the passively suspended car.

(Courtesy of Auto Motor und Sport.)

introduced a similar hydraulic 5 Hz system for the 1991 model year Q45, badged as Q45a when carrying the ‘Full Active Suspension’ (FAS). This only remained on the market until the 1996 model year, when it was withdrawn the year before the face-lifted model was introduced. Nissan cited low take-up of the FAS option as the reason for its deletion, with internet rumours suggesting only 15% of US customers subscribing to it. Around the same time, Toyota showcased similar technology as part of a well-integrated *tour de force* on the UZZ32 Soarer (Sato et al., 1992) and subsequently on the 1992 Corolla. Neither vehicle sold in large volumes, suggesting the difficulty with full authority active suspension is not the technicalities but the commercial reality of persuading customers that the value is worth the cost.

Recently, Mercedes has introduced the system many engineers dreamed of in the early 1980s by adding a previewing feed-forward aspect to their active suspension, marketing it as ‘Magic Body Control’ in an attempt to distance it from damper control systems (Figure 8.3). Previewing is a good way to circumvent bandwidth limitations. It is easy to demonstrate conceptually in multibody systems (MBS) models if a deterministic road profile (e.g. a flat road with a cosine step at a certain distance) is employed, by using a location offset in looking up the road profile function. However, in a more general model with an arbitrary road surface — a scanned physical surface, for example — then the preview is more difficult to achieve. Visual terrain mapping of the kind used by the Mercedes system is more or less impossible to implement inside general purpose multibody software. Mercedes’ own publicity material suggests the camera previews to about 15 m ahead of the vehicle but does not describe the spatial resolution. If such a preview were desired with an existing road profile then it could conceivably be implemented by conceptually extending the vehicle body component the requisite distance ahead of the vehicle and attaching a large number of small contact blocks to it. The contact properties for the blocks would be set to generate very little force so as not to influence the solution in any



**FIGURE 8.3**

Mercedes Magic Body Control (top) with road preview compared to an unspecified configuration (bottom) – the improved absolute pitch control is very apparent.

*(Courtesy of Mercedes.)*

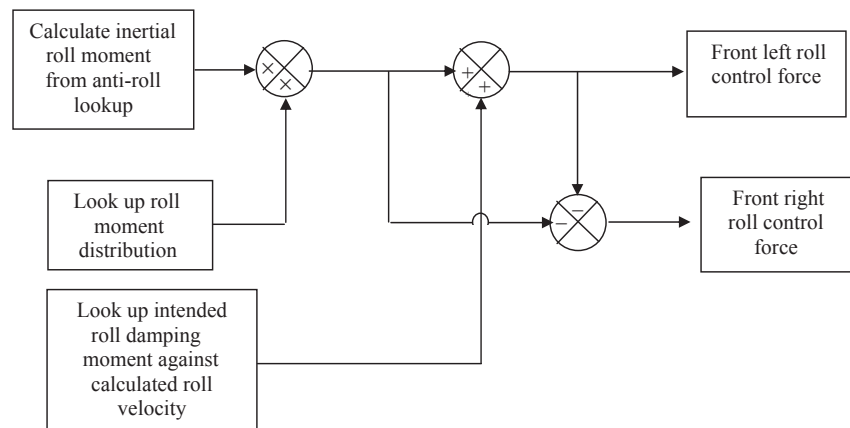
meaningful way. Interrogating the contact elements for their penetration will make a preview of the road profile available to control algorithm.

Partial active systems, working only against body roll, have included Citroën's 'Activa' system fitted to the Xantia (not to be confused with the two concept cars of the same name) from 1995 until 2000, when it was replaced by the C5 model that reverted to the adaptive roll stiffness/damping system known as 'Hydractive II'. Land Rover's 'Active Cornering Enhancement' is a hydraulically actuated fast roll levelling system, working around 5 Hz and available on Land Rover Discoveries from the 1999 Model year to date, and has made its way upmarket onto Range Rover models in a reversal of the normal trend. BMW's Active Roll Stabilisation system is conceptually similar and available on X5, X6 and X7 series platforms.

Active suspension systems and their cousins, continuously variable damper systems, are easy in concept to implement in MBS models. Action/reaction force pairs are introduced into each vehicle suspension unit in addition to the normal spring, damper and anti-roll bar forces. The magnitude of these forces is controlled by control laws in a similar fashion to that in the real vehicle. For example, a system (Figure 8.4) might use an open-loop roll moment method by applying forces proportional to the output from a lateral accelerometer. A further modification might be the addition of a roll damping term based on relative velocities of left and right suspension units, with a lookup table (spline) for use at different vehicle speeds.

Table 8.1 shows a sample of an MSC ADAMS command file implementing such a system on the front suspension of a vehicle.

Active suspension in principle allows the control of individual tyre reaction forces in a manner decoupled from platform orientation. As well as allowing a level platform during braking and cornering events (preserving suspension travel for disturbance events and improving the angle at which the tyre is presented to the road), it allows the redistribution of roll moment reaction on a moment-by-moment basis. The mechanisms discussed in Chapter 7 for inducing over- and



**FIGURE 8.4**

Simplified active suspension model.

understeer can thus be harnessed and used to improve turn-in behaviour, yaw damping and so on.

Note that in the example, no attempt has been made to filter the output down to any particular bandwidth. Filtering is a lengthy topic in its own right and is best not approached glibly. Many software environments allow the use of filtering techniques within their postprocessing tools. However, for simulating active systems, some kind of filtering is almost always necessary at run-time. Whether to simulate the limited bandwidth of an actuation device or to smooth noisy data, some form of filter must be implemented while the model is running.

Some MBS codes have access to a sophisticated run-time filtering library, such as those set up in the Matlab product environment. Others, like MSC ADAMS and Simpack, lack such run-time tools but do have the ability to enter a generic transfer function command. The filter transfer function itself must be arrived at using some external filter design tool. Particular caution is needed since some software environments use descending order for the terms in the transfer function while some use ascending order, as illustrated in Eqn (8.5), which represents a 1 Hz 2nd order Butterworth filter in the Laplace domain as used in Matlab and MSC ADAMS:

$$g(s) = \frac{39.884}{s^2 + 8.8869s + 39.884} = \frac{39.884}{39.884 + 8.8869s + s^2} \quad (8.5)$$

Matlab Description	MSC.ADAMS Description
--------------------	-----------------------

More sophisticated filtering is possible using higher order transfer functions and using the more general state-space modelling methods, which are available in most MBS environments. The interested reader is referred to 'Digital Filtering in the Time and Frequency Domain' (Blinchikoff and Zverev, 2001) for detailed discussion of

**Table 8.1** MSC ADAMS Command File Sample for Simplified Active Suspension

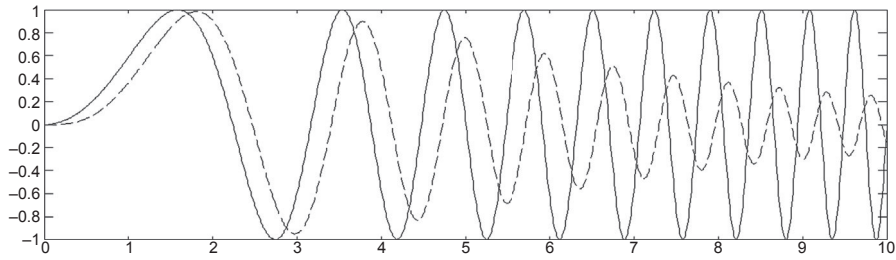
```
data_element create variable &
variable_name = inertial_sprung_roll_moment &
function = "(VARVAL(latacc)* (mass)*", &
    "(DZ(m_body_CG,base) - ", &
    "(AKISPL(varval(fl_suspension_position),0,front_anti_roll_spline) ", &
    "* (front_track)/2)* varval(fl_suspension_load)/varval(total_load)", &
    "+AKISPL(varval(fr_suspension_position),0,front_anti_roll_spline) ", &
    "* (front_track)/2)* varval(fr_suspension_load)/varval(total_load)", &
    "+AKISPL(varval(rl_suspension_position),0,rear_anti_roll_spline) ", &
    "* (rear_track)/2)* varval(rl_suspension_load)/varval(total_load)", &
    "+AKISPL(varval(rr_suspension_position),0,rear_anti_roll_spline) ", &
    "* (rear_track)/2)* varval(rr_suspension_load)/varval(total_load)", &
    ")))"

data_element create variable &
variable_name = roll_moment_load_front &
function = "varval(inertial_sprung_roll_moment)*", &
    "AKISPL(varval(latacc),0,roll_moment_distribution_spline)", &
    "/(front_track)"

data_element create variable &
variable_name = roll_damping_load &
function = "( ", &
    "(VZ(fl_damper_top,fl_damper_bottom) + ", &
    "VZ(rl_damper_top,rl_damper_bottom)) - ", &
    "(VZ(fr_damper_top,fr_damper_bottom) + ", &
    "VZ(rr_damper_top,rr_damper_bottom)) ", &
    ")* ((front_track)+(rear_track))/2* ", &
    "* AKISPL(varval(velocity),0,roll_damping_velocity_spline)"

force create direct single_component_force &
single_component_force_name = front_left_active_force &
i_marker_name = .fl_damper_top &
j_marker_name = .fl_damper_bottom &
function = "(+varval(roll_moment_load_front)+varval(roll_damping_load)", &
    "* STEP(TIME,0.0,0.0,1.0,1.0)"

force create direct single_component_force &
single_component_force_name = front_right_active_force &
i_marker_name = .fr_damper_top &
j_marker_name = .fr_damper_bottom &
function = "(- varval(roll_moment_load_front) - varval(roll_damping_load)", &
    "* STEP(TIME,0.0,0.0,1.0,1.0)"
```

**FIGURE 8.5**

2-pole Butterworth filter described in Eqn (8.5) applied to signal in Eqn (8.6) (solid) – resulting signal in dashed line.

filtering methods and their repercussions. The most important thing to recognise is that almost any form of real-time/run-time filtering introduces some form of phase delay. Using the example transfer function in Eqn (8.5), an input chirp signal in Matlab defined by Eqn (8.6) produces an output as shown in Figure 8.5.

$$x = \sin\left(2\pi \cdot \frac{\text{time}^2}{10}\right) \quad (8.6)$$

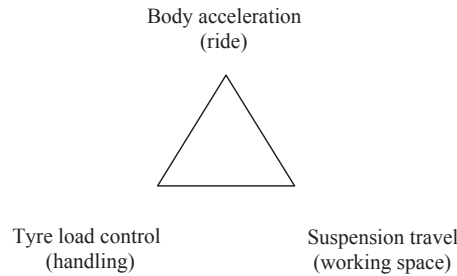
Note that the amplitude attenuation is very gentle but that a phase delay has been introduced even at frequencies well below the notional cut-off frequency. Many software packages offer ‘zero phase shift’ Butterworth filtering by running the filter forwards and backwards on logged data, but a moment’s thought should reveal this is unavailable in real-time applications in the absence of time travel technology.

The 2-pole Butterworth filter represents a linear second-order mechanical system and so its intelligent use can be very helpful in avoiding modelling complexity. The interested reader is encouraged to study its formulation and implementation in different forms – pole/zero and Laplace polynomial – as a launch-pad to a deeper understanding of real-time filtering issues and complexities.

Adaptive damping logic can be implemented in a similar fashion to active suspension inside an MBS model, scaling a damper spline or adding a scaled ‘damper variation’ spline to a ‘minimum damping’ spline according to a controller demand. The 1988 Lancia Thema 8.32 is believed by the authors to be the first production implementation of an adaptive damping system, with two-state dampers switching rapidly between ‘soft’ and ‘hard’ settings according to an algorithm that examined ride accelerations and driver inputs.

Active suspensions and continuously variable dampers seek to address the traditional three-way trade-off in suspension calibration (Figure 8.6). Given enough working space it is possible to improve both ride and handling simultaneously, as evidenced by desert race vehicles.

However, in road cars with suspension travel limited by prosaic packaging constraints, generally ride is achieved at the expense of handling or vice versa. Crolla comments (Crolla, 1995) that good control of the suspension dampers delivers

**FIGURE 8.6**

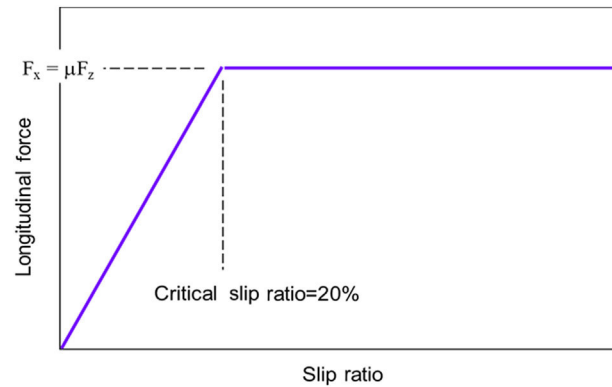
Suspension trade-offs.

many of the benefits of active control without the power consumption. Thus it seems to the authors to be the ‘thinking man’s’ solution to improved suspension calibration. It also offers the prospect of software control for calibration of suspension dampers and making inroads into development times. The recent proliferation of adaptive damping systems suggests Crolla’s observations were indeed correct.

Adaptive damping systems also offer an intriguing possibility for logistical convenience for companies making use of so-called ‘platform engineering’, when common hardware is used across a diverse range of products. The Volkswagen group of companies, — making vehicles badged Skoda, Seat, VW and Audi on the same hardware platforms — have a range of dampers that are physically interchangeable. When considered in conjunction with the engine and body style variants this might make something like 20 dampers with unique internals but an interchangeable vehicle interface — which is to say 19 opportunities to fit the wrong damper. A damper that communicates with the vehicle and is instructed by the vehicle to select a different set of parameters could in concept become a single part serving the entire platform family. This certainly brings logistical savings and, given the remorselessly downward trend in electronics costs, there must logically arrive some tipping point when the cost of the adaptive hardware is offset by the savings in logistics costs (which, in contrast, are not changing a great deal).

### 8.2.2 Brake-based systems

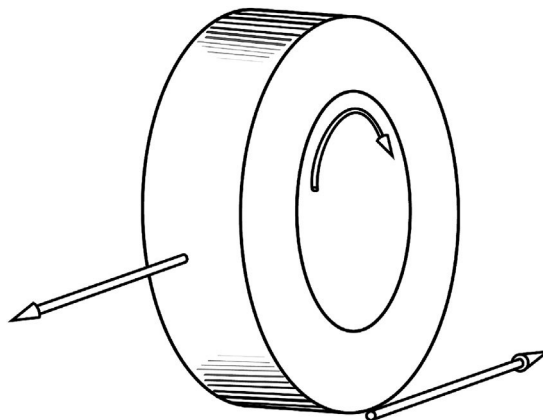
Other systems are less comprehensive but can have large benefits. In 1978, Mercedes introduced the Bosch ‘Anti-Blockier System’ as an option on the Mercedes S-Class. This system, universally abbreviated to ABS (albeit with some contortions to translate the German expression for ‘anti-lock’ into English), releases, holds and reapplies brake pressure in individual wheel brakes in order to retain directional stability and steering control with very little sacrifice in braking capability. Although technically it is possible for a skilled driver to outbrake anti-lock braking system (ABS), in truth the level of rehearsal this requires means it is unlikely in a road car in any realistic situation. Highly rehearsed circuit driving is an area where skilled drivers may improve over ABS performance and so circuit cars rarely use ABS.

**FIGURE 8.7**

A simplified slip ratio versus force curve for the following ABS discussions.

Described in some detail in Limpert's work (1999) is the function of a vehicle ABS system. The main variable is the brake pressure. In the work by Ozdalyan (1998) a slip control model was initially developed as a precursor to the implementation of an ABS model. In order to discuss ABS operation, a simplified tyre slip characteristic can be used. Such a characteristic is shown in Figure 8.7. It can be seen to be made up of two linear portions and has a constant friction coefficient,  $\mu$ .

ABS operation is essentially the interplay between two opposing torques, the frictional torque from the brake rotor and the spin-up torque resulting from friction between the tyre and the ground as illustrated in Figure 8.8. At any speed substantially above zero, the brake friction torque is broadly independent of speed but the tyre friction torque varies significantly with slip ratio (which is in essence a measure

**FIGURE 8.8**

Anti-lock braking system operation is the balancing of two torques against each other.

of the sliding velocity of the contact patch, as discussed in Chapter 5). Spin-up torque also varies substantially with surface condition, varying by an order of magnitude between dry, smooth road and slightly wet ice.

The angular acceleration of a wheel,  $\dot{\Omega}_{\text{wheel}}$ , can be simply described with Newton's Second Law.

$$\dot{\Omega}_{\text{wheel}} = \frac{T_B - T_S}{I_{\text{wheel}}} \quad (8.7)$$

Where  $T_B$  is brake torque,  $T_S$  is spin-up torque and  $I_{\text{wheel}}$  is the mass moment of inertia in the rolling direction. If this wheel is connected to a single mass representing the vehicle then the linear deceleration of the wheel is given by:

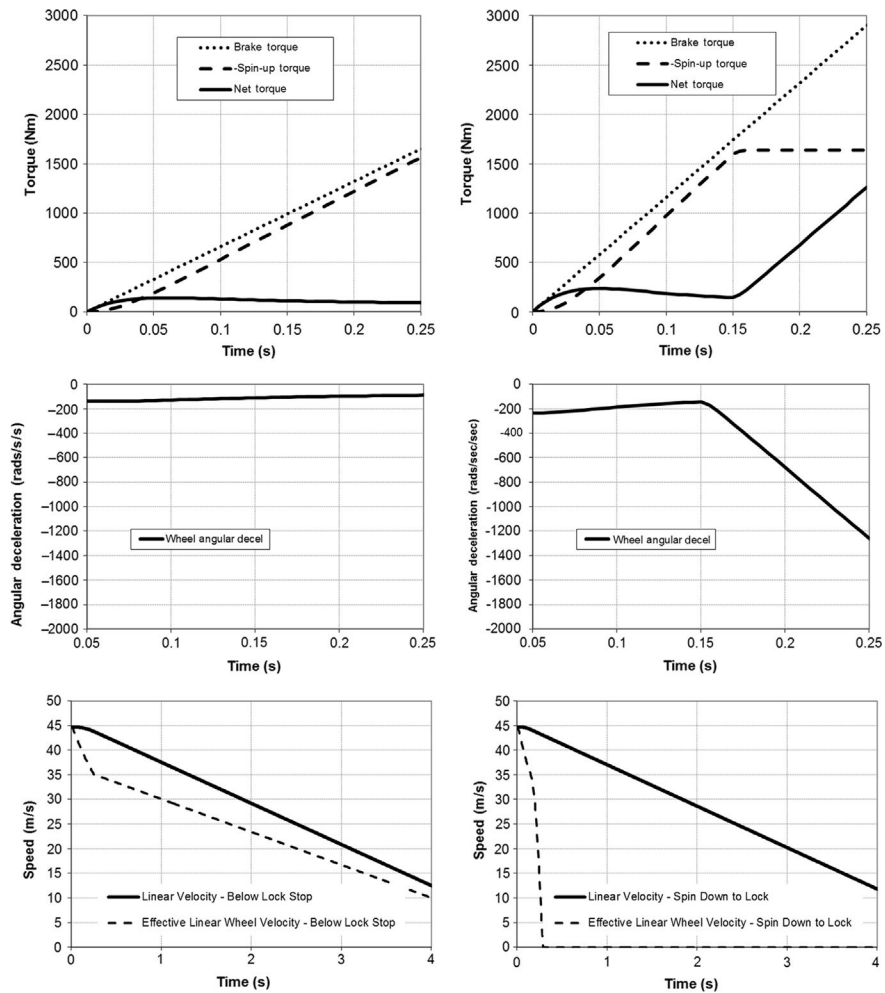
$$A_{\text{vehicle}} = \frac{T_S}{R_1 m} \quad (8.8)$$

The above equations can be readily implemented in, say, MS-Excel to investigate their behaviour. Ramping on brake torque to a certain level, the wheel can be made to hold at 19.9% slip, as shown in Figure 8.9. A period of initial deceleration of the wheel can be seen, followed by a settling of the effective linear wheel velocity (its angular velocity multiplied by its radius) to its steady state value of approximately 80% of the vehicle linear velocity during the stop. A more aggressive brake torque ramp to a higher level causes the wheel to rapidly decelerate. The rapid nature of the deceleration can be explained by considering the net torque on the wheel during the non-locked stop in comparison to the locking stop — the saturation of the spin-up torque means the full additional brake torque is applied unopposed to the wheel, and it is very large in comparison to the net torque prior to locking.

Considering Figure 8.9, it can be seen that monitoring wheel angular deceleration gives a clear signature for a wheel entering a lock event. It could be imagined that some software might watch for a value below -400 rad/s and intervene when this is detected.

For real tyres, in Figure 8.10 it can be seen that on initial application of the brakes the brake force rises approximately linearly with slip ratio depending on the wheel load. If the braking is severe the slip ratio increases past the point where the optimum brake force is generated. To prevent the slip ratio increasing further to the point where the wheel is locked, an ABS system will then cycle the brake pressure on and off maintaining peak braking performance and a rolling wheel to assist manoeuvres during the braking event.

In the Ozdalyan (1998) model, the brake pressure is found by integrating the rate of change of brake pressure, this having set values for any initial brake application or subsequent application during the ABS cycle phase. The key ingredient of such a system is the ability to control brake pressure in one of three modes, often described as 'hold, dump and pump'. Hold is fairly self-explanatory, the wheel cylinder pressure is maintained regardless of further demanded increases in pressure from the driver's pedal. 'Dump' is a controlled reduction in pressure, usually at a predetermined rate and 'pump' is a controlled increase in pressure, again usually at a

**FIGURE 8.9**

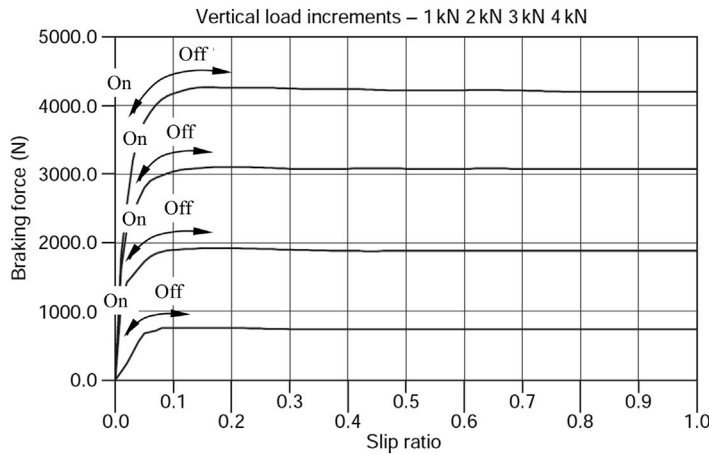
A 'below lock' stop (left) in contrast to a 'locked wheel' stop (right); note linear velocities are indistinguishable between the two (bottom graphs).

predetermined rate. Implementation of these changing dump, pump and hold states requires care to ensure no discontinuities in the brake pressure formulation.

To explore their influence in an ABS system, we can return to the simplified formulation used earlier and imagine four ascending levels of hardware complexity, as shown in Figure 8.11.

Using the hold valve alone is ineffective for controlling wheel speed, resulting in either a slight slowing of the wheel lock event or a reduction in vehicle deceleration, depending on whether it is triggered early or late (Figure 8.12).

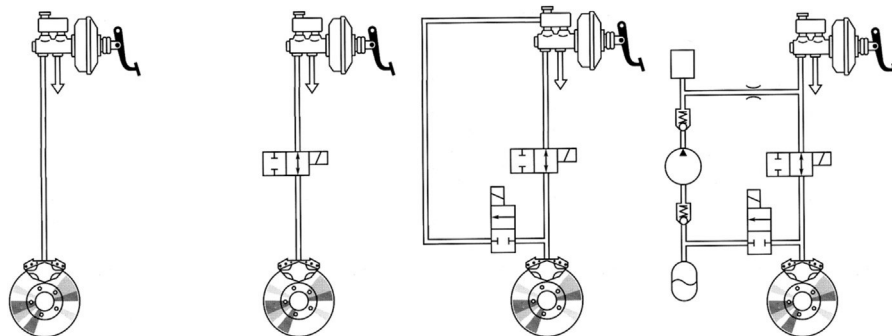


**FIGURE 8.10**

Principle of a brake slip control model.

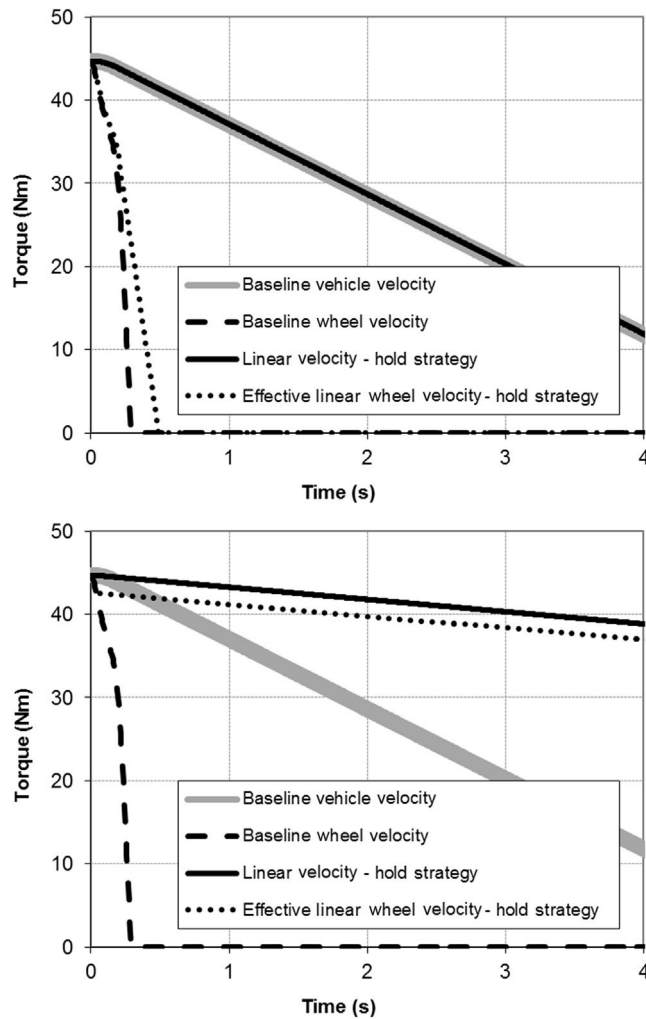
Using a dump valve to bleed off excess pressure at a predetermined rate until the wheel spins up again is substantially more successful in principle. This strategy might be called Hold-Dump-Hold and is shown in Figure 8.13. However, although successful at preventing wheel lock, it does not optimise stopping distance and it does not deal well with low-to-high friction transitions.

Closing the dump valve and reopening the hold valve reconnects the master cylinder to the system and allows the reintroduction of pressure at a controlled rate. This is illustrated in Figure 8.14. Note that for the simplified friction model and idealised system shown here the strategy manages to discover the ideal brake

**FIGURE 8.11**

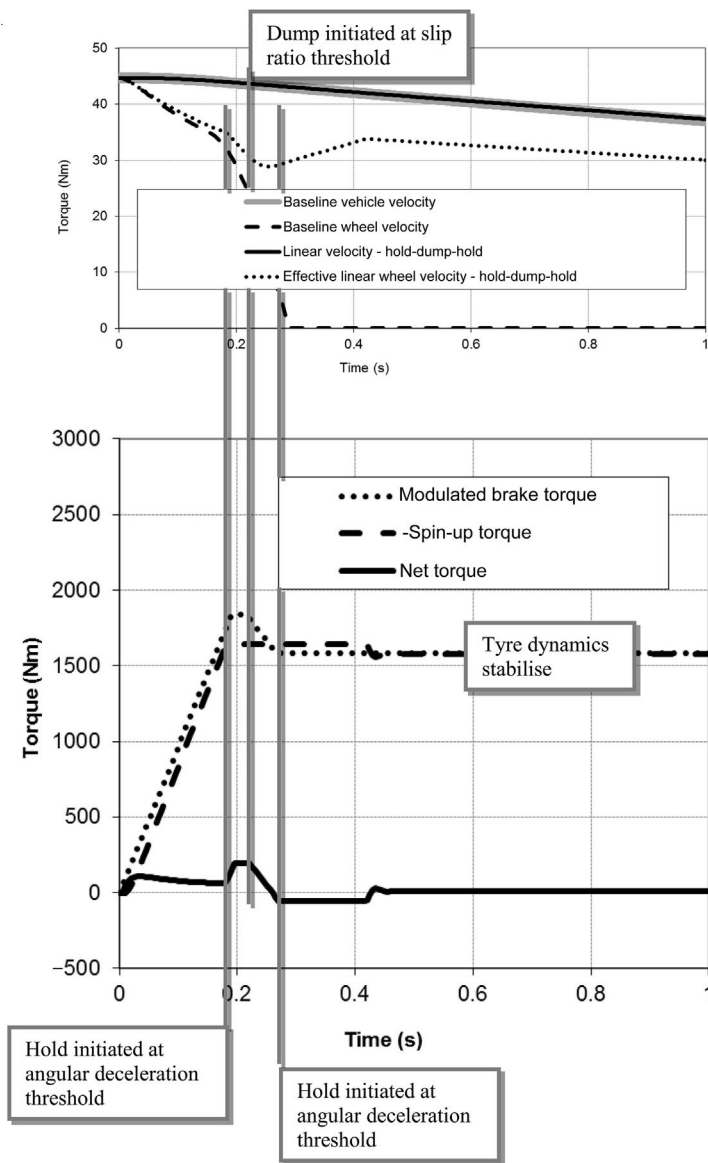
Ascending levels of brake hardware complexity — from left to right: simple push through, hold valve, dump and hold valves, dump/pump/hold valving. Illustrations based on those in Driving-safety Systems.

(Bosch, 1999.)

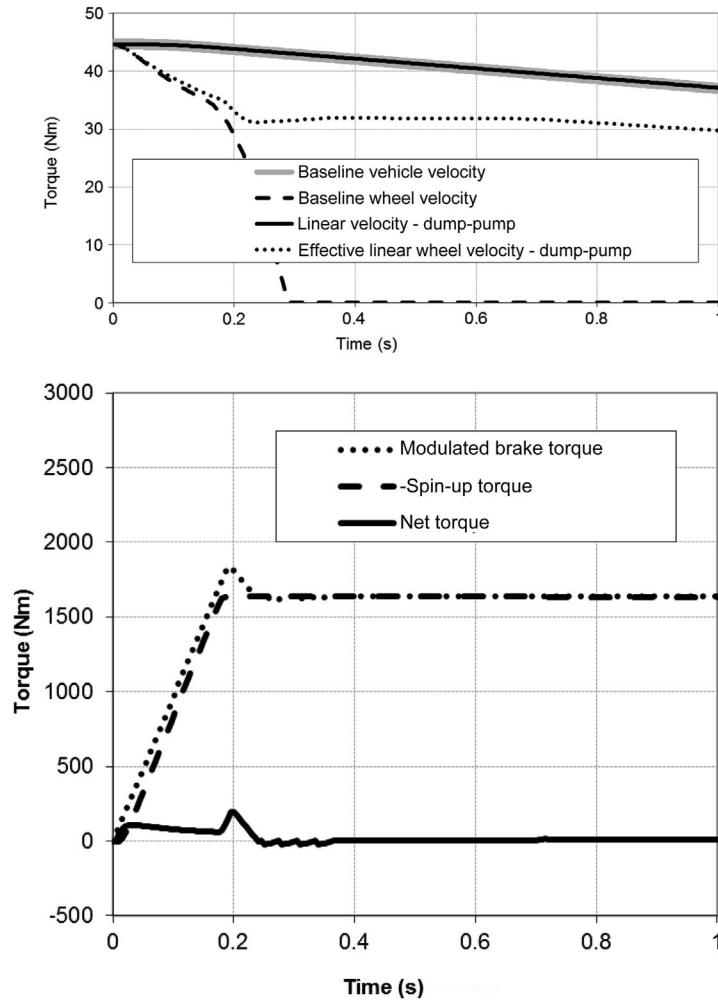
**FIGURE 8.12**

A comparison of unsuccessful hold-only strategies — too late (top) and too early (bottom); the hold threshold varies by 5% between the two.

pressure to prevent wheel lock and achieve deceleration that is indistinguishable from the baseline system; in real systems the nonlinear friction characteristics of the tyre combined with transport delays in the system mean that even on a smooth, consistent surface the system continues to cycle with the familiar ABS ‘pulsing’. It can be readily seen that repeated cycles deplete the fluid from the master cylinder and so a pumped return is often employed. Both the cycling and the pumped return can give haptic sensations that surprise operators.

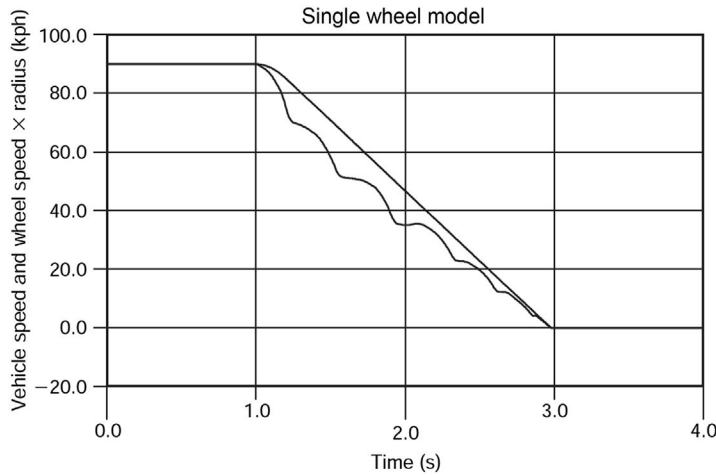
**FIGURE 8.13**

A hold-dump-hold strategy with the simplified 2 DoF vehicle braking model.

**FIGURE 8.14**

A hold-dump-hold-pump strategy with the simplified 2 DoF vehicle braking model.

The modelling in MBS of more realistic ABS algorithms (van der Jagt et al., 1989) is more challenging as the forward velocity and hence slip ratio is not directly available for implementation in the model; this of course is also the challenge to implementing the algorithms in real life. The implementation of such a model allows the angular velocity of the wheel to be factored with the rolling radius to produce an output commonly referred to as ‘wheel speed’ by practitioners in this area. A plot of wheel speed is compared with vehicle speed in Figure 8.15, where the typical oscillatory nature of the predicted wheel speed reflects the cycling of the brake pressure during the activation of the ABS model in this vehicle simulation.

**FIGURE 8.15**

Plot of vehicle speed and wheel speed during from ABS braking simulation.

In 1995 Mercedes introduced the Bosch ‘Electronic Stability Program’ (ESP) on the Mercedes S-Class. This system applies the brakes asymmetrically and without the driver’s foot applied to the brake pedal, in order to exert restoring yaw moments on the vehicle. While previous ABS strategies had an element of open-loop control to minimise the build-up of yaw moment for asymmetric braking situations, ESP proceeds to the next logical step of closed-loop control of yaw rate. Desired yaw rate is mapped through testing on the vehicle and compared with actual yaw rate. Conceptually this is similar to the driver control model described in Chapter 7. However, a fundamental difference is that the brake actuators are not always controlled as proportional devices. To suppress understeer, brake torque is typically applied at the inside rear wheel in a manner substantially proportional to the magnitude of the understeer. For oversteer, a different strategy is employed. First, a threshold is set below which no intervention occurs and second, the system introduces a defined amount of brake torque for variable lengths of time to trim oversteer. It is also typical that engine torque will be reduced under these conditions. This gives rise to the typical ‘lumpy’ sensation of the operation of ESP and the belief for many drivers that it is a relatively crude system. In truth, it is extremely elaborate, but the requirement not to burn out the brakes during spirited driving means this ‘threshold’ control is necessary.

Both ABS and ESP control can be difficult to model for reasons of the discontinuities described. In the real world, finite hydraulic bandwidth imposes its own smoothing on the system but the operation of ABS systems remains subject to a repeated series of ‘impact’ noises during sharp hydraulic transients. Only the very latest electrohydraulic braking technology succeeds in avoiding this. Cosimulation between multiple modelling codes is the answer suggested by the industry for all

these problems although this is not preferred for a variety of reasons, described in Chapter 6. Another alternative is so-called ‘hardware-in-the-loop’ simulation that uses a real physical brake system with pressure sensors to feed the behaviour of the brake system back into the model. The model computes the effects on the vehicle and feeds vehicle state information out to ‘virtual sensors’ that replace the real sensors on the vehicle, making its decisions and affecting the real brake system to complete the loop. Although potentially accurate, this type of modelling is more suited to confirmation work as shown on the right-hand side of Figure 7.45.

The biggest difficulty in real-world systems is the accurate discernment of the vehicle speed under conditions when the wheel speeds are only loosely connected to it and each other. This is usually referred to as the ‘reference speed’ problem. Another significant difficulty is the need to control the vehicle by varying longitudinal slip ratio when the actuator is a torque-control device. On ice, for example, 10 Nm may be enough to lock the wheel whereas on dry pavement, 10 Nm will produce barely any change in slip ratio. For both these reasons, ABS and ESP strategies are more complex than might first be imagined; differences in effectiveness and implementations largely come down to the sophistication of the reference speed calculation and friction estimation.

### 8.2.3 Active steering systems

In 1986, Nissan launched their High Capacity Active Steering (HICAS) system on the rear axle of the R31 Skyline Coupé. Honda followed with a mechanical system on the Prelude and Mitsubishi were close behind in the Lancer, using an electronically controlled system like the Nissan. Nissan used the notion of a compliant sub-frame with its location controlled by a hydraulic actuator, while Honda and Mitsubishi essentially duplicated the front steering system at the rear of the vehicle, using a central rack. The objectives of the systems were clear and there is a great deal of literature published on the subject (Ro and Kim, 1996). Four-wheel steer seeks to control the body slip angle and rate, and hence has a profound effect on driver impressions of the vehicle. It also has the comparatively trivial effect of tightening the turning circle at low speeds. Honda has subsequently dropped four-wheel steer, declaring that ‘advances in tyre technology have rendered it unnecessary’. Mitsubishi kept it until 1999 on their 3000 GTO model, while Nissan continue with a development now called Super-HICAS system on the Skyline.

A difficulty with four-wheel steer is that, while it makes the vehicle feel excellent in the linear region (through enhanced body slip angle control), as the handling limit approaches the flat nature of the tyre side-force-versus-slip-angle curve means that its ability to improve vehicle control disappears. In this sense, it is possibly the worst type of system — enhancing driver confidence without actually improving limit capability. Four-wheel steer is easily simulated in MBS modelling, with an additional part to represent the rear steering rack and forces applied to it according to a control law as with other systems.

BMW announced 'Active Front Steer' in conjunction with ZF for the 2004 model-year 5-Series. Consideration of the behaviour of competition drivers, particularly rally drivers, suggests the potential for this system is high and that it offers a much more continuous control than brake-based systems. Although onerous, the potential failure modes have clearly been overcome. It also offers the chance to overcome the problem of increased control sensitivity at high speeds by reducing the yaw rate gain progressively with speed. It does not suffer any of the problems of rear-wheel steering in terms of limit control.

There are several patents for an 'Active Toe Control' system for application to all four wheels. This is primarily an on-centre modifier for the vehicle and is intended to complement the torque distribution systems they are also known for. Such a system is postulated in Lee et al (1999) and has been prototyped on a research vehicle. The interaction between such a system and a torque distribution system is explored in (He et al., 2003).

Modelling active steer systems is in principle identical to modelling a driver. An external force is applied to the steering system — either the rack or column in response to an algorithm. The algorithm is typically similar to that in use for a brake-based stability system in that it will calculate a target yaw rate and then use the steering system to attempt to achieve that yaw rate.

#### 8.2.4 Active camber systems

Milliken Research Associates produced an active camber Corvette in the mid-1980s with closed-loop control of yaw rate following the success of the 'camber racer', the latter recently run again at Goodwood on modern motorcycle tyres. Mercedes have also produced the F400 Carving that runs an active camber system. Although attractive in function, the increased package requirements in the wheelhouse make active camber something that will probably not be applied to road cars.

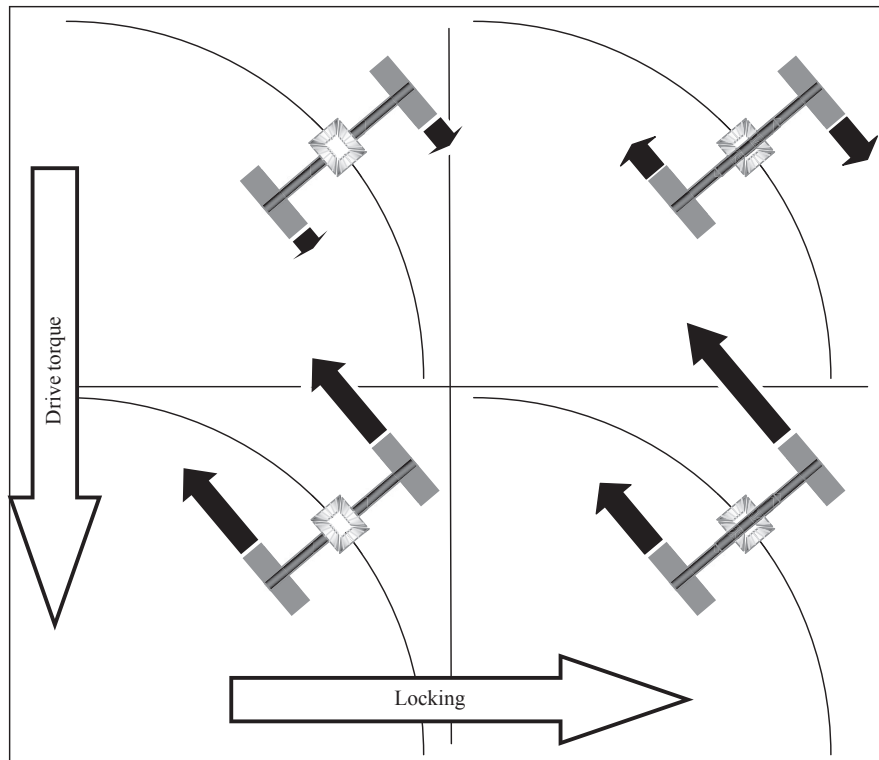
#### 8.2.5 Active torque distribution

Nissan's Skyline R32 Coupé had the somewhat bewilderingly named ATTESA-ETS (Advanced Total Traction Engineering System for All-Electronic Torque Split) system in 1989. The Porsche 959 had similar technology in the same year, although without the lengthy acronym. Since then, systems of torque redistribution according to handling (as against traction) priorities have remained elite in production cars. The author's (Harty) work at Prodrive revived interest with several low-cost Active Torque Dynamics (ATD) systems suitable for a wide variety of vehicles in the period 1999–2004, but commercial restrictions prevent discussion of which, if any, achieved production status. In essence, traditional driveline technology systems seek to minimise wheelspin, whereas these systems seek to connect drive torque distribution to vehicle handling via closed-loop feedback. From a modelling point of view, they are simple to implement in that some state variables are declared

(typically a target yaw rate and current state, similar to ESP systems) and the actuation forces are implemented accordingly.

There exists a large amount of confusion surrounding torque distribution, which frequently exercises students. It is commonly declared, with some confidence, that ‘a locked differential produces a 50/50 torque split’. The origins of this statement are unknown and certainly do not bear scrutiny. Where a traditional open differential of the bevel or epicyclic type does indeed produce a torque split of known proportions, a locked differential – in essence a shaft – produces an equality of speed outputs. How these speeds turn into torques is governed entirely by the wheel to which they are attached – it is a function of vertical load, friction coefficient and slip state. The latter is influenced by vehicle states and, of course, the differential locking condition. These comments apply whichever differential is under consideration.

Figure 8.16 shows a comparison at the rear axle of a vehicle in four different states, combinations of an open and locked differential, applied drive torque and no applied drive torque. The tractive force arrows shown are dimensionally representative of torques in the half shafts, since no other mechanism is applying a force to the tyre.



**FIGURE 8.16**

Locked differentials do not result in equal torque distribution – this is a common myth.



In the upper left quadrant of Figure 8.16, in the presence of an open differential and the absence of driveline torque, the only forces present at the contact patch are the rolling resistance. The outer wheel may be expected to have a slightly higher rolling resistance due to its higher load, associated with weight transfer as discussed in Chapter 7. The individual wheel rotational speeds are related to their individual radius of turn and rolling radius.

In the lower left quadrant of Figure 8.16, in the presence of an open differential and with driveline torque, the forces at the contact patch may be expected to be the sum of the rolling resistance and drive torque. When the drive torque is large and neglecting any frictional torque in the differential, the forces are broadly equal; this is the fundamental purpose of the differential. The individual wheel speeds are the sum of the wheel speeds from the upper left quadrant and the wheel speeds required to deliver the slip ratio, which in turn delivers the required reaction torque to the sideshaft such that it is in equilibrium. Note that a lightly laden inner wheel and a high drive torque may exceed the tyre's ability to generate such a reaction torque, in which case no equilibrium state will be reached and the sideshaft (and hence wheel) rotational speed will continue to increase until torque is interrupted — typically by an attentive driver, an engine speed limiter or a traction control system.

In the upper right quadrant of Figure 8.16, with a locked differential imposing equal rotational speeds across the axle and in the absence of driveline torque, it can be seen that something approximating the average of the speeds from the upper left quadrant is imposed on both wheels. Since this is slower than the free-rolling speed for the outer wheel, it constitutes a negative slip ratio and produces a rearward-acting force at the contact patch. Since it is faster than the free-rolling speed for the inner wheel, it constitutes a positive slip ratio and produces a forward-acting force at the contact patch. Together these produce a yaw moment on the car in opposition to the turn; it is common experience that 'stiff differentials blunt turn-in'. This effect can be used in active systems to introduce an amount of yaw damping to a vehicle. It works with both cross-axle differentials and with the centre differential in an all-wheel-drive vehicle.

In the lower right quadrant of Figure 8.16, with a locked differential imposing equal rotational speeds across the axle and in the presence of driveline torque, positive slip ratios are induced on both wheels. The slip ratios are such that, while satisfying the constraint of equal speeds in both shafts, the total reaction torques sum to the applied shaft torques. The weight difference between the two wheels results in more torque for a given slip ratio, as described in Chapter 5. The geometric effects of the turn tend to slightly increase the slip ratio on the inner wheel but for realistic turn radii these effects are relatively small compared to weight transfer effects. The net result tends to add yaw moment with driveline torque for a cross-axle differential. This effect is readily observable in both front- and rear-wheel drive vehicles when a comparison can be made with and without cross-axle locking.

*A locked differential does not result in a 50/50 torque split under most imaginable circumstances. The persistence of the myth that it does is somewhat baffling.*

A fully locked drive axle has an unfortunate tendency to spin up both wheels simultaneously, which robs the vehicle of side force on the drive axle by broadly collapsing the cornering stiffness of both rear tyres simultaneously. From the discussions of anti-lock brake control, it can be seen that the runaway phenomenon in the presence of excess drive torque can be very rapid indeed. For rear-wheel-drive vehicles in particular, locked differentials spin the vehicle while open differentials spin a wheel. It can be imagined that spinning the vehicle is a rather more onerous failure mode and it is true that limited slip differentials (which may be regarded as some way along the spectrum to locked) have a reputation for being difficult to drive at the limit. A control system for a proportional limited slip differential that can recognise the vehicle states and reduce the differential locking in response to excess yaw rate is extremely effective, in the author's experience, at limiting vehicle departure behaviour both on split friction and uniform friction surfaces.

Driveline devices that impose a left-right speed difference have also been constructed. The Mitsubishi Active Yaw Control, Super Active Yaw Control and Super All Wheel Control (AYC, S-AYC and S-AWC respectively) use a clutch and additional gear wheels to deliver a limited slip differential in which the 'fully locked' state results in a fixed speed ratio between one sideshaft and the other — typically differing by the order of 10%. The requirement for symmetric control results in a second clutch and a second pair of gear wheels. Honda's Super Handling All Wheel Drive (SH-AWD) and BMW's Dynamic Cornering Performance systems use similar arrangements in principle.

Such a device might be known generically as an 'asymmetric overspeeding differential' and can be represented in a multibody model using the topology shown in Figure 8.17. A three-element coupling equation represents the open differential, an

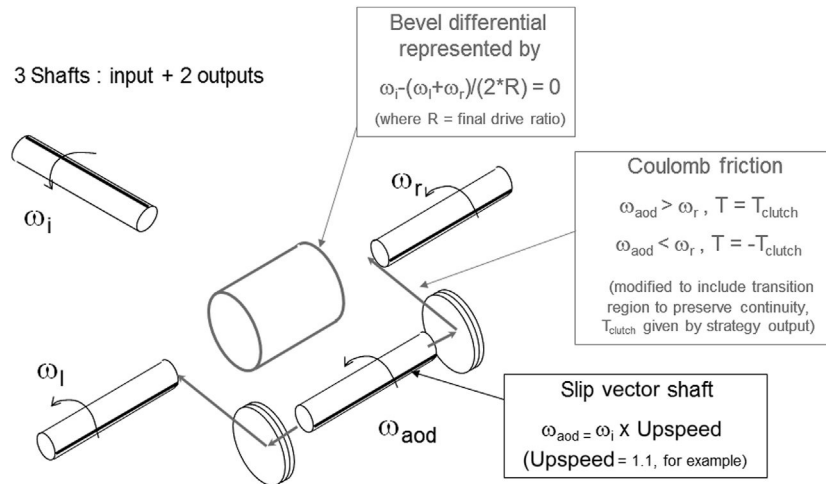


FIGURE 8.17

A modelling scheme for an asymmetric overspeeding differential — right shaft speedup.

additional spin degree of freedom represents the AOD coupling shaft. An additional mass is created, constrained to spin at some multiple of the input speed. In the example shown, a frictional torque using the Coulomb model is then engaged which is scaled by a solution variable  $T_{\text{clutch}}$ , which in turn is given as the output from some control strategy. When  $T_{\text{clutch}}$  is high, the upsped AOD shaft is effectively locked to the right-hand shaft. The conventional bevel differential enforces the speed relationship in the model, ensuring that the left-hand shaft is downsped by the same amount to preserve the average speed relationship. Additional details can be added for frictional bias torques and so on, to suit the level of available data. Despite being widely described as Torque Vectoring, this approach is really ‘slip vectoring’; the torque distribution is not really enforced. Nevertheless on homogenous surfaces the effect is to deliver a yaw moment that may be used advantageously to manage the vehicle states.

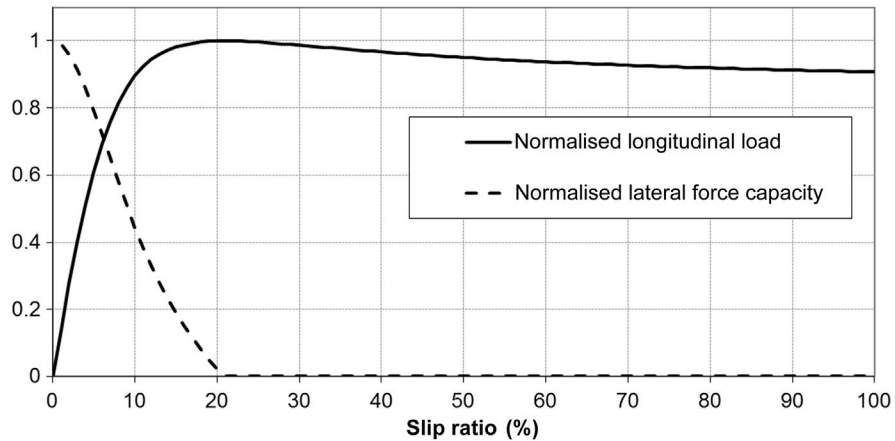
For the rear axle, any use of an asymmetric device tends to reduce the capacity of the axle as a whole and thus it is easier to add yaw moment than to subtract it. The opposite is true for the front wheels. To the author’s knowledge there has only been one front axle asymmetric overspeeding device, the Honda Active Torque Transfer System (ATTS) fitted to the Prelude SH in the late 1990s. ‘SH’ allegedly stood for ‘Super Handling’.

A similar principle can be applied between the axles. The Honda SH-AWD system runs an epicyclic overspeed — extremely similar to the electric overdrive unit on 1970s Triumphs in principle — on the propeller shaft to the rear axle. If, say, a 10% overspeed is invoked then the centre differential, when open, simply processes to accommodate the now 10% difference in overall axle ratios. If the centre differential were to be fully locked, the rear axle would have a 10% overspeed compared to the front. This overspeed has an interesting effect on the lateral behaviour of the car; in order to understand it we must return to the tyre characteristics discussed in Chapter 5. To illustrate the effects, a sample tyre longitudinal force curve is presented in [Figure 8.18](#). The curve is normalised to simplify the illustration. A notional ‘lateral capacity fraction’ can be calculated by presuming a uniform maximum frictional force is available and subtracting the longitudinal force such that the Pythagorean sum of lateral and longitudinal is unity. This sounds complex in words but is simple mathematically:

$$\frac{\hat{F}_y}{\mu F_z} = \sqrt{1 - \frac{F_x}{\mu F_z}} \quad (8.9)$$

For a slip ratio beyond that at which maximum longitudinal force is generated, the lateral capacity is presumed zero. Normalised longitudinal force and this notional ‘lateral capacity fraction’ are plotted together in [Figure 8.18](#) against longitudinal slip ratio.

Examining this graph closely, it can be seen for two similarly laden tyres that one with 10% slip ratio has approximately half the lateral capacity of one with 0% slip ratio — a slip ratio difference of 10% between them. If these are the outboard tyres on a vehicle it can be seen that the ability to exert a yaw moment on the vehicle is

**FIGURE 8.18**

Normalised Longitudinal Force and Lateral Force Capacity against Slip Ratio.

substantial — this is the effect of slip vectoring and makes a rear-wheel-drive car yaw tail out under power without gross wheelspin. What is interesting is that if in a four-wheel drive driveline, the same slip ratio difference is kept but the slip ratios are now 10% and 20%, then the relative difference in lateral force capacity is of the same order; this means the yaw moment on the vehicle is broadly independent of the mean slip level between 5% and 15% — a substantial torque window. It means that absolute speed control is not vital for interaction with yaw moment, only the inter-axle speed difference. If the mean speed comes up further, to say 15% on one wheel and 25% on the other, lateral force capacity broadly converges and so the consequence of a gross excess driveline torque does not destabilise the vehicle but rather uniformly removes yaw moment as both tyres overspeed.

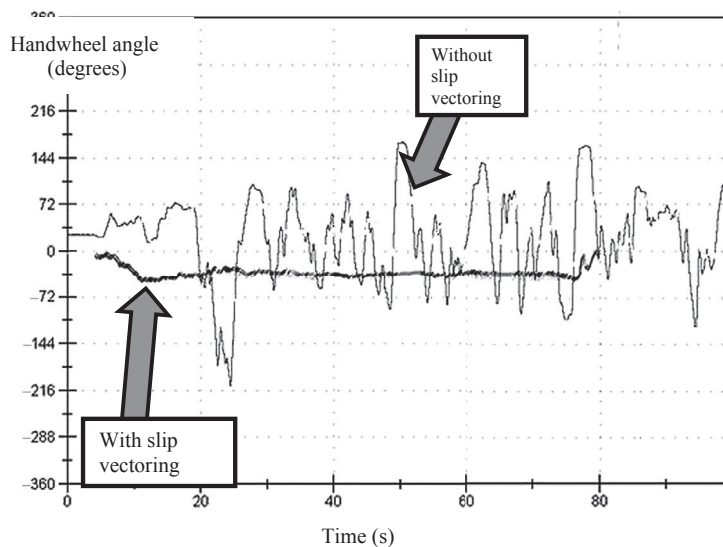
It is generally true that manipulating side forces via lateral force capacity gives a larger yaw moment than by exerting it directly left–right on the vehicle since generally wheelbase exceeds track. For this reason it is rarely necessary to run such a large speed difference as a left–right device; the first Prodrive prototype slip-vectoring vehicle ran an 11% speed difference but the next and subsequent vehicles dropped to a more typical 5% or so — the values used by other manufacturers.

It can be seen in the Honda-style configuration that the overspeed at the rear can have the effect of reducing lateral force capacity at the rear axle — the Prodrive ATD systems also used this effect — and thus can add yaw moment. When coupled with an epicyclic centre differential having a notional front-biased torque split, a simple proportional clutch across it delivers a front torque bias when open and a rearward torque bias when closed. Using this, a simple linear yaw rate demand model can generate a target yaw rate curve, with the control law for the centre differential responding to insufficient yaw rate by tightening and an excess yaw rate by slackening from its current position. In this manner, the driveline can be set up never to destabilise the car — in other words to decouple the steering from the throttle.

Overall driving impressions of such systems are, on the whole, ‘remarkably unremarkable’ with the drive experience uncorrupted by drive torque.

One demonstration of the Prodrive system involved wedging the steering wheel with the driver’s leg and driving a continuous drift on a 60-m snow circle, managing the line of the car using throttle inputs but leaving the car to look after its yaw rate on the rough, rutted surface of the snow circle by controlling the centre differential. The task of managing the car was so undemanding that the author could fold his arms and eat an apple while doing it. A comparison data set with four-wheel drives but no closed-loop torque vectoring is shown in [Figure 8.19](#); the duration is some 70 s, showing this was no cherry-picked small data segment. At a speed of around 60 mph, this represented of the order of a mile of continuous control. The uncontrolled car had a sophisticated all-wheel-drive system (including hydraulic centre differential and asymmetric rear axle) and a substantially faster steering ratio than the controlled car, but crucially no loop closure around vehicle yaw rate. It is noted with interest that since 2007 the Mitsubishi S-AWC system has had yaw rate feedback added compared to the AYC and S-AYC systems, in which it was absent.

Nothing about the learning with geared slip-vectoring systems precludes the same ideas being applied with individually controlled wheel motors. In particular, in-wheel motors such as those produced by Protean Electric Ltd at the time of writing offer high fidelity control and very accurate speed sensing, allowing, for example, a left and right motor to be ‘software coupled’ for relative speed and managed for mean torque to mimic an asymmetric overspeeding functionality that



**FIGURE 8.19**

A comparison between the presence and absence of closed-loop slip vectoring driving on a 200-m snow circle in Arvidsjaur, Sweden – author (Harty) driving both vehicles.

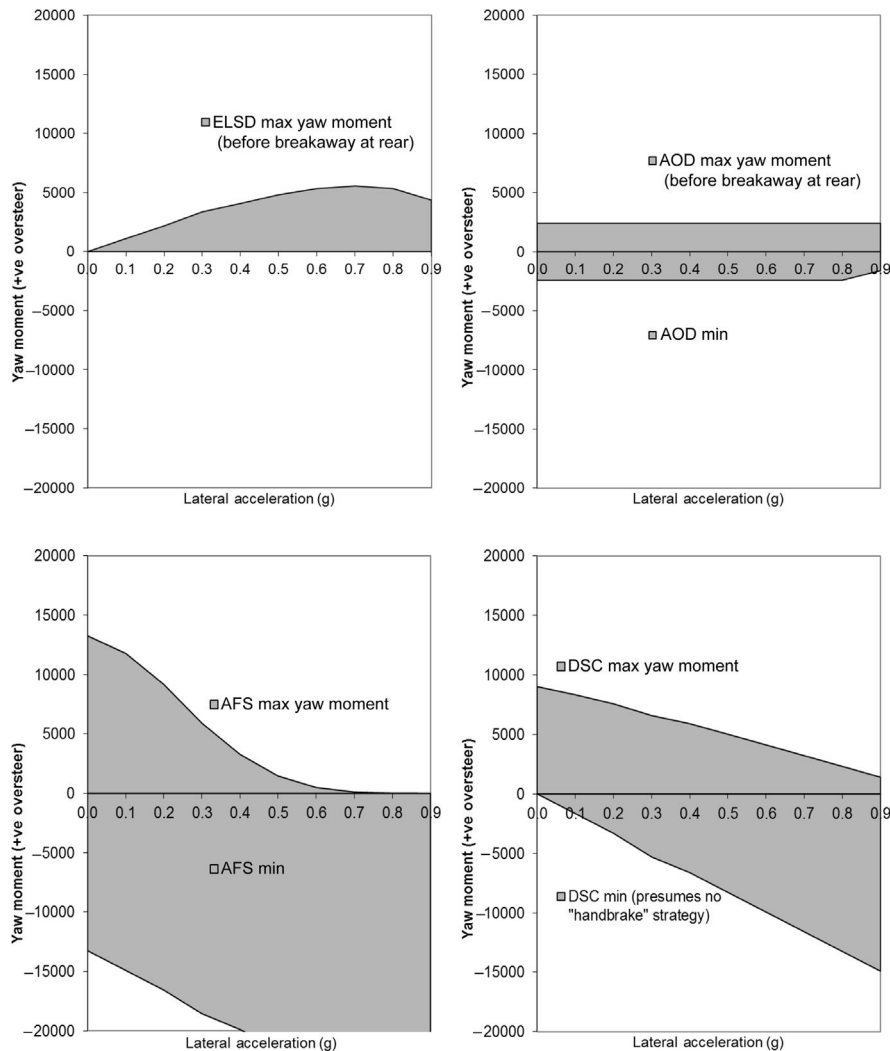
needs no knowledge of friction coefficient. Key to unlocking these benefits is the realisation that what is in play is ‘slip vectoring’ and not ‘torque vectoring’.

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### 8.3 Which active system?

The North American market is currently resisting complexity in its best-selling light truck segment (Pickups and SUVs), since the appeal from the vehicle manufacturer’s perspective is the low cost of production, giving margins that are allowing the vehicle manufacturers to remain in business during lean times. Nevertheless, competition from offshore competitors is making the US consumer expectations for vehicle performance and quality increase steadily. In Europe, an already sophisticated consumer is expecting steadily improving dynamic performance from their vehicle, while in Asia the predilection for ‘gadgets’ on vehicles has seen this market lead the way in terms of satellite navigation and so on. All three major world vehicle markets have their own reason for needing to improve vehicle dynamic performance over and above the level that can be achieved using conventional, passive technology. However, no-one wants to be first to market with a system that initially gives them a cost penalty and runs the risk of delivering benefits that the customer does not notice — and is therefore not prepared to pay for.

This appears to have been a lesson learned by the Japanese manufacturers — notably Toyota with their remarkable Soarer, who have largely withdrawn from the active system war and concentrated significantly on hybrid drivelines. In deciding which active system to employ, some note needs to be taken of the fact that different systems have different levels of authority under different driving regimes. In terms of systems that intervene directly — that is to say by generating ground plane forces, as distinct from modifying vertical forces in the manner of an active suspension — then different systems have differing strengths. [Figure 8.20](#) presents the available ‘understeer and oversteer’ moments — which is to say moments out of the turn and into the turn, respectively — for a number of different systems. The characteristics are arrived at by considering wheel loads for a point mass vehicle with a finite footprint (wheelbase and track) and a centre of gravity above the ground. Each system is then imagined to modify the ground plane forces up to the friction limit or system capacity limit, whichever is the lower. For the electronically controlled limited slip differential, tractive power is presumed added to remove the presence of braking slip from the inside wheel. The different ‘authority envelopes’ can be clearly seen. Oversteer moment is available to correct an understeer error state and vice versa. Thus it can be seen that the steering is by far the most powerful actuator on the car for controlling oversteer, but brake-based stability systems are a close second. Steering is ineffective at suppressing understeer in any normally balanced car. Of particular interest is the somewhat limited envelope of the asymmetric overspeeding differential despite its complexity. In particular, in the oversteer case when understeer moment is needed, it is typically reduced in capacity by unloading of the inside rear wheel.

**FIGURE 8.20**

An illustrative authority envelope comparison between an electronic limited slip differential (ELSD) (top left) and asymmetric overspeeding differential (AOD) (top right), a full authority steering system (AFS) (bottom left) and a brake-based system (bottom right) with no 'handbrake' mode.

Any 'full spectrum' dynamic intervention system is likely to need more than one set of actuators. For this reason, predictive work is needed on a case-by-case basis to sort the useful from the gimmick in terms of vehicle dynamics controls. The author's expectations for the use of active systems on vehicles are given in [Table 8.2](#). Three driving regimes are identified, which may be described as 'normal', 'spirited' and

**Table 8.2** Which Active System – and Why?

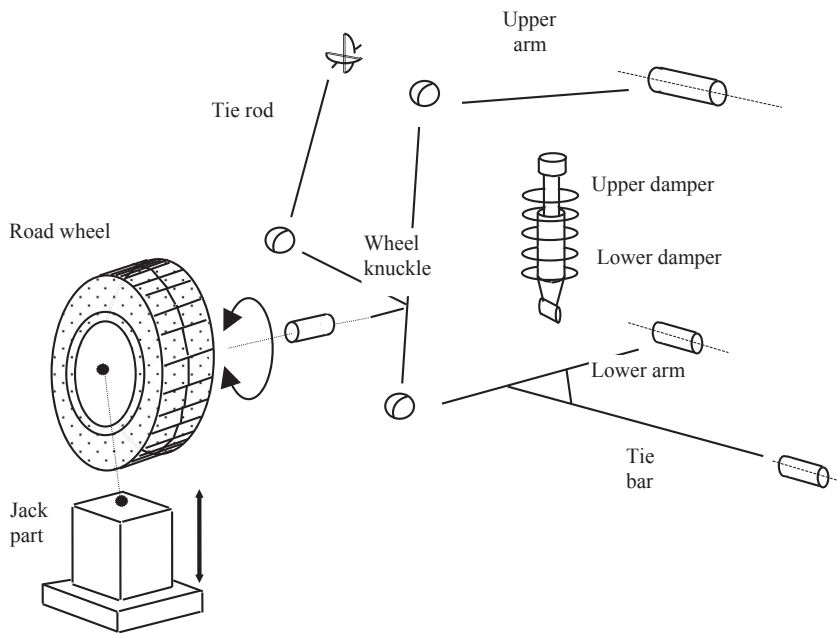
Name	Lateral Acceleration	Steering Rate	Guiding Principles	Applicable Active System
Normal	0–0.3 g	0–400°/s	<ul style="list-style-type: none"> <li>• To steer the car, steer the wheels.</li> <li>• Ride matters.</li> </ul>	Active rear toe, active front steer, adaptive dampers.
Spirited	0.3–0.6 g	400–700°/s	<ul style="list-style-type: none"> <li>• Intelligently combine drive and steer forces to deliver control without retardation.</li> <li>• Control body motion.</li> </ul>	Active torque distribution, <i>active rear toe</i> , <i>active front steer</i> , <i>adaptive dampers</i>
Emergency	0.6 g+	700+°/s	<ul style="list-style-type: none"> <li>• Use brakes to reduce kinetic energy of the vehicle.</li> <li>• Minimise wheel load variation for maximum grip and control.</li> </ul>	Brake-based system, <i>adaptive dampers</i> , active anti-roll bars.
<i>Italics denote duplication – i.e. reuse.</i>				

‘accident avoidance’. These may be loosely classified as being 0 to 0.3 g lateral acceleration, 0.3 to 0.6 g lateral acceleration and over 0.6 g lateral acceleration respectively. They are also characterised by driver inputs, particularly in terms of handwheel rates – 0 to 400°/s, 400 to 700°/s and 700 to 1200°/s, respectively. These classifications are entirely arbitrary and based on empirical observations. The upper bound is observed to be broadly independent of the nature of the driver, which is to say that motorsport drivers are not observed to be able to put in higher handwheel rates than the rest of us. It’s fair to say they have a better idea what handwheel rate to choose than many of us, though.

The results in Table 8.2 are not in any sense definitive but the authors considered views in the light of the available evidence. It is hoped that the readers of this text will be part of the group of engineers discovering just how good these judgements are.



# Vehicle Model System Schematics and Data Sets



**FIGURE A.1**

Front suspension components.

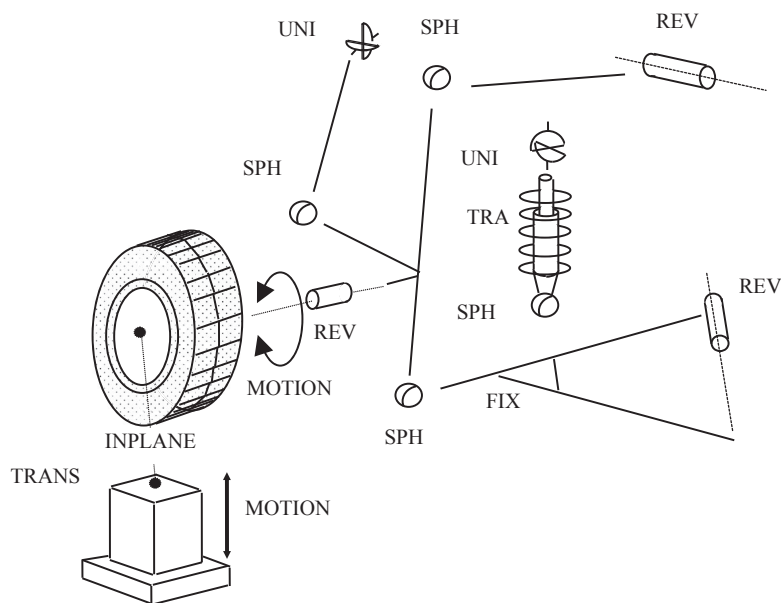
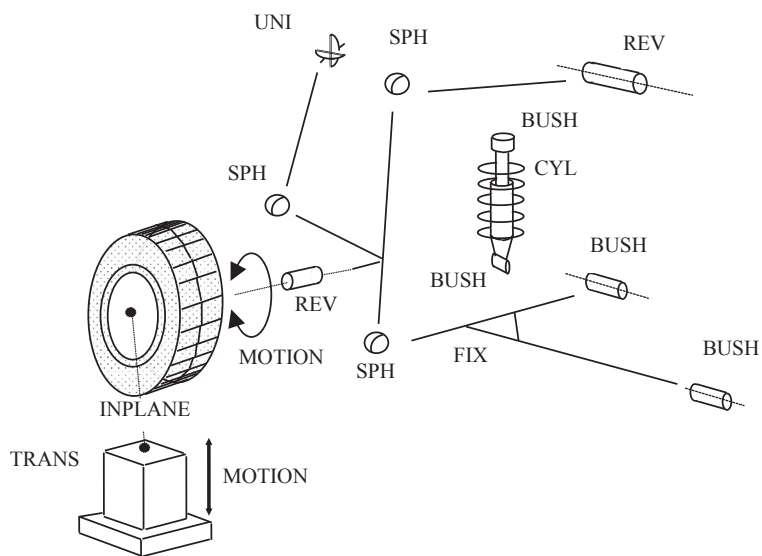


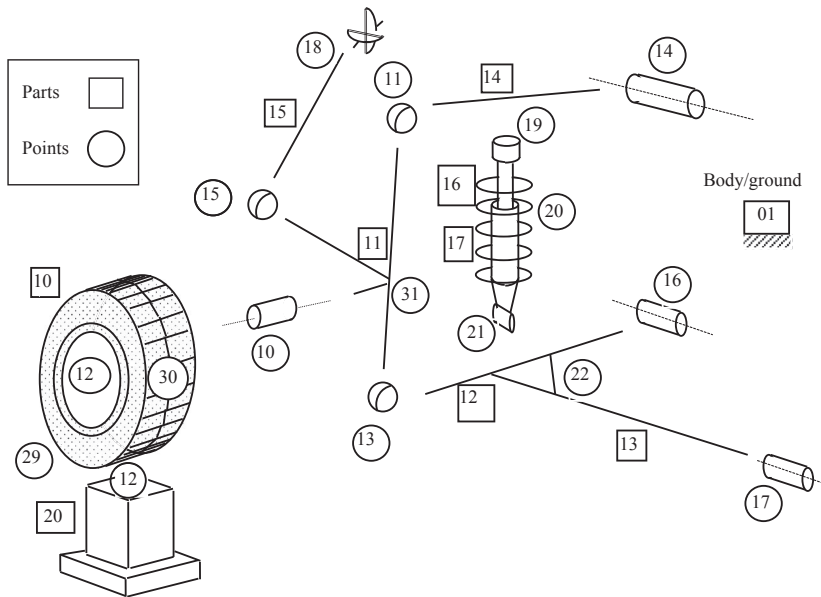
FIGURE A.2

Front suspension with joints.



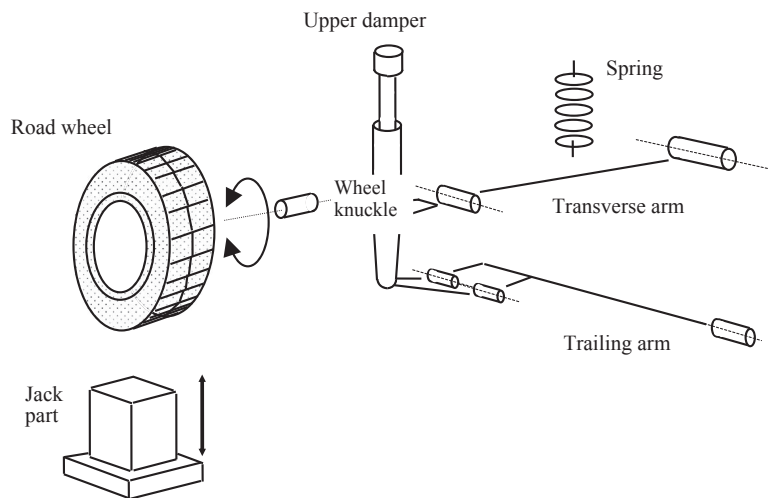
**FIGURE A.3**

Front suspension with bushes.



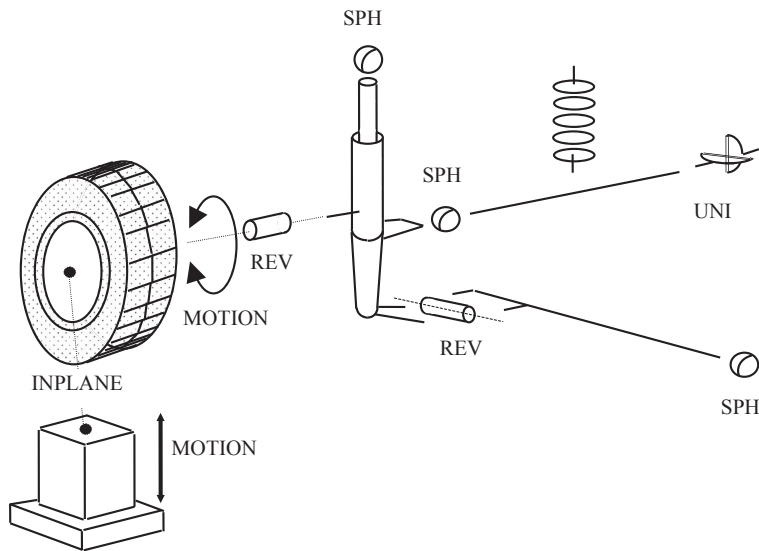
**FIGURE A.4**

Front suspension numbering convention.

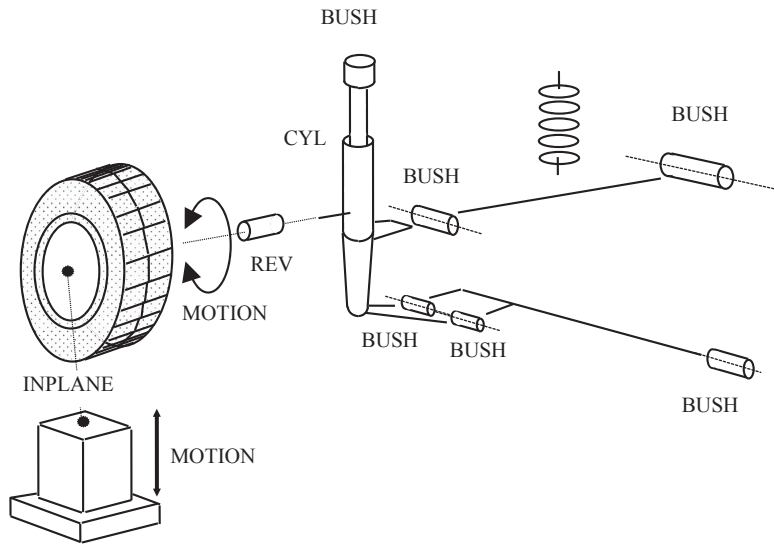


**FIGURE A.5**

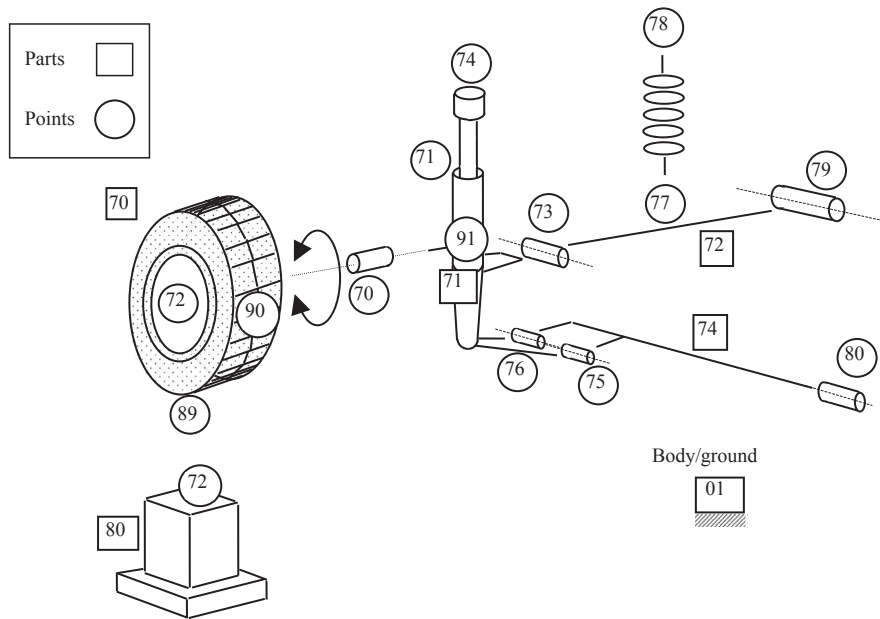
Rear suspension components.



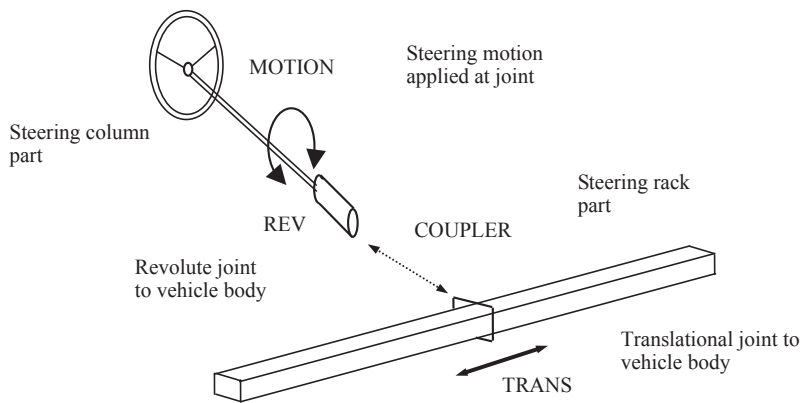
**FIGURE A.6**  
Rear suspension with joints.



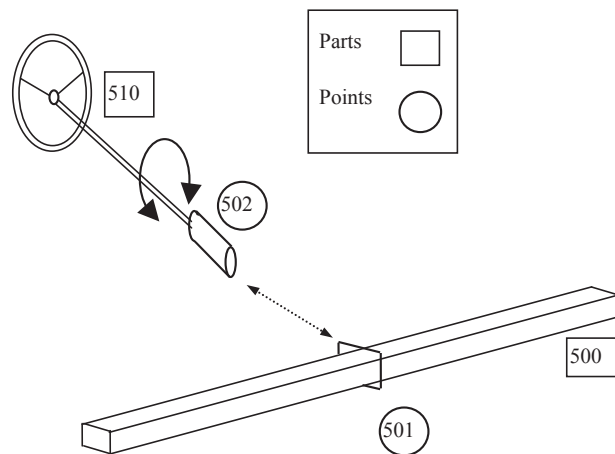
**FIGURE A.7**  
Rear suspension with bushes.

**FIGURE A.8**

Rear suspension numbering convention.

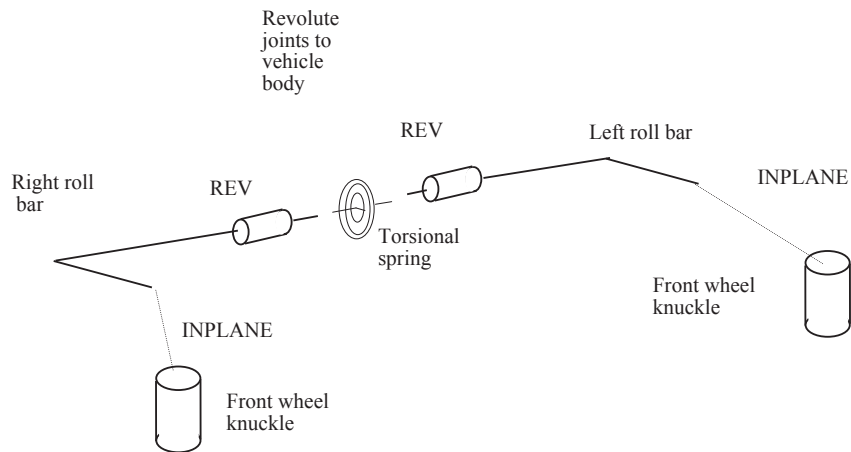
**FIGURE A.9**

Steering system components and joints.



**FIGURE A.10**

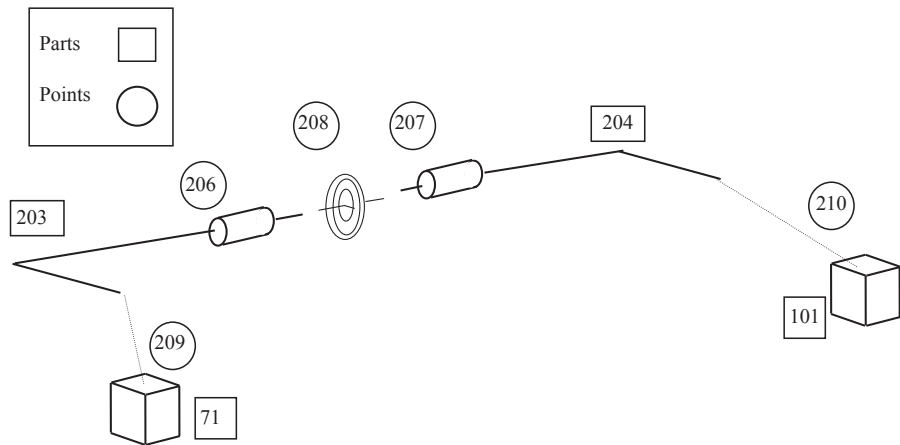
Steering system numbering convention.



**FIGURE A.11**

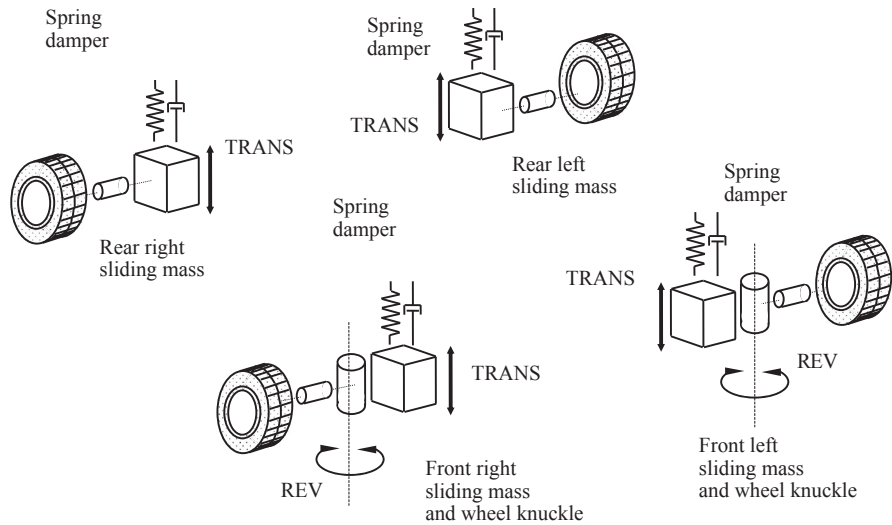
Front roll bar system components and joints.





**FIGURE A.14**

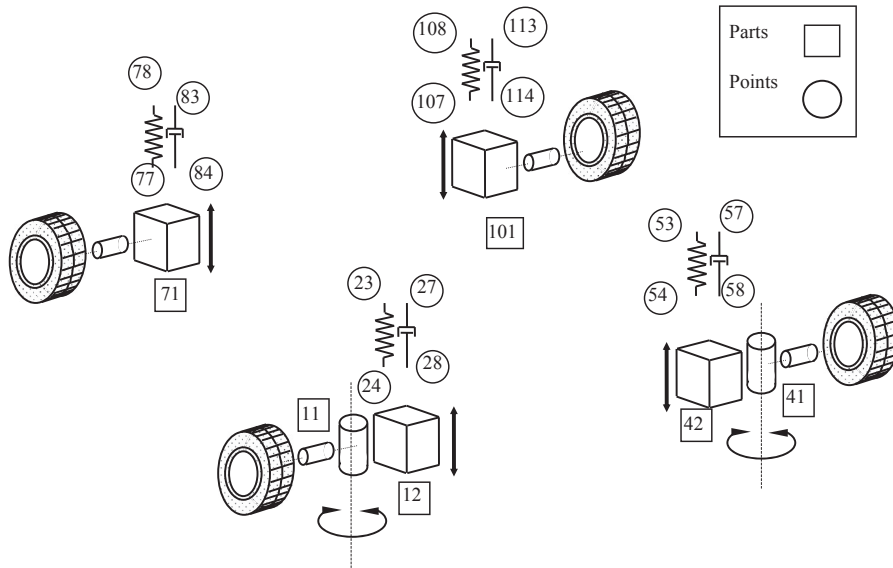
Rear roll bar system numbering convention.



**FIGURE A.15**

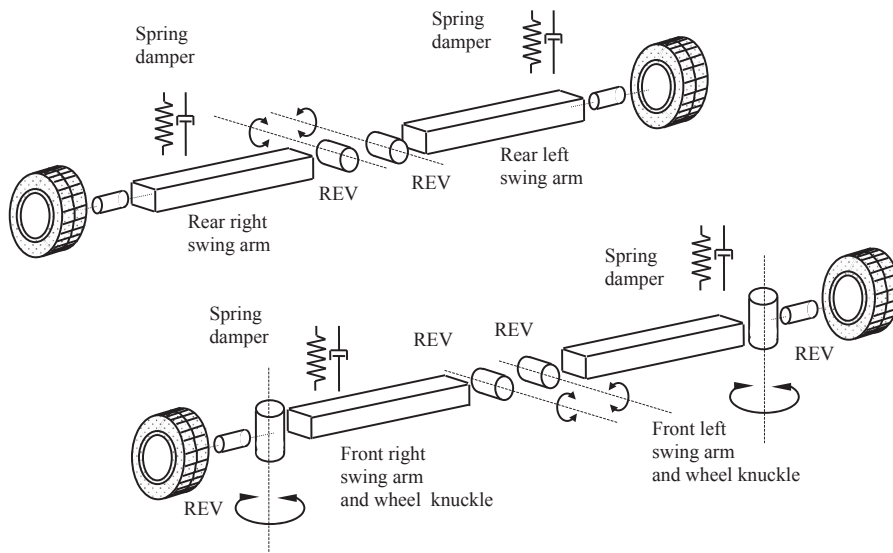
Lumped mass model suspension components and joints.





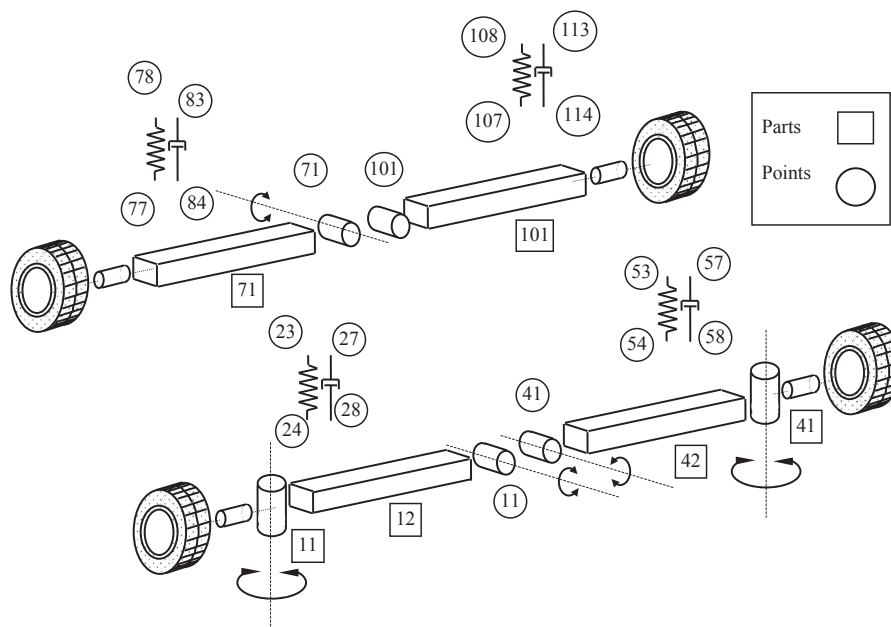
**FIGURE A.16**

Lumped mass model suspension numbering convention.



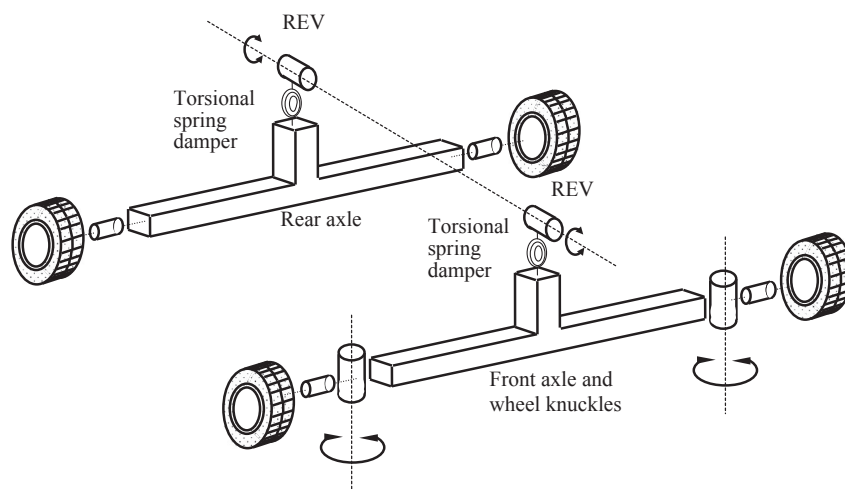
**FIGURE A.17**

Swing arm model suspension components and joints.



**FIGURE A.18**

Swing arm model suspension numbering convention.



**FIGURE A.19**

### Roll stiffness model suspension components and joints.

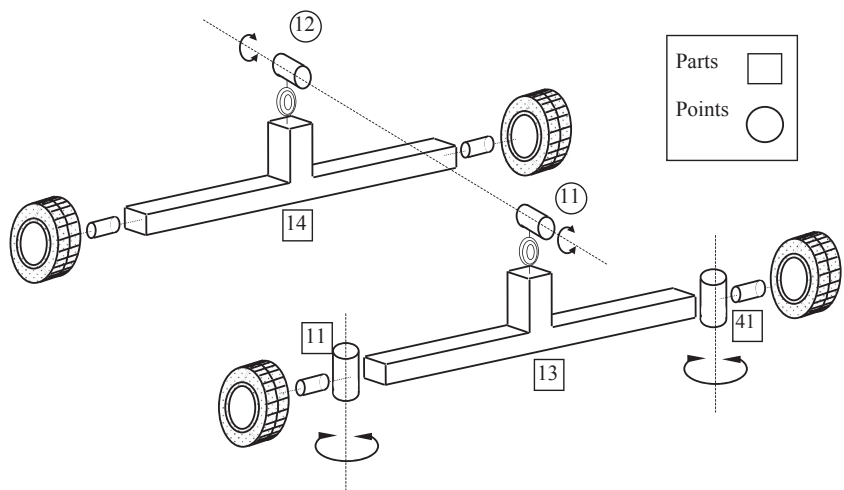


FIGURE A.20

Roll stiffness model suspension numbering convention.

Table A.1 Road Wheel Mass and Moment of Inertia Data						
Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
		Marker		Ix	Iy	Iz
Front right wheel	10	10	21.204	577.59E3	577.59E3	931.077E3
Front left wheel	40	40 <sup>a</sup>	21.204	577.59E3	577.59E3	931.077E3
Rear right wheel	170	70 <sup>a</sup>	21.204	577.59E3	577.59E3	931.077E3
Rear left wheel	100	100 <sup>a</sup>	21.204	577.59E3	577.59E3	931.077E3

<sup>a</sup> The wheel parts are generated automatically by the TIRE statement. The centre of mass is taken to be at the location of the J marker for each tyre, i.e., locations 10, 40, 70, 100.

Table A.2 Front Right Suspension Mass and Moment of Inertia Data						
Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
		Marker		Ix	Iy	Iz
Wheel knuckle	11	1100	11.678	65.647E3	120.541E3	77.691E3
Lower arm	12	1200	3.4405	37.856E3	0.348E3	37.933E3
Tie bar	13	1300	2.64	0.023E3	3.876E3	3.876E3
Upper arm	14	1400	2.187	11.520E3	4.071E3	15.218E3
Tie rod	15	1500	0.575	3.876E3	0.023E3	3.876E3
Upper damper	16	1600	0.389	63.948E3	63.948E3	0.822E3
Lower damper	17	1700	6.215	107.364E3	107.364E3	4.972E3

**Table A.3** Front Left Suspension Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker			lx	ly	lz	
Wheel knuckle	41	4100	11.678	65.647E3	120.541E3	77.691E3
Lower arm	42	1200	3.4405	37.856E3	1.348E3	37.933E3
Tie bar	43	1300	2.64	0.023E3	3.876E3	3.876E3
Upper arm	44	1400	2.187	11.520E3	4.071E3	15.218E3
Tie rod	45	1500	0.575	3.876E3	0.023E3	3.876E3
Upper damper	46	1600	0.389	63.948E3	63.948E3	0.822E3
Lower damper	47	1700	6.215	107.364E3	107.364E3	4.972E3

**Table A.4** Rear Right Suspension Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker			lx	ly	lz	
Wheel knuckle	71	7100	12.036	165.994E3	196.457E3	34.224E3
Transverse arm	72	7200	6.424	101.389E3	15.215E3	113.237E3
Trailing arm	74	7400	4.322	12.826E3	190.372E3	199.393E3
Upper damper	73	7300	0.982	1.003E3	1.003E3	0.1E3

**Table A.5** Rear Left Suspension Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker			lx	ly	lz	
Wheel knuckle	101	10,100	12.036	165.994E3	196.457E3	34.224E3
Transverse arm	102	10,200	6.424	101.389E3	15.215E3	113.237E3
Trailing arm	104	10,400	4.322	12.826E3	190.372E3	199.393E3
Upper damper	103	10,300	0.982	1.003E3	1.003E3	0.1E3

**Table A.6** Body, Rollbars and Steering Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker			lx	ly	lz	
Body	200	20,000	1427.3	379.0E6	2235.0E6	2269.0E6
Front right roll bar	201	20,100	1.4	10.0	10.0	20.0
Front left roll bar	202	20,200	1.4	10.0	10.0	20.0
Rear right roll bar	203	20,300	1.4	10.0	10.0	20.0
Rear left roll bar	204	20,400	1.4	10.0	10.0	20.0
Steering rack	500	50,000	0.48	13,357.0	48.0	13,357.0
Steering column	510	51,000	2.2	24.0E3	24.0E3	40,762.0

**Table A.7** Lumped Mass Model Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker				Ix	Iy	Iz
Right wheel knuckle	11	1100	11.678	65.647E3	120.541E3	77.691E3
Left wheel knuckle	41	4100	11.678	65.647E3	120.541E3	77.691E3
Front right mass	12	1200	15.447	86.818E3	159.445E3	102.765E3
Front left mass	42	4200	15.447	86.818E3	159.445E3	102.765E3
Rear right mass	71	7100	23.764	280.209E3	402.062E3	346.854E3
Rear left mass	101	10,100	23.764	280.209E3	402.062E3	346.854E3

**Table A.8** Swing Arm Model Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker				Ix	Iy	Iz
Right wheel Knuckle	11	1100	11.678	65.647E3	120.541E3	77.691E3
Left wheel Knuckle	41	4100	11.678	65.647E3	120.541E3	77.691E3
Front right arm	12	1200	15.447	86.818E3	159.445E3	102.765E3
Front left arm	42	4200	15.447	86.818E3	159.445E3	102.765E3
Rear right arm	71	7100	23.764	280.209E3	402.062E3	346.854E3
Rear left arm	101	10,100	23.764	280.209E3	402.062E3	346.854E3

**Table A.9** Roll Stiffness Model Mass and Moment of Inertia Data

Description	Part ID	CM	Mass (kg)	Mass Moments of Inertia (kgmm <sup>2</sup> )		
Marker				Ix	Iy	Iz
Right wheel knuckle	11	1100	11.678	65.647E3	120.541E3	77.691E3
Left wheel knuckle	41	4100	11.678	65.647E3	120.541E3	77.691E3
Front axle	13	1300	30.894	173.636E3	318.89E3	205.53E3
Rear axle	14	1400	47.528	560.418E3	804.124E3	693.708E3

**Table A.10** Front Right Suspension Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
1100	966.1	743.9	165.9						
1200	1006.0	525.0	113.0	12D	−13.5D	0D			
1300	722.0	444.0	115.0						
1400	1064.0	500.0	566.0	28.5D	9D	0D			
1500	1129.0	534.0	189.0						
1600	954.4	509.6	497.3	163.66D	164.61D	0D			
1700	954.4	509.6	497.3	163.66D	164.61D	0D			
10	966.1	743.9	165.9	0D	90D	0D			
11	986.6	639.3	572.1						
13	962.2	703.1	90.7						
14	973.0	417.8	548.4	119.05D	91.03D	0.0D			
15	1106.2	673.9	171.7						
16	1050.0	348.3	137.0	−75.95D	90.87D	−172.78D			
17	474.0	332.0	115.0	−66.15D	90D	0.0D			
18	1145.0	392.0	196.0				1245.0	392.0	196.0 <sup>a</sup>
19	945.9	480.4	607.7	163.66D	164.61D	0.0D			
20	954.4	509.6	497.3	163.66D	164.61D	0.0D			
21	987.0	620.0	76.5	−89.72D	98.11D	0.0D			
22	984.2	562.3	113.9						
23	945.9	480.4	607.7						
24	965.5	547.4	354.1						
25	945.9	480.4	607.7						
26	954.4	509.6	497.3						
27	945.9	480.4	607.7						
28	987.0	620.7	76.5						
31	966.1	693.1	165.9						
32	965.3	547.4	354.1						
33	945.9	480.4	607.7	163.66D	164.61D	0D			

<sup>a</sup> The ZP orientation is only applied to the marker which belongs to the rack part and is used for the universal joint connecting the rack to the tie rod.

**Table A.11** Front Left Suspension Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
4100	966.1	−743.9	165.9						
4200	1006.0	−525.0	113.0	12D	−13.5D	0D			
4300	722.0	−444.0	115.0						
4400	1064.0	−500.0	566.0	28.5D	9D	0D			
4500	1129.0	−534.0	189.0						
4600	954.4	−509.6	497.3	−343.66D	164.61D	−180D			
4700	954.4	−509.6	497.3	−343.66D	164.61D	−180D			
40	966.1	- 743.9	165.9	0D	90D	0D			
41	986.6	−639.3	572.1						
43	962.2	−703.1	90.7						
44	973.0	- 417.8	548.4	−299.05D	91.03D	−180.0D			
45	1106.2	−673.9	171.7						
46	1050.0	−348.3	137.0	−104.05D	90.87D	−7.22D			
47	474.0	−332.0	115.0	−113.85D	90D	−180D			
48	1145.0	−392.0	196.0				1245.0	−392.0	196.0 <sup>a</sup>
49	945.9	−480.4	607.7	−343.66D	164.61D	−180.0D			
50	954.4	−509.6	497.3	−343.66D	164.61D	−180D			
51	987.0	−620.0	76.5	89.72D	81.89D	0.0D			
52	984.2	−562.3	113.9						
53	945.9	−480.4	607.7						
54	965.5	−547.4	354.1						
55	945.9	−480.4	607.7						
56	954.4	−509.6	497.3						
57	945.9	−480.4	607.7						
58	987.0	−620.7	76.5						
61	966.1	−693.1	165.9						
62	965.3	−547.4	354.1						
63	945.9	−480.4	607.7	−343.66D	164.61D	180D			

<sup>a</sup> The ZP orientation is only applied to the marker which belongs to the rack part and is used for the universal joint connecting the rack to the tie rod.

Table A.12 Rear Right Suspension Geometry									
Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
7100	3732	725	167						
7200	3743	493	112	0D	4.55D	0D			
7300	3701.02	525.66	606.36	163.46D	166.09D	0D			
7400	3630	587	70						
70	3732	725	167	0D	90D	0D			
71	703.6	534.4	569.6	163.46D	166.09D	0D			
72	3732	725	167						
73	3747	706	109.5	88.809D	90D	−4.3376D			
74	3701	525.7	606.4	163.46D	166.09D	−195D			
75	3695.4	684.6	48.1	56.715D	97.334D	0.0D			
76	3803.4	613.7	34.2	56.716D	97.334D	0.0D			
77	3743	496	84.0						
78	3743	475	265.8						
79	3737	225	146	88.809D	90D	−4.3376D			
80	3203	600	144.5	75.0491D	98.3D	0D			
81	3703.5	533	573						
82	3693	499	714						
83	3693.3	499.7	715.9						
84	3729.7	622.2	200						
85	3701	525.7	606.4	163.46D	166.09D	0D			
86	3703.6	534.4	569.6						
91	3732	684.6	167						



**Table A.13** Rear Left Suspension Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\Psi$	$\theta$	$\phi$	X	Y	Z
10100	3732	−725	167						
10200	3743	−493	112	−180D	4.55D	−180D			
10300	3701.02	−525.66	606.36	−343.46D	166.09D	−180D			
10400	3630	−587	70						
100	3732	−725	167	0D	90D	0D			
101	3703.6	−534.4	569.6	163.46D	166.09D	0D			
102	3732	−725	167						
103	3747	−706	109.5	91.191D	90D	−175.6624D			
104	3701	−525.7	606.4	−343.46D	166.09D	15D			
105	3695.4	−684.6	48.1	123.284D	97.334D	180D			
106	3803.4	−613.7	34.2	123.284D	97.334D	180D			
107	3743	−496	84.0						
108	3743	−475	265.8						
109	3737	−225	146	91.191D	90D	−175.634D			
110	3203	−600	144.5	−255.049D	98.3D	180D			
111	3703.5	−533	573						
112	3693	−499	714						
113	3693.3	−499.7	715.9						
114	3729.7	−622.2	200						
115	3701	−525.7	606.4	−343.46D	166.09D	180D			
116	3703.6	−534.4	569.6						
121	3732	−684.6	167						

**Table A.14** Body, Rollbars and Steering Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
20000	2150.4	0.0	452.0						
20100	1264	263	87	0D	90D	0D			
201	1264	263	87	0D	90D	0D			
203	1264	0.0 87	0D	90D	0D				
204	966.1	743.9	165.9						
20200	1264	-263	87	0D	90D	0D			
202	1264	-263	87	0D	90D	0D			
203	1264	0.0	87	0D	90D	0D			
205	966.1	-743.9	165.9						
20300	4142	508	268	0D	90D	0D			
206	4142	508	268	0D	90D	0D			
208	4142	0	268	0D	90D	0D			
209	3732	725	167						
20400	4142	-508	268	0D	90D	0D			
207	4142	-508	268	0D	90D	0D			
208	4142	0	268	0D	90D	0D			
210	3732	-725	167						
50000	1145	0	196						
501	1145	0	196	180D	90D	0D			
51000	1964	353	787.6				1698.6	348.6	604.4
502	1964	353	787.6				1698.6	348.6	604.4
5101	2072	353	631						
5102	1961	543	788						
5103	1967	163	788						
5104	1145	338	222						

**Table A.15** Lumped Mass Model Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
1100	966.1	743.9	165.9						
1200	966.1	500	165.9						
4100	966.1	-743.9	165.9						
4200	966.1	-500	165.9						
7100	3732	500	167						
10100	3732	-500	167						
11	966.1	500	165.9						
41	966.1	-500	165.9						
39	966.1	743.9	165.9						
69	966.1	-743.9	165.9						
99	3732	500	167						
129	3732	-500	167						

**Table A.16** Swing Arm Model Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
1100	966.1	743.9	165.9						
1200	966.1	500	165.9						
4100	966.1	-743.9	165.9						
4200	966.1	-500	165.9						
7100	3732	500	167						
10,100	3732	-500	167						
39	966.1	743.9	165.9						
69	966.1	-743.9	165.9						
11	966.1	-1361	358.6						
41	966.1	1361	358.6						
71	3732	-576.5	206.2						
101	3732	576.5	206.2						

**Table A.17** Roll Stiffness Model Geometry

Point Id.	Coordinates (mm)			Euler Angles (deg)			ZP (mm)		
	X	Y	Z	$\psi$	$\theta$	$\phi$	X	Y	Z
1100	966.1	743.9	165.9						
4100	966.1	-743.9	165.9						
1300	966.1	0.0	165.9						
1400	3732	0.0	167						
39	966.1	743.9	165.9						
69	966.1	-743.9	165.9						
11	966.1	0.0	27.4						
12	3732	0.0	47.4						

**Table A.18** Spring Data

The springs are defined using the SPRINGDAMPER statement.

**Front Spring:**

Stiffness  $K = 31.96 \text{ N/mm}$   
 Free length  $L = 426 \text{ mm}$

**Rear Spring:**

Stiffness  $k = 60.8 \text{ N/mm}$   
 Free length  $L = 253 \text{ mm}$

**Table A.19** Front Damper Data

The dampers are defined using the SFORCE statement. The dampers are nonlinear.  
The data provided shows the variation of force in the damper with velocity.

Velocity (mm/s)	-5000, -3150, -2870, -2450, -2205, -1925, -1610, -1260, -910, -630, -470, -400, -350, -300, -250, -230, -200, -190, -160, -120, -80, -55, -40, -20, -10, -1, -0.1, 0, 0.3, 3, 30, 40, 60, 80, 100, 200, 250, 400,490, 770, 1050, 1330, 1820, 2060, 2485, 2590, 2730, 2835, 2940, 3080, 5000
Force (N)	10425, 5800, 5200, 4400, 4000, 3600, 3200, 2800, 2400, 2000, 1800, 1700, 1600, 1500, 1400, 1350, 1310, 1290, 1200, 1000, 700, 400, 210, 80, 40, 4, 0.4,0, -1, -10, -100, -123, -150, -182, -200, -260, -300, -400, -500, -800, -1200, -1600, -2400, -2800, -3400, -3500, -3600, -3700, -3800, -4000, -7840

**Table A.20** Rear Damper Data

The dampers are defined using the SFORCE statement. The dampers are nonlinear.  
The data provided shows the variation of force in the damper with velocity.

Velocity (mm/s)	-5000, -3150, -2800, -2450, -2100, -1750, -1400, -1050, -700, -560, -500, -450, -400, -350, -300,-250, -200, -150, -100, -50, -25, -5, -1, -0.1, 0, 0.1, 1, 5, 25,50,100, 200, 300, 400, 500, 700, 1050, 1400, 1750, 2100, 2450, 2800, 3150, 5000
Force(N)	7352,3652,3120,2635,2193,1855,1518,1180,927,843,800,773,722, 686,658,596, 560,488,329,154,77,15.4,3.08,0.308,0,-0.126,-1.26, -6.3, -31.5,-63,-126,-153.25, -180.5,-208,-235,-253,-380,-675,-970, -1349, -1788,-2277,-2867,-6567

**Table A.21** Roll Bar Data

Front Roll Bar:	
Torsional stiffness	Kt = 490E3 Nmm/rad
Rear roll bar:	
Torsional stiffness	Kt = 565E3 Nmm/rad

**Table A.22** Front Suspension Bush Data

The following linear values are used to define the stiffness and damping in the bushes. For each bush data is listed as:

$K = k_x, k_y, k_z$  Stiffness (N/mm)

$KT = k_{tx}, k_{ty}, k_{tz}$  Torsional stiffness (Nmm/rad)

$C = c_x, c_y, c_z$  Damping (Ns/mm)

$CT = c_{tx}, c_{ty}, c_{tz}$  Torsional Damping (Nmms/rad)

**Lower Arm Mount Bush (location 12 and 42)**

$K = 7825, 7825, 944$

$KT = 2.5E6, 2.5E6, 500$

$C = 35, 35, 480$

$CT = 61000, 61000, 40$

**Tie Bar Bush Mount**

$K = 5723, 5723, 6686$

$KT = 543000, 543000, 500$

$C = 400, 400, 300$

$CT = 18400, 18400, 4$

**Upper Damper Mount**

$K = 14353, 14353, 10000$

$KT = 120000, 120000, 400$

$C = 400, 400, 300$

$CT = 1200, 1200, 40$

**Lower Damper Mount**

$K = 6385, 6385, 550$

$KT = 355000, 355000, 400$

$C = 640, 640, 50$

$CT = 35000, 35000, 40$

<p><b>Table A.23</b> Rear Suspension Bush Data</p> <p>The following linear values are used to define the stiffness and damping in the bushes. For each bush data is listed as:</p> <p>K = <math>k_x</math>, <math>k_y</math>, <math>k_z</math> Stiffness (N/mm)  KT = <math>k_{tx}</math>, <math>k_{ty}</math>, <math>k_{tz}</math> Torsional stiffness (Nm/rad)  C = <math>c_x</math>, <math>c_y</math>, <math>c_z</math> Damping (Ns/mm)  CT = <math>c_{tx}</math>, <math>c_{ty}</math>, <math>c_{tz}</math> Torsional Damping (Nmms/rad)</p>
<p><b>Rear Trailing Link to Hub Bushes</b></p> <p>K = 10500,10500,870  KT = 2.8E5,2.8E5,67500  C = 1000,1000,100  CT = 25000,25000,40</p>
<p><b>Rear Trailing Link to Body Bush</b></p> <p>K = 660,660,175  KT = 260300,260300,40000  C = 100,100,50  CT = 25000,25000,40</p>
<p><b>Rear Upper Damper Mount</b></p> <p>K = 540,1300,532  KT = 58915,180750,670  C = 200,200,70  CT = 5800,5800,67</p>
<p><b>Rear Lower Arm to Body Mount</b></p> <p>K = 10800,3420,840  KT = 790000,380000,400  C = 1000,400,100  CT = 88000,40000,40</p>
<p><b>Rear Lower Arm to Hub Bush</b></p> <p>K = 5540,5540,515  KT = 210540,210540,400  C = 800,800,50  CT = 25000,25000,40</p>

# Fortran Tyre Model Subroutines

## B.1 Interpolation tyre model subroutine

```

SUBROUTINE TIRSUB ( ID, TIME, TO, CPROP, TPROP, MPROP,
&                  PAR, NPAR, STR, NSTR, DFLAG,
&                  IFLAG, FSAE, TSAE, FPROP )
C
C   This program is part of the CUTyre system - M Blundell, Feb 1997
C   This version is based on an interpolation approach using measured
C   tyre test data which is include in SPLINE statements. The model is referred
to as the
C   Limited version based on the limited testing where camber and slip
are varied
C   independently.
C
C   The coefficients in the model assume the following units:
C   slip angle: degrees
C   camber angle: degrees
C   Fz (load): kg
C   Fy and Fx: N
C   Tz : Nm
C
C   Note this subroutine is developed to not account for offsets
C   twice. The offsets are include for slip interpolation
C   but for camber the offset at zero camber is subtracted.
C
C Inputs:
C
C   INTEGER          ID, NPAR, NSTR
C   DOUBLE PRECISION TIME, TO
C   DOUBLE PRECISION CPROP(*), TPROP(*), MPROP(*), PAR(*)
C   CHARACTER*80     STR(*)
C   LOGICAL          DFLAG, IFLAG, ERRFLG
C
C Outputs:
C
C   DOUBLE PRECISION FSAE(*), TSAE(*), FPROP(*), ARRAY(3)
C

```

C Local Variables:

C

```
DOUBLE PRECISION SLIP, ALPHA, DEFL, DEFLD
DOUBLE PRECISION R2, CZ, CS, CA, CR, DZ, AMASS, WSPIN
```

C

```
DOUBLE PRECISION GAMMA, CG, RALPHA, RGAMMA, FZL, TZL, TZLA, TZLG
DOUBLE PRECISION CFY, DFY, EFY, SHFY, SVFY, PHIFY, TZLGO, TZLG1
DOUBLE PRECISION CTZ, DTZ, ETZ, BTZ, SHTZ, SVTZ, PHITZ
DOUBLE PRECISION CFX, DFX, EFX, BFX, SHFX, SVFX, PHIFX
```

C

```
INTEGER IORD
DOUBLE PRECISION ZERO, ONE, SCFACT, DELMAX, FYA, FYG, FYGO, FYG1
DOUBLE PRECISION FX, FY, FZ, FX1, FX2, TY, TZ, H, ASTAR, SSTAR
DOUBLE PRECISION U, FZDAMP, FZDEFL, WSPNMX, DTOR, RTOD
LOGICAL ERFLG
```

C

```
PARAMETER (ZERO=0.0)
PARAMETER (ONE=1.0)
PARAMETER (IORD=0)
PARAMETER (WSPNMX=5.0D-1)
PARAMETER (DTOR=0.017453292)
PARAMETER (RTOD=57.29577951)
```

C

C

C EXECUTABLE CODE

C

C

C Extract data from input arrays

C

```
SLIP = CPROP(1)
DEFL = CPROP(4)
DEFLD = CPROP(5)
WSPIN = CPROP(8)
```

C

```
AMASS = MPROP(1)
```

C

```
R2 = TPROP(2)
CZ = TPROP(3)
CS = TPROP(4)
CA = TPROP(5)
CR = TPROP(7)
DZ = TPROP(8)
U = TPROP(11)
```

C

```
RALPHA = CPROP(2)
RGAMMA = CPROP(3)
CG = TPROP(6)
```



```

      ALPHA=RALPHA*RTOD
      GAMMA=RGAMMA*RTOD
C
C   Initialize force values
C
      FX = 0.D0
      FY = 0.D0
      FZ = 0.D0
      TY = 0.D0
      TZ = 0.D0
C
      IF(DEFL .LE. 0.D0) THEN
         GOTO 1000
      ENDIF
C
C   Calculate normal loads due to stiffness (always .LE. zero)
C
      FZDEFL = -DEFL*CZ
C
C   Calculate normal loads due to damping
C
      FZDAMP = - 2.D0*SQRT(AMASS*CZ)*DZ*(DEFLD)
C
C   Calculate total normal force (fz)
C
      FZ      =  MIN (0.0D0, (FZDEFL + FZDAMP) )
C
C   Convert to kg and change sign
C
      FZL = -FZ/9.81
C
C   Calculate critical longitudinal slip value
C
      SSTAR = ABS(U*FZ/(2.D0*CS))
C
C   Compute longitudinal force
C
      IF(ABS(SLIP) .LE. ABS(SSTAR)) THEN
         FX = -CS*SLIP
      ELSE
         FX1 = U*ABS(FZ)
         FX2 = (U*FZ)**2/(4.D0*ABS(SLIP)*CS)
         FX = -(FX1-FX2)*SIGN(1.0D0,SLIP)
      ENDIF
C
C   Compute lateral force
C

```

```

      CALL CUBSPL (ALPHA,FZL,100,0,ARRAY,ERRFLG)
      FYA=ARRAY(1)
      CALL CUBSPL (0,FZL,300,0,ARRAY,ERRFLG)
      FYG0=ARRAY(1)
      CALL CUBSPL (GAMMA,FZL,300,0,ARRAY,ERRFLG)
      FYG1=ARRAY(1)
      FYG=FYG1-FYG0
      FY=FYA+FYG
C
C      Compute self aligning moment
C
      CALL CUBSPL (ALPHA,FZL,200,0,ARRAY,ERRFLG)
      TZLA=ARRAY(1)
      CALL CUBSPL (0,FZL,400,0,ARRAY,ERRFLG)
      TZLG0=ARRAY(1)
      CALL CUBSPL (GAMMA,FZL,400,0,ARRAY,ERRFLG)
      TZLG1=ARRAY(1)
      TZLG=TZLG1-TZLG0
      TZL=TZLA+TZLG
C
C      Convert to Nmm
C
      TZ = TZL*1000.0
C
C      Copy the calculated values for FX, FY, FZ, TY & TZ to FSAE
C      and TSAE arrays
C
1000 FSAE(1) = FX
      FSAE(2) = FY
      FSAE(3) = FZ
C
      TSAE(1) = 0.0
      TSAE(2) = 0.0
      TSAE(3) = TZ
C
      FPROP(1) = 0.0
      FPROP(2) = 0.0
C
      RETURN
      END

```

---

## B.2 Magic formula tyre model (version 3) subroutine

```

      SUBROUTINE TIRSUB ( ID, TIME, TO, CPROP, TPROP, MPROP,
&                        PAR, NPAR, STR, NSTR, DFLAG,
&                        IFLAG, FSAE, TSAE, FPROP )
C

```

```

C   This program is part of the CUTyre system - M Blundell, Feb 1997
C   This version is based on the Magic Formula tyre model (Version 3).
C   Coefficients are for TYRE B
C
C   The coefficients in the model assume the following units:
C   slip angle: radians
C   camber angle: radians
C   slip ratio %
C   Fz (load): N
C   Fy and Fx: N
C   Tz : Nm
C   Note sign changes between Paceka formulation and SAE convention
C   If camber is not included set A5,A10,A13,A14,A15,A16
C   and C6,C10,C13,C16,C17,C18,C19,C20 to zero
C
C Inputs:
C
C   INTEGER          ID, NPAR, NSTR
C   DOUBLE PRECISION TIME, T0
C   DOUBLE PRECISION CPROP(*), TPROP(*), MPROP(*), PAR(*)
C   CHARACTER*80     STR(*)
C   LOGICAL          DFLAG, IFLAG
C
C Outputs:
C
C   DOUBLE PRECISION FSAE(*), TSAE(*), FPROP(*)
C
C Local Variables:
C
C   DOUBLE PRECISION SLIP, ALPHA, DEFL, DEFLD
C   DOUBLE PRECISION R2, CZ, CS, CA, CR, DZ, AMASS, WSPIN
C
C
C   DOUBLE PRECISION GAMMA,CG,RALPHA,RGAMMA,FXP,FZP,FYP,TZP
C   DOUBLE PRECISION A0,A1,A2,A3,A4,A5,A6,A7,A8,A9,A10,A11,A12,A13
C   DOUBLE PRECISION A14,A15,A16,A17,SLIPCENT
C   DOUBLE PRECISION C0,C1,C2,C3,C4,C5,C6,C7,C8,C9,C10,C11,C12,C13
C   DOUBLE PRECISION C14,C15,C16,C17,C18,C19,C20
C   DOUBLE PRECISION CFY,DFY,EFY,SHFY,SVFY,PHIFY
C   DOUBLE PRECISION CTZ,DTZ,ETZ,BTZ,SHTZ,SVTZ,PHITZ
C   DOUBLE PRECISION CFX,DFX,EFX,BFX,SHFX,SVFX,PHIFX,DUMTZ,DUMFY
C
C   INTEGER          IORD
C   DOUBLE PRECISION ZERO, ONE, SCFACT, DELMAX
C   DOUBLE PRECISION FX, FY, FZ, FX1, FX2, TY, TZ, H, ASTAR, SSTAR
C   DOUBLE PRECISION U, FZDAMP, FZDEFL, WSPNMX, DTOR, RTOD
C   LOGICAL          ERFLG

```

```

C
PARAMETER      (ZERO=0.0)
PARAMETER      (ONE=1.0)
PARAMETER      (IORD=0)
PARAMETER      (WSPNMX=5.0D-1)
PARAMETER      (DTOR=0.017453292)
PARAMETER      (RTOD=57.29577951)

```

```

C
C      Define Pacejka Coefficients
C

```

```

A0=.103370E+01
A1=-.224482E-05
A2=.132185E+01
A3=.604035E+05
A4=.877727E+04
A5=0.0
A6=.458114E-04
A7=.468222
A8=.381896E-06
A9=.516209E-02
A10=0.00
A11=-.366375E-01
A12=-.568859E+02
A13=0.00
A14=0.00
A15=0.00
A16=0.00
A17=.379913

```

```

C
C
C0=.235000E+01
C1=.266333E-05
C2=.249270E-02
C3=-.159794E-03
C4=-.254777E-01
C5=.142145E-03
C6=0.00
C7=.197277E-07
C8=-.359537E-03
C9=.630223
C10=0.00
C11=.120220E-06
C12=.275062E-02
C13=0.00
C14=-.172742E-02
C15=.544249E+01
C16=0.00
C17=0.00

```

```

C18=0.00
C19=0.00
C20=0.00
C
C
C EXECUTABLE CODE
C
C
C Extract data from input arrays
C
SLIP  = CPROP(1)
DEFL  = CPROP(4)
DEFLD = CPROP(5)
WSPIN = CPROP(8)
C
AMASS = MPROP(1)
C
R2     = TPROP(2)
CZ     = TPROP(3)
CS     = TPROP(4)
CA     = TPROP(5)
CR     = TPROP(7)
DZ     = TPROP(8)
U      = TPROP(11)
C
C Convert sign on alpha
C
RALPHA = CPROP(2)
RGAMMA = CPROP (3)
CG = TPROP (6)
ALPHA=-RALPHA
GAMMA=RGAMMA
C
C Initialize force values
C
FX = 0.00
FY = 0.00
FZ = 0.00
TY = 0.00
TZ = 0.00
C
IF(DEFL .LE. 0.00) THEN
  GOTO 1000
ENDIF
C
C Calculate normal loads due to stiffness (always .LE. zero)
C
FZDEFL = -DEFL*CZ

```

```

C
C   Calculate normal loads due to damping
C
FZDAMP = - 2.D0*SQRT(AMASS*CZ)*DZ*(DEFLD)
C
C   Calculate total normal force (fz)
C
FZ      = MIN (0.0D0, (FZDEFL + FZDAMP) )
C
C   Convert to kN and change sign
C
FZP = -FZ
C
C   Compute longitudinal force
C
IF(ABS(SLIP) .LE. ABS(SSTAR)) THEN
    FX = -CS*SLIP
ELSE
    FX1 = U*ABS(FZ)
    FX2 = (U*FZ)**2/(4.D0*ABS(SLIP)*CS)
    FX = -(FX1-FX2)*SIGN(1.0D0,SLIP)
ENDIF
C
C   Compute lateral force
C
CFY=A0
SHFY=A8*FZP+A9+A10*GAMMA
DFY=(A1*FZP+A2)*(1-A15*GAMMA**2)*FZP
IF(ALPHA+SHFY.LT.0.0)THEN
    DUMFY=-1.0
ELSE
    DUMFY=1.0
ENDIF
EFY=(A6*FZP+A7)*(1-(A16*GAMMA+A17)*DUMFY)
BFY=((A3*SIN(2*ATAN(FZP/A4)))*(1-A5*ABS(GAMMA)))/(CFY+DFY)
SVFY=A11*FZP+A12+(A13*FZP**2+A14*FZP)*GAMMA
PHIFY=(1-EFY)*(ALPHA+SHFY)+(EFY/BFY)*ATAN(BFY*(ALPHA+SHFY))
FYP=DFY*SIN(CFY*ATAN(BFY*PHIFY))+SVFY
C
C   Change sign
C
FY=FYP
C
C   Compute self aligning moment
C
CTZ=C0
SHTZ=C11*FZP+C12+C13*GAMMA
DTZ=(C1*FZP**2+C2*FZP)*(1-C18*GAMMA**2)

```

```

      IF (ALPHA+SHTZ.LT.0.0) THEN
        DUMTZ=-1.0
      ELSE
        DUMTZ=1.0
      ENDIF
      ETZ=(C7*FZP**2+C8*FZP+C9)*(1-(C19*GAMMA+C20)*DUMTZ)
      ETZ=ETZ/(1-C10*ABS(GAMMA))
      BTZ=((C3*FZP**2+C4*FZP)*(1-C6*ABS(GAMMA))*EXP(-C5*FZP))/(CTZ+DTZ)
      SVTZ=C14*FZP+C15+(C16*FZP**2+C17*FZP)*GAMMA
      PHITZ=(1-ETZ)*(ALPHA+SHTZ)+(ETZ/BTZ)*ATAN(BTZ*(ALPHA+SHTZ))
      TZP=DTZ*SIN(CTZ*ATAN(BTZ*PHITZ))+SVTZ
C
C   Convert to Nmm and change sign
C
      TZ = TZP*1000.0
C
C   Copy the calculated values for FX, FY, FZ, TY & TZ to FSAE
C   and TSAE arrays
C
1000 FSAE(1) = FX
      FSAE(2) = FY
      FSAE(3) = FZ
      TSAE(1) = 0.0
      TSAE(2) = 0.0
      TSAE(3) = TZ
      FPROP(1) = 0.0
      FPROP(2) = 0.0
C
      RETURN
      END

```

---

## B.3 The Harty tyre model subroutine

### TYR501

```

C MDI TYR501 : Concept Tyre Model
C
C
C   A Quick & Dirty Tyre Model which plugs in as the FIALA
C   model does, with a "TIRE" statement.
C
C   Unlike FIALA, critical slip angle is broadly independent
C   of load and initial cornering stiffness is strongly
C   load dependent.
C
C   These attributes better represent a modern radial tyre
C   than does either the FIALA or University of Arizona
C   model.
C

```

```

C The model does handle comprehensive slip. Lateral force
C generation is zero at peak longitudinal force slip ratio
C (typically about 20%) but returns to a value around one
C tenth of the peak lateral force as the wheel progresses
C beyond that limit. This may result in poor post-spin
C performance. The force generated with locked wheels is
C aligned with the wheel plane; this is incorrect.
C
C Longitudinal force generation is assumed to be symmetric
C for tractive and braking slip. This is not generally
C true beyond the critical slip ratio for real tyres but
C is reasonable up to that point. This tyre will over
C estimate longitudinal forces for tractive slip and
C slightly underestimate them for braking slip in the
C post-critical regions.
C
C -- 29th December 2000 --
C
C Camber thrust is included as for the motorcycle tire
C model using "taut string" logic. Lateral migration of
C the contact patch is also included, as for the motorcycle
C tyre model.
C
C Aligning Torque calculation includes the lateral force
C due to camber. This is not quite right as the camber
C force mechanism has no pneumatic trail associated with
C it. Pay attention if using this for motorcycle work;
C consider reworking it so that TZ does not include the
C camber force. The form of the aligning torque is a
C bit poor and would benefit from some more thought;
C pneumatic trail collapses linearly with lateral force.
C
C -- 10th January 2001 --
C
C Unsuitable Aligning Moment behaviour substantially improved
C for motorcycle use.
C
C --
C
C Relaxation Length is externally imposed as with the
C Fiala tyre.
C
C Tyre Data is taken from the tyre parameter file (.tpf)
C but note that not all the data is used. The other
C parameters are passed in via the UPARAMETERS argument
C on the TIRE statement inside an ADAMS deck.
C

```



```

C The model is quite empirical and has no basis in any sort
C of established fact or theory. It may or may not bear a
C passing resemblance to "Maltyre", a Malcolm Burgess model
C implemented at Lotus to the same end. I don't care, I
C did it all myself without a grown-up to help with the
C pointy bits.
C
C -- 24th April 2001 --
C
C Banner and zero parameter check added in IFLAG loop.
C
C -- 7th July 2001 --
C
C Improved representation of behaviour outside friction
C circle. Correct differentiation between lock and
C wheelspin in terms of force vector.
C
C -- 6th October 2004 --
C
C Improved aligning moment form - was significantly too high.
C Uses passed in parameter for Pneumatic Trail on-centre.
C Note that passed-in parameter can be negative, giving
C pneumatic "lead".
C
C -- 10th March 2006 --
C
C Pneumatic lead introduced for camber forces to match
C measured motorcycle data. Minor error in limit camber
C clipping corrected
C
C -- 30th March 2006 --
C
C Minor error with form of camber clipping (asymmetric)
C corrected.
C
C
C -- 16th February 2009 --
C
C Damian made the mistake of letting someone else have a go at
C his model and so I am attempting to migrate it to TYR501
C since the TIRSUB routine will become defunct at the next release.
C
C (Teena Gade)
C
C -- 7th September 2009 --
C
C Migration to TYR501 completed by DAH after Teena did all the
C nasty bits getting the right data into the right place.
C

```

```

C -- 22nd May 2013 --
C
C Sign error in forces carried over from original MSC TYR501
C sample file has led to erratic behaviour of TYR501 until
C pinned down, now fixed. Not that FORCES and TORQUE are the
C variables which actually deliver forces back to the solution.
C VARINF is associated with VPG Tire, a mode of usage I have
C never successfully invoked. VARINF information is of unknown
C provenance and should be used without checking.
C
C Also uncovered some strange behaviour of original TYR501
C that didn't allow it to run backwards - fixed with velocity
C sign check just before FORCES is returned. VARINF not
C corrected.
C
C Resulting bug with rolling resistance fixed, works
C correctly (has it ever done before?)
C
C --
C
C (c) DAH 24 Oct 1999-2013
C
      SUBROUTINE TYR501( NDEV, ISWTCH, JOBFLG, IDTYRE,
+                      TIME, DIS, TRAMAT, ANGTWC, VEL, OMEGA, OMEGAR,
+                      NDEQVR, DEQVAR, NTPAR, TYPARR,
+                      NCHTDS, CHTDST, ROAD, IDROAD,
+                      NROPAR, ROPAR, NCHRDS, CHRDS,
+                      FORCES, TORQUE, DEQINI, DEQDER, TYRMOD,
+                      NVAR, VARINF, NWORK, WRKARR,
+                      NIWORK, IWRKAR, IERR )
C
C Inputs:
      INTEGER          NDEV
      INTEGER          ISWTCH
      INTEGER          JOBFLG
      INTEGER          IDTYRE
      DOUBLE PRECISION TIME
      DOUBLE PRECISION DIS(3)
      DOUBLE PRECISION TRAMAT(3,3)
      DOUBLE PRECISION ANGTWC
      DOUBLE PRECISION VEL(3)
      DOUBLE PRECISION OMEGA(3)
      DOUBLE PRECISION OMEGAR
      INTEGER          NDEQVR
      DOUBLE PRECISION DEQVAR(NDEQVR)
      INTEGER          NTPAR
      DOUBLE PRECISION TYPARR(NTPAR)

```

```

      INTEGER          NCHTDS
      CHARACTER*256    CHTDST
      INTEGER          IDROAD
      INTEGER          NROPAR
      DOUBLE PRECISION ROPAR(NROPAR)
      INTEGER          NCHRDS
      CHARACTER*256    CHRST
C Outputs:
      DOUBLE PRECISION FORCES(3)
      DOUBLE PRECISION TORQUE(3)
      DOUBLE PRECISION DEQINI(NDEQVR)
      DOUBLE PRECISION DEQDER(NDEQVR)
      CHARACTER*256    TYRMOD
      INTEGER          NVAR
      DOUBLE PRECISION VARINF(NVAR)
      INTEGER          NWORK
      DOUBLE PRECISION WRKARR(NWORK)
      INTEGER          NIWORK
      INTEGER          IWRKAR(NIWORK)
      INTEGER          IERR
C
C
C
C Local Variables:
C
C Locals:
      INTEGER I
C      DOUBLE PRECISION C_SLIP
      DOUBLE PRECISION C_ALPHA
      DOUBLE PRECISION C_GAMMA
      DOUBLE PRECISION U1
      DOUBLE PRECISION U0
      DOUBLE PRECISION GAIN
      DOUBLE PRECISION R_LEN
      DOUBLE PRECISION URAD(3)
      DOUBLE PRECISION U
      DOUBLE PRECISION F(6)
      DOUBLE PRECISION FCP(3)
      DOUBLE PRECISION TCP(3)
      INTEGER          ARRPTR
      INTEGER          UMODE
      DOUBLE PRECISION RAD(3)
      DOUBLE PRECISION RADIUS
      INTEGER          NROAD
      DOUBLE PRECISION RCP(3)
      DOUBLE PRECISION RNORM(3)
      DOUBLE PRECISION SURFAC

```

```

      DOUBLE PRECISION CN
      DOUBLE PRECISION RDR
C     DOUBLE PRECISION CRR
      DOUBLE PRECISION CPMTX(3,3)
      DOUBLE PRECISION VCPLON
      DOUBLE PRECISION VCPLAT
      DOUBLE PRECISION VCPVRT
      DOUBLE PRECISION VLON
C     DOUBLE PRECISION ALPHA
      DOUBLE PRECISION ALPHA_L
      DOUBLE PRECISION KAPPA
      DOUBLE PRECISION KAPPA_L
      DOUBLE PRECISION GAMMA
      DOUBLE PRECISION FRCRAD
      DOUBLE PRECISION FRCVRT
      DOUBLE PRECISION FRCLON
      DOUBLE PRECISION FRCLAT
      DOUBLE PRECISION TRQALN
      DOUBLE PRECISION FZMAG
C
C -----
C -- Carried across from tirsuB --
      DOUBLE PRECISION FX, FY, FZ, TX, TY, TZ

      DOUBLE PRECISION SLIP, ALPHA, DEFL, DEFLD
      DOUBLE PRECISION R1, R2, CZ, CS, C_MX, CR, DZ, AMASS, WSPIN

      DOUBLE PRECISION ALPHA_C, Ay, By, R_LOAD, dB_dFz, B
      DOUBLE PRECISION SLIP_C, Ax, SLIP_M, FR_ELLIP, CP_LEN
      DOUBLE PRECISION LSLIP, USLIP, UNLRAD

      DOUBLE PRECISION THRSH, CAMB_C, CAMB_INC, A_INT, B_INT, C_INT
      DOUBLE PRECISION SLIPSQ, TAN_ALPHA_SQ, DIVISOR
      DOUBLE PRECISION FX1, FY1, FX2, FY2, ABSLIP, PTRAILC, PNOFFSET
      DOUBLE PRECISION PLEAD, FY_CAMBER, FYWAS
C -----
C
C
C Scaling parameters
C     DOUBLE PRECISION SCLRR
      DOUBLE PRECISION SCLFY
      DOUBLE PRECISION SCLMX
      DOUBLE PRECISION SCLMZ

C Drift array parameters

      DOUBLE PRECISION PLYFRC
      DOUBLE PRECISION CONFRC

```

```

        DOUBLE PRECISION PLYTRQ
        DOUBLE PRECISION CONTRQ

        LOGICAL          ERRFLG

C
C Road Declarations:
        INTEGER          MAXDIV
        PARAMETER        (MAXDIV = 10)
        INTEGER          N_T_SHAPE
        DOUBLE PRECISION T_SHAPE(2, MAXDIV)
        DOUBLE PRECISION EFFVOL
        DOUBLE PRECISION EFFPEN
        CHARACTER*256     ERRMSG
        CHARACTER*80      ERRTMP

C
        DOUBLE PRECISION STARTUP
        DOUBLE PRECISION OFF_GRND

C
C Useful Parameters:
        DOUBLE PRECISION ZERO_VAL
        PARAMETER        (ZERO_VAL = 0.00)
        DOUBLE PRECISION ONE
        PARAMETER        (ONE = 1.00)
        DOUBLE PRECISION ZERLIM
        PARAMETER        (ZERLIM = 1.0E-10)
        DOUBLE PRECISION TFULL
        PARAMETER        (TFULL = 0.5)
        DOUBLE PRECISION NO_FRC
        PARAMETER        (NO_FRC = 448)
        DOUBLE PRECISION WSPNMX
        PARAMETER        (WSPNMX = 5.0D-1)
        INTEGER          DYNAMIC
        PARAMETER        (DYNAMIC = 1)
        INTEGER          STATIC
        PARAMETER        (STATIC = 0)
        INTEGER          IORD
        PARAMETER        (IORD=0)
        INTEGER          IMODE

C
        include 'ac_tir_jobflg.inc'
        include 'abg_varptr.inc'
        include 'tyrHarty_501.inc'

C Functions:
        DOUBLE PRECISION DOT
        EXTERNAL          DOT

C

```

```

EXTERNAL      ROAD

LOGICAL        STAFLG
SAVE          STAFLG

DATA  STAFLG  /.FALSE./

      IERR = 0
C Read the tire property file during initialization:

      IF ( JOBFLG .EQ. INIT ) THEN
      CALL USRMES( .TRUE.,
+ ' ', 0,
+ 'INFO_NOPAD' )
      CALL USRMES( .TRUE.,
+ ' ', 0,
+ 'INFO_NOPAD' )
      CALL USRMES( .TRUE.,
+ '*****', 0,
+ 'INFO_NOPAD' )
      CALL USRMES( .TRUE.,
+ 'TYR501 Harty Model: Compiled 12 Aug 2013', IDTYRE,
+ 'INFO_NOPAD' )
      CALL USRMES( .TRUE.,
+ '*****', 0,
+ 'INFO_NOPAD' )
      CALL USRMES( .TRUE.,
+ ' ', 0,
+ 'INFO_NOPAD' )
      CALL USRMES( .TRUE.,
+ ' ', 0,
+ 'INFO_NOPAD' )
      CALL RPF501( NCHTDS, CHTDST, IDTYRE, NTPARR, TYPARR )
      ENDIF

C Set DEQINI:
      IF ( JOBFLG .EQ. INQUIRE ) THEN
      DEQINI(1) = 0.0D0
      DEQINI(2) = 0.0D0
      ENDIF

      IF (JOBFLG .NE. ENDSIM) THEN

C Decode TYPARR Array:
      UMODE = NINT( TYPARR( use_mode ) )

      UNLRAD = TYPARR( unloaded_radius )
      TIREW  = TYPARR( width )
      TIREK  = TYPARR( vertical_stiffness )

```

```

TIREC  = TYPARR( vertical_damping )
CR      = TYPARR( rolling_resistance )
CA      = TYPARR( calpha )
C_GAMMA = TYPARR( cgamma )
U_MIN   = TYPARR( umin )
U_MAX   = TYPARR( umax )
R_LEN   = TYPARR( relaxation_length )
ALPHA_C = TYPARR( alpha_critical )
Ay      = TYPARR( curvature_factor_angle )
By      = TYPARR( scale_factor_lateral )
R_LOAD  = TYPARR( rated_load )
dB_dFz  = TYPARR( scale_factor_dim )
SLIP_C  = TYPARR( slip_ratio_critical )
Ax      = TYPARR( curvature_factor_ratio )
PTRAILC = TYPARR( pneum_trailing_scaling )
PLEAD   = TYPARR( pneumatic_lead_camber )
THRSH   = TYPARR( limit_camber_onset_fric )

C
N_T_SHAPE = NINT( TYPARR( n_shape ) )

C
IF ( JOBFLG .EQ. INIT .OR. JOBFLG .EQ. RESET ) THEN
C   -- Debug only - check we're getting what we think --
C -----
C   WRITE(*,*) 'UNLRAD ', UNLRAD, ', TIREK ', TIREK
C   WRITE(*,*) ', TIREW ', TIREW
C   WRITE(*,*) 'TIREC', TIREC, ', CR ', CR
C   WRITE(*,*) 'CA ', CA
C   WRITE(*,*) 'C_GAMMA ', C_GAMMA, 'U_min ', U_min
C   WRITE(*,*) 'U_max ', U_max, 'R_LEN ', R_LEN
C   WRITE(*,*) 'ALPHA_C ', ALPHA_C, 'Ay ', Ay
C   WRITE(*,*) 'By ', By, 'R_LOAD ', R_LOAD
C   WRITE(*,*) 'dB_dFz ', dB_dFz
C   WRITE(*,*) 'SLIP_C ', SLIP_C
C   WRITE(*,*) 'Ax ', Ax
C   WRITE(*,*) 'PTRAILC ', PTRAILC, 'PLEAD ', PLEAD, 'THRSH ', THRSH
C -----
C
C
C   ENDIF
C

C=====
C
C -- All this is standard TYR501 stuff
C   - dynamic or static
C   - soft start to calculations

```

```

C      - road/tyre interaction including profile
C      - states for the tyre model
C
C=====

C Initialize mode (STATIC or DYNAMIC)
      IMODE = DYNAMIC
      IF ( ISWTCH .EQ. 0 ) IMODE = STATIC

C Set flag for quasi-static analyses

      IF ( ISWTCH .EQ. 2 ) STAFLG = .TRUE.

C Setup Smoothing Function:
C
C The MDI tire models include a feature for smoothing the
C tire forces around time=0.0. So, for example, if there's
C some initial slip angle at time=0.0, the lateral force
C builds up slowly instead of acting like a step input.
C This helps the integrator get started. UMODE comes
C from the tire property file.

      IF(UMODE .GE. 2 .AND.(.NOT.STAFLG) )THEN
        CALL STEP(TIME,ZERO_VAL,ZERO_VAL,TFULL,ONE,0,
+          STARTUP,ERRFLG)
      ELSE
        STARTUP = ONE
      ENDIF

C Setup The Tire Carcase (Cross Section) Shape
C for use by the durability tire road contact
C model:

      IF (N_T_SHAPE.EQ.0) THEN
        T_SHAPE(1,1) = 0.D0
        T_SHAPE(2,1) = 0.D0
      ELSE
        ARRPTR = SHAPE
        DO I=1,N_T_SHAPE
          T_SHAPE(1,I)=TYPARR(ARRPTR)
          T_SHAPE(2,I)=TYPARR(ARRPTR+1)
          ARRPTR = ARRPTR + 2
        ENDDO
      ENDIF

C Offset rolling radius - this is in the original code but
C I don't know why.
C
C      UNLRAD = UNLRAD + SCLRR

```



```

C Call ROAD routine
C
C The road routine calculates the local road normal, the
C road contact point (contact patch location), the
C local surface coefficient of friction and the tire's
C vertical deflection. The STI passes in the name of
C the subroutine to be called. Hence "ROAD" is just a
C placeholder.

      CALL ROAD(JOBFLG, IDTYRE,
&             TIME, DIS, TRAMAT,
&             IDROAD, NROPAR, ROPAR, NCHRDS, CHRST,
&             N_T_SHAPE, T_SHAPE, UNLRAD, TIREW,
&             NROAD, EFFVOL, EFFPEN, RCP,
&             RNORM, SURFAC, IERR, ERRMSG )

C Call the TIRE Kinematics Routine (ACTCLC):
C
C The ACTCLC routine calculates the slip angle (ALPHA),
C inclination (camber) angle (GAMMA), longitudinal slip
C (KAPPA), the longitudinal (VCPLON) and lateral (VCPLAT)
C slip velocities, the longitudinal velocity of wheel
C center (VLON), the vertical velocity of the wheel center
C normal to the road (VCPVRT), the unit vector directed
C from the wheel center to the contact patch (URAD) expressed
C in global coordinates, and the transformation
C matrix from SAE contact patch coordinates (CPMTX) to
C global (ground part) coordinates.
C
C Calculate the tire kinematics if:
C   The tire is in contact with road (e.g. not flying)
C
C   - and -
C
C   The job is normal execution or differencing for
C   derivatives.
C
      IF(
.        NROAD .EQ. 1 .AND. IERR .NE. 3
.        .AND.
.        (
.          JOBFLG .EQ. NORMAL .OR.
.          JOBFLG .EQ. DIFF
.        )
.      ) THEN

```

```

        RADIUS = UNLRAD - EFFPEN

        CALL ACTCLC( TRAMAT, VEL, OMEGA, OMEGAR, RADIUS, RNORM,
&                  VLON, VCPLON, VCPLAT, VCPVRT,
&                  ALPHA, GAMMA, KAPPA,
&                  URAD, CPMTX)

C
C Lag The slip angle to for tire relaxation effects:
C
C d( Alpha_lagged )/dt = (VLON/Relaxation_Length)*( Alpha - Alpha_lagged )
C
C If the relaxation length is less than 1e-4 Meters, then don't lag the
C slips.
C
      IF ( R_LEN .LT. 1D-4 .OR.
&        IMODE .EQ. STATIC ) THEN

        ALPHA_L = ALPHA
        KAPPA_L = KAPPA

      ELSE

        GAIN      = ABS(VLON)/R_LEN
        ALPHA_L   = DEQVAR(1) + DEQINI(1)
        KAPPA_L   = DEQVAR(2) + DEQINI(2)
        DEQDER(1) = GAIN*(ALPHA - ALPHA_L)
        DEQDER(2) = GAIN*(KAPPA - KAPPA_L)

      ENDIF

C=====
C
C -- End of the Standard TYR501 Stuff --
C
C=====

C -- Now the tyre modelling proper can start --
C All forces calculated in SAE reference frame and transformed to TYDEX
format
C afterwards - ease of continuity with previous model (also true of
reference
C TYR501 model provided by MSC)

```

```

C -- SAE Vertical Force like original tirsuB calculations --

C      Normal Loads; simple calculations as with sample tirsuB.f;
C      Penetrations to hub are not accounted for.

C -- Calculate normal loads due to stiffness (always .LE. zero) --

      FZDEFL = -EFFPEN*TIREK

C -- Calculate normal loads due to damping --

      FZDAMP = -VCPVRT*TIREC

C -- Note the startup modification that was present in the tirsuB
model is
C      no longer needed --

C -- Sum for total normal force --

      FZ = MIN (0.000, (FZDEFL + FZDAMP) )

      IF ( IMODE .EQ. DYNAMIC ) THEN

C      Coefficient of friction as function of combined slip:

      U = U_MAX+SQRT(KAPPA_L**2+(TAN(ALPHA_L))**2)*(U_MIN-U_MAX)

C      Modify coefficient of friction based on road surface
C      factor:

      U = U * SURFAC

C Longitudinal Loads

C -- We're working in percent --

      SLIP=KAPPA*100

      IF(ABS(SLIP) .LE. ABS(SLIP_C)) THEN

```

```

C -- Exponential Rise (1-e^-x) below critical slip ratio --
      FX = (1-EXP(-Ax*ABS(SLIP)/SLIP_C))*U*ABS(FZ)*SIGN(1.0D0,SLIP)

      ELSE

C -- Linear Decay to Sliding Friction above critical slip ratio --
      FX = ABS(FZ)*(1-EXP(-Ax))*U*SIGN(1.0D0,SLIP)

      ENDIF

C Lateral force and aligning torque (FY & TZ)

C -- Scale Factor Diminished with Load FZ --
      B = By+(ABS(FZ)-R_LOAD)*dB_dFz

C -- We're working in degrees --
      ALPHA_L=ALPHA_L*45/ATAN(1.0)

C -- Don't let alpha go beyond 80 - the TAN functions go kinda wild --
      IF(ALPHA_L.GT.80.) THEN
        ALPHA_L = 80.0
      ENDIF
      IF(ALPHA_L.LT.-80.) THEN
        ALPHA_L = -80.0
      ENDIF

      IF(ABS(ALPHA_L) .LE. 1.D-10) THEN

        FY = 0.D0
        TZ = 0.D0

      ELSE IF( ABS(ALPHA_L) .LE. ALPHA_C ) THEN

C -- As for longitudinal forces, Exponential Rise (1-e^-x) below
C   critical slip angle --

C -- This line contains an even number of minus-sign errors --
      FY = (1-EXP(-Ay*ABS(ALPHA_L)/ALPHA_C))
+        *U*B*FZ*SIGN(1.0D0,ALPHA_L)

```

```

ELSE

C -- As for longitudinal forces, Linear Decay to Sliding Friction
C   above critical slip ratio --

C   FY = FZ*U*B*SIGN(1.0D0,SLIP)*(1-(ABS(ALPHA_L)-ALPHA_C)/800)

C -- Simplified - ADAMS handles transition from static to sliding
C   friction in the calling routine --

      FY = FZ*(1-EXP(-Ay))*U*B*SIGN(1.0D0,ALPHA_L)

ENDIF

C Aligning Torque based on intermediate FY excluding camber force.

C -- Contact Patch Length --

R1=UNLRAD
R2=TIREW

CP_LEN = (R1**2 - (R1-ABS(FZ)/TIREK)**2)**0.5 * 2.0

IF(ABS(ALPHA_L) .GT. 1.D-10) THEN

  IF( ABS(ALPHA_L) .LE. ALPHA_C ) THEN

    TZ = -FY*CP_LEN/6*(1-ABS(ALPHA_L)/ALPHA_C)*PTRAILC

C -- Divisor is because lever arm is not the entire contact patch
length. --

C -- Parameter PTRAILC should be set to 1.0 for tyres with recetangular
C contact patches (i.e. car tyres) and 0.5 for tyres with elliptical
C contact patches (i.e. motorcycle tyres.) --

ELSE

  TZ = 0.0

ENDIF

ENDIF

```

```

C    -- Add camber force to FY - "Taut String" --

C    DAH Sign of Camber Component Changed 13-11-00
C    FY = FY - FZ * TAN(GAMMA)

C    CAMBER=GAMMA

C    -- "Clipped" Camber model - improved limit behaviour --
C    DAH 10-01-00

C    -- THRSH represents aggression of departure at limit; high value
C    implies high limit & aggressive departure, lower value implies
C    progression.

C    -- was hard-coded, now user parameter
C    THRSH=0.8

    IF (ABS(GAMMA) .LT. ATAN(THRSH*U/C_GAMMA)) THEN

C    -- Camber term now held separate for aligning moment calculation

        FY_CAMBER = - FZ * TAN(GAMMA) * C_GAMMA

    ELSE

        CAMB_C=ATAN(THRSH*U/C_GAMMA)
        CAMB_INC=ABS(GAMMA)-CAMB_C

        A_INT=(1/(1-THRSH*C_GAMMA))/(COS(ATAN(THRSH*U/C_GAMMA)))**2
        B_INT=-(1-THRSH)*U*C_GAMMA

C    -- Needed when C_GAMMA is not equal to unity --

        C_INT= - FZ * TAN(CAMB_C) * C_GAMMA /
&            (
&            SIGN(1.,CAMB_C)*FZ*B_INT*(1-EXP(-A_INT*CAMB_INC)) -
&            SIGN(1.,CAMB_C)*THRSH*U*FZ*C_GAMMA
&            )
C    MUX=C_INT

        FY_CAMBER =SIGN(1.,GAMMA)*FZ*B_INT*(1-EXP(-A_INT*CAMB_INC)) -
&            SIGN(1.,GAMMA)*THRSH*U*FZ*C_GAMMA * C_INT

    ENDIF

```

```

FY = FY + FY_CAMBER
FYWAS = FY

C    Mitigate FY depending on "Friction Ellipse"

FR_ELLIP = (FX/(FZ*U))**2 + (FY/(FZ*U*B))**2

X_SIGN = SIGN(1.0D0,FX)
Y_SIGN = SIGN(1.0D0,FY)

IF ( FR_ELLIP .GT. 1.0 ) THEN
    LSLIP=50.0
    USLIP=100.0
    ABSLIP=ABS(SLIP)

C    -- Friction Ellipse treatment for comprehensive slip below
C    critical slip ratio - revised over previous calculations
C    to preserve ratio of FX, FY but bring them inside the
C    friction ellipse --

DIVISOR=1 + (FY/FX)**2/B**2

FX1 = ( (U*FZ)**2 / DIVISOR )**0.5 * X_SIGN

C    -- Alternative term; longitudinal force is preserved at the
C    expense of lateral; seems intuitively more correct but
C    produces apparently poorer results. --
C    FX1 = FX

FY1 = ( ( 1-(FX1/(FZ*U))**2 ) * (FZ*U*B)**2 )**0.5*Y_SIGN

C    -- Revised formulation for highest slip ratios arrived at
C    by consideration of contact patch velocity. Gives pleasing

```

```

C      results for wheels locked and wheels spinning cases. Note
C      conversion of ALPHA from degrees back to radians for
C      this calculation and SLIP back from percent. --

      SLIPSQ = (SLIP/100)**2
      TAN_ALPHA_SQ=(TAN(ALPHA_L*ATAN(1.0)/45))**2

      DIVISOR=( 1 + TAN_ALPHA_SQ/SLIPSQ )

      FX2 = ( (U*FZ)**2 / DIVISOR )**0.5 * X_SIGN
      FY2 = ((1-(FX2/(FZ*U))**2)*(FZ*U)**2)**0.5*Y_SIGN

C      -- Smear between two models using slip ratio --

      CALL STEP(ABSLIP,LSLIP,FX1,USLIP,FX2,IORD,FX,ERRFLG)
      CALL STEP(ABSLIP,LSLIP,FY1,USLIP,FY2,IORD,FY,ERRFLG)
C      FX=FX1
C      FY=FY1

C      -- Mitigate Camber forces too, for subsequent aligning moment
calculations

C      CALL STEP(ABSLIP,LSLIP,FY1,USLIP,FY2,IORD,FY_CAMBER,ERRFLG)

C      Is this right? Doesn't it significantly corrupt aligning torque for
C      a locking wheel at a high slip angle?

      FY_CAMBER=FY_CAMBER*FY/FYWAS

      ENDIF

C C -- The real MUX and MUY; all others are for debug only --
C C      MUX = (FX/(FZ*U))
C C      MUY = (FY/(FZ*U*B))

C      Rolling resistance moment (TY) as FIALA Tyre:

C      IF ( OMEGAR .GE. 0.0 ) THEN
C
C      TY = -CR * FZ

```



```

C
C   ELSE
C
C       TY = CR * FZ
C
C   ENDIF

C   No need for loop above - velocity change below takes care of it

TY = CR * FZ


C   Compute righting moment due to lateral Contact Patch Shift (TX)
C   Use CA as "shape factor" to add to or subtract righting moment
C   from ADAMS' Toroidal assumption.  CA > 1 = fatter than toroid
C   CA < 1 = more like blade.


C   Lateral Contact Patch Shift clips at tyre extremity
C
C   Add aligning torque based on lateral offset of contact patch and
C   longitudinal forces to give "stand up under braking" behaviour for
C   motorcycles or tramlining for cars.


PNOFFSET = 2 * GAMMA * R2/2 * (CA - 1)
IF (ABS(PNOFFSET) .LT. R2/2 ) THEN
    TX = FZ * PNOFFSET
    TZ = TZ - FX * PNOFFSET
ELSE
    TX = FZ * R2/2 * SIGN(1.,PNOFFSET)
    TZ = TZ - FX * R2/2 * SIGN(1.,PNOFFSET)
ENDIF


C   Measured data shows evidence of significant "pneumatic lead" on
C   camber force data, aligning moment further modified to reflect
this.
C   Real data shows small dependency on load, some dependency on
camber
C   angle at low cambers; constant lead formulation neglects load
C   dependency and may overestimate torques at small cambers.
However,
C   camber forces are low and so torques are low too.


TZ = TZ + FY_CAMBER*PLEAD

```

```

        ELSE
c  For static equilibrium zero the forces.

        FX = ZERO_VAL
        FY = ZERO_VAL
        TX = ZERO_VAL
        TY = ZERO_VAL
        TZ = ZERO_VAL

ENDIF

C=====
C
C  -- After the tyre model giving forces & moments in SAE co-ordinates,
C  the long and arduous business of giving them back to ADAMS. Is this
C  *really* progress? --
C
C  -- Below here, all is standard TYR501 code except for sign mapping in
C  FCP and TCP --
C
C=====

C  Apply the start-up transient smoothing and force the
C  all other tire forces to zero when the vertical force goes to
C  zero (e.g. when the tire is flying).

        FZMAG = DABS(FZ)

        CALL STEP(FZMAG,ZERO_VAL,ZERO_VAL,NO_FRC,ONE,
+             0,OFF_GRND,ERRFLG)

        IF( IMODE .EQ. DYNAMIC ) THEN

            FCP(1) = -FX
            FCP(2) = FY
            FCP(3) = FZ

            TCP(1) = TX
            TCP(2) = TY
            TCP(3) = TZ

```

```

ELSE

    FCP(1) = 0.0
    FCP(2) = 0.0
    FCP(3) = FZ

    TCP(1) = 0.0
    TCP(2) = 0.0
    TCP(3) = 0.0

ENDIF

C Transform the contact patch forces and moments to hub
C coordinates:
C
C Inputs:
C   FCP tire forces at contact patch in SAE contact patch
C   coordinates.
C
C   TCP Tire moments (torques) at contact patch in SAE
C   contact patch coordinates.
C
C   RAD Tire radius vector express in global (ground)
C   coordinates.
C
C   CPMTX Transformation from SAE contact patch coordinates
C   to ground.
C
C Outputs:
C
C   F tire forces at hub in global coordinates
C   T Tire moments (torques) at hub (wheel center) in
C   global coordinates.
C
C   {F} = [CPMTX]{FCP}
C
C   {T} = [CPMTX]{TCP} + {RAD} X {F}
C
C   CALL SVEC( RAD, URAD, RADIUS, 3 )
C   CALL XCP2HB(FCP, TCP, RAD, CPMTX, F(1), F(4) )

C Transformation of forces/torques from global to wheelcarrier
C axes
C   CALL M3T1(FORCES, TRAMAT, F(1) )
C   CALL M3T1(TORQUE, TRAMAT, F(4) )
C
C   XVEL=VEL(1)
C   IF (XVEL .GE. 0.0) THEN
C       WRITE(*,*) 'Reverse'
C       FORCES(1)=-FORCES(1)

```

```

        TORQUE(2)=-TORQUE(2)
C      WRITE(*,*) FORCES
C      WRITE(*,*) TORQUE
      ELSE
C      WRITE(*,*) 'Forwards'
C      WRITE(*,*) FORCES
C      WRITE(*,*) TORQUE
      ENDIF

C *****
C Assigning output quantities to VARINF array
C *****

      IF(NVARS .GE. 75) THEN

C Contact Patch Forces/Torques
C VARINF in ISO coordinates, FCP and TCP in SAE coordinates
C - Note the VARINF array is for examination inside A/View and
C does not represent the forces passed back to the solver
C - Arrays FORCES and TORQUE are the ones which influence
C the solution

        VARINF(FX_ISO_PTR) = FCP(1)
        VARINF(FY_ISO_PTR) = -FCP(2)
        VARINF(FZ_ISO_PTR) = -FCP(3)

        VARINF(MX_ISO_PTR) = TCP(1)
        VARINF(MY_ISO_PTR) = -TCP(2)
        VARINF(MZ_ISO_PTR) = -TCP(3)

C Derivatives of state variables

        CALL COPYD(VARINF(DUDT_PTR), DEQDER(1), 2)

C Slip quantities

C Kinematic:

        VARINF(SLIPX_PTR) = KAPPA
        VARINF(SLIPI_PTR) = -ALPHA

C Dynamic:
        VARINF(SLIPX_D_PTR) = KAPPA_L
        VARINF(SLIPI_D_PTR) = -ALPHA_L

```

## C Friction coefficients

```

VARINF(MUXTYR_PTR) = DABS(FRCLON/FCP(3))
VARINF(MUYTYR_PTR) = DABS(FRCLAT/FCP(3))

```

## C Tire characteristics

```

VARINF(PT_PTR) = 0.D0
VARINF(MZR_PTR) = 0.D0
VARINF(S_PTR) = 0.D0
VARINF(SIGKPO_PTR) = 0.D0
VARINF(SIGALO_PTR) = 0.D0
VARINF(MGYR_PTR) = 0.D0
VARINF(SVYKAP_PTR) = 0.D0
VARINF(SVX_PTR) = 0.D0
VARINF(SVY_PTR) = 0.D0

```

## C Contact Point

```

CALL COPYD(VARINF(RCP1_PTR), RCP, 3)

```

## C Road Normal

```

CALL COPYD(VARINF(RNORM1_PTR), RNORM, 3)

```

## C Surface Friction

```

VARINF(SURFAC1_PTR) = SURFAC
VARINF(SURFAC2_PTR) = SURFAC
VARINF(SURFAC3_PTR) = SURFAC

```

## C Tire kinematics

```

VARINF(CAMB_PTR) = GAMMA
VARINF(EFFPEN_PTR) = EFFPEN
VARINF(VCPVRT_PTR) = VCPVRT
VARINF(RADIUS_PTR) = RADIUS
VARINF(VCPLOM_PTR) = VCPLOM
VARINF(VCPLOT_PTR) = VCPLOT
VARINF(VLOM_PTR) = VLOM
ELSE
CALL ZERO(VARINF(1), NVAR)
IERR = 2
ERRMSG = 'TYR501: Incorrect Dimension on VARINF Array'
ENDIF

```

```

C *****
C Use these values if tire is FLYING
C          ^^^^^

      ELSE
        CALL ZERO(VARINF(1), NVAR)
        IF(NVAR .GE. 75) THEN
          CALL COPYD(VARINF(DUDT_PTR), DEQDER(1), 2)
          VARINF(RADIUS_PTR) = UNLRAD
        ELSE
          CALL ZERO(VARINF(1), NVAR)
          IERR = 2
          ERRMSG = 'TYR501: Incorrect Dimension on VARINF Array'
        ENDIF
      ENDIF
    ELSE
      CALL ZERO(VARINF(1), NVAR)
    ENDIF

C Error Handling

      IF(IERR .EQ. 0) THEN
        TYRMOD = 'TYR501 -> Harty Tyre Model '
      ELSE
        TYRMOD = ERRMSG(1:256)
      ENDIF
C
      RETURN

      END

```

## RPF501

```

      SUBROUTINE RPF501( NCHTDS, CHTDST, TIR_ID, NTYPAR, TYPARR )

```

```

c Copyright (C) 2000-1999
c By Mechanical Dynamics, Inc. Ann Arbor, Michigan
c
c All Rights Reserved, This code may not be copied or
c reproduced in any form, in part or in whole, without the
c explicit written permission of Mechanical Dynamics, Inc.
c
c DESCRIPTION:
c
c Reads property file for user tire model based on
c the Fiala Tire model and initializes the
c tire parameter array (TYPARR).

```

c

c ARGUMENT LIST:

c

name	type	storage	use	description
NCHTDS	I.S.	-	R	Number of characters in tire property file name.
CHTDST	C.A.		R	Tire property file name
TIR_ID	I.S.	1	R	Tire GFORCE id
NTYPAR	I.S.	1	R	Dimension of TYPARR
TYPARR	D.A.	NTYPAR	E	Tire parameter array

c \*\*\* Legend: I integer S scalar R referenced  
 c D double precision A array E evaluated  
 c C character

C Inputs:

INTEGER NCHTDS  
 CHARACTER\*(\*) CHTDST  
 INTEGER TIR\_ID  
 INTEGER NTYPAR

C Outputs:

DOUBLE PRECISION TYPARR( NTYPAR )

C Locals:

C Units conversions:

CHARACTER\*(12) UNITS(5)  
 DOUBLE PRECISION CV2MDL(5)  
 DOUBLE PRECISION CV2SI(5)  
 DOUBLE PRECISION FCVSI  
 DOUBLE PRECISION LCVSI  
 DOUBLE PRECISION MCVSI  
 DOUBLE PRECISION ACVSI  
 DOUBLE PRECISION TCVSI

c Fiala Property File Map

INCLUDE 'tyrHarty\_501.inc'

C RTO variables:

INTEGER RETURN\_VAL  
 DOUBLE PRECISION TMPREAL

C Shape Array RTO Stuff:

```
DOUBLE PRECISION TMP1, TMP2
INTEGER          N_NODES
INTEGER          ARRPTR
CHARACTER*80     FORM
INTEGER FLEN
CHARACTER*80     TABLE
INTEGER TLEN
```

```
LOGICAL          ERRFLG
CHARACTER*80     MESSAG
```

c+-----\*

c

c Open the file:

```
CALL RTO_OPEN_FILE_F2C ( CHTDST, NCHTDS, RETURN_VAL )
```

```
ERRFLG = RETURN_VAL .EQ. 0
```

```
MESSAG = 'Harty Tyre 501: No Error opening tire property file.'
```

```
CALL ERRMES ( ERRFLG, MESSAG, TIR_ID, 'STOP' )
```

c Read [UNITS] block from property file:

```
c Parameters in the property file may be given in any consistent
c set of units. The [UNITS] block identifies those units.
c During evaluation, however, SI Units are used. So as parameters
c are read from the property file they are converted
c to SI units.
```

c SI unit system.

c LENGTH = meter

c FORCE = newton

c ANGLE = radians

c MASS = kg

c TIME = second

```
c UNITS(1)-> FORCE UNITS(2)-> MASS UNITS(3)-> LENGTH
```

```
c UNITS(4)-> TIME UNITS(5)-> ANGLE
```

```
CALL ATRTOU( TIR_ID, UNITS )
```

```
CALL ACUNFN( UNITS, CV2MDL, CV2SI )
```

```
FCVSI = CV2SI(1)
```

C Force Conversion

```
MCVSI = CV2SI(2)
```

C Mass Conversion

```
LCVSI = CV2SI(3)
```



```

C   Length Conversion
      TCVSI = CV2SI(4)
C   Time Conversion
      ACVSI = CV2SI(5)
C   Angle Conversion

C***** TIRPRP POPULATION *****

c Read [MODEL] block:

      CALL RTO_READ_REAL_F2C
      (
      .   'MODEL', 5, 'USE_MODE', 8,
      .   TYPARR( USE_MODE ), RETURN_VAL
      . )

      ERRFLG = RETURN_VAL .EQ. 0
      CALL ERRMES( ERRFLG ,
      .   'Harty Tyre 501: No Use_mode?'
      .   ,TIR_ID,'STOP')

c Read [DIMENSION] block:
      CALL RTO_READ_REAL_F2C
      (
      .   'DIMENSION', 9, 'UNLOADED_RADIUS', 15,
      .   TMPREAL, RETURN_VAL
      . )

      ERRFLG = RETURN_VAL .EQ. 0
      CALL ERRMES( ERRFLG,
      .   'Harty Tyre 501: No UNLOADED_RADIUS?'
      .   ,TIR_ID,'STOP')

      TYPARR( UNLOADED_RADIUS ) = TMPREAL * LCVSI

      CALL RTO_READ_REAL_F2C
      (
      .   'DIMENSION', 9, 'WIDTH', 5,
      .   TMPREAL, RETURN_VAL
      . )

      ERRFLG = RETURN_VAL .EQ. 0
      CALL ERRMES( ERRFLG, 'Harty Tyre 501: No WIDTH?'
      .   ,TIR_ID,'STOP')

      TYPARR( WIDTH ) = TMPREAL * LCVSI

```

```

c Read [PARAMETER] block

      CALL RTO_READ_REAL_F2C
      . (
      . 'PARAMETER', 9, 'VERTICAL_STIFFNESS', 18,
      . TMPREAL, RETURN_VAL
      . )

      ERRFLG = RETURN_VAL .EQ. 0
C-----
      CALL ERRMES( ERRFLG, 'Harty Tyre 501: No VERTICAL_STIFFNESS?'
      . ,TIR_ID,'STOP')

      TYPARR( VERTICAL_STIFFNESS ) = TMPREAL * (FCVSI / LCVSI)

      CALL RTO_READ_REAL_F2C
      . (
      . 'PARAMETER', 9, 'VERTICAL_DAMPING', 16,
      . TMPREAL, RETURN_VAL
      . )

      ERRFLG = RETURN_VAL .EQ. 0
      CALL ERRMES( ERRFLG, 'Harty Tyre 501: No VERTICAL_DAMPING?'
      . ,TIR_ID,'STOP')

      TYPARR( VERTICAL_DAMPING ) = TMPREAL * (FCVSI * TCVSI / LCVSI)

      CALL RTO_READ_REAL_F2C
      . (
      . 'PARAMETER', 9, 'ROLLING_RESISTANCE', 18,
      . TMPREAL, RETURN_VAL
      . )

      ERRFLG = RETURN_VAL .EQ. 0
      CALL ERRMES( ERRFLG,
      . 'Harty Tyre 501: No ROLLING_RESISTANCE?',
      . TIR_ID,'STOP')

      TYPARR( ROLLING_RESISTANCE ) = TMPREAL

      CALL RTO_READ_REAL_F2C
      . (
      . 'PARAMETER', 9, 'CMX', 3,
      . TMPREAL, RETURN_VAL
      . )

```

```

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'rpf501: CMX undefined.',
. TIR_ID,'STOP')

TYPARR( CMX ) = TMPREAL

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'CGAMMA', 6,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No CGAMMA?',
. TIR_ID,'STOP' )

TYPARR( CGAMMA ) = TMPREAL * (FCVSI / ACVSI)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'UMIN', 4,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No UMIN?',
. TIR_ID,'STOP')

TYPARR( UMIN ) = TMPREAL

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'UMAX', 4,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No UMAX?',
. TIR_ID,'STOP')

TYPARR( UMAX ) = TMPREAL

```

```

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'RELAXATION_LENGTH', 17,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No RELAXATION_LENGTH?',
. TIR_ID,'STOP')

TYPARR( RELAXATION_LENGTH ) = ABS(TMPREAL * LCVSI)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'ALPHA_CRITICAL', 14,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No ALPHA_CRITICAL?',
. TIR_ID,'STOP')

TYPARR( ALPHA_CRITICAL ) = ABS(TMPREAL * ACVSI)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'CURVATURE_FACTOR_ANGLE', 22,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No CURVATURE_FACTOR_ANGLE?',
. TIR_ID,'STOP')

TYPARR( curvature_factor_angle ) = ABS(TMPREAL)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'SCALE_FACTOR_LATERAL', 20,
. TMPREAL, RETURN_VAL
. )

```

```

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No SCALE_FACTOR_LATERAL?',
. TIR_ID,'STOP')

TYPARR( SCALE_FACTOR_LATERAL ) = ABS(TMPREAL)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'RATED_LOAD', 10,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No RATED_LOAD?',
. TIR_ID,'STOP')

TYPARR( rated_load ) = ABS(TMPREAL * MCVSI)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'SCALE_FACTOR_DIM', 16,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No SCALE_FACTOR_DIM?',
. TIR_ID,'STOP')

TYPARR( scale_factor_dim ) = ABS(TMPREAL)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'SLIP_RATIO_CRITICAL', 19,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No SLIP_RATIO_CRITICAL?',
. TIR_ID,'STOP')

```

```

TYPARR( slip_ratio_critical ) = ABS(TMPREAL)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'CURVATURE_FACTOR_RATIO', 22,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No CURVATURE_FACTOR_RATIO?',
. TIR_ID,'STOP')

TYPARR( curvature_factor_ratio ) = ABS(TMPREAL)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'PNEUM_TRAILING_SCALING', 22,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No PNEUM_TRAILING_SCALING?',
. TIR_ID,'STOP')

TYPARR( pneum_trailing_scaling ) = ABS(TMPREAL)

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'PNEUMATIC_LEAD_CAMBER', 21,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No PNEUMATIC_LEAD_CAMBER?',
. TIR_ID,'STOP')

TYPARR( pneumatic_lead_camber ) = ABS(TMPREAL * LCVSI)

```

```

CALL RTO_READ_REAL_F2C
. (
. 'PARAMETER', 9, 'LIMIT_CAMBER_ONSET_FRIC', 23,
. TMPREAL, RETURN_VAL
. )

ERRFLG = RETURN_VAL .EQ. 0
CALL ERRMES( ERRFLG,
. 'Harty Tyre 501: No LIMIT_CAMBER_ONSET_FRIC?',
. TIR_ID, 'STOP')

TYPARR( limit_camber_onset_fric ) = ABS(TMPREAL * ACVSI)

n_nodes = 0
arrptr = shape

C READ [SHAPE] BLOCK IF IT EXISTS:

CALL RTO_START_TABLE_READ_F2C
. (
. 'SHAPE', 5, FORM, FLEN, RETURN_VAL
. )

IF ( RETURN_VAL .EQ. 1 ) THEN

800    CONTINUE

CALL RTO_READ_TABLE_LINE_F2C( TABLE, TLEN, RETURN_VAL )
if ( return_val .eq. 1.and. tlen .gt. 3 ) then

    call act_line_parse (table, tmp1, tmp2, tlen)

    if ( n_nodes .lt. max_shape .and. tlen .eq. 2) then
        n_nodes = n_nodes + 1
        typarr( arrptr ) = tmp1
        typarr( arrptr + 1) = tmp2
        arrptr = arrptr + 2
    else

        if ( n_nodes .gt. max_shape) then
CALL ERRMES( .true.,
. 'Harty Tyre 501: Shape table has more than 10 nodes',
. TIR_ID, 'STOP' )
endif

```

```

        if (tlen .ne. 2) then
            CALL ERRMES( .true.,
                . 'Harty Tyre 501: Error parsing line of SHAPE table',
                . TIR_ID, 'STOP' )
        endif

    endif

    goto 800

endif

typarr( n_shape ) = n_nodes

else

call usrmes( .true.,
    . 'Harty Tyre 501: No shape table. Cylinder will be used'
    . ,tir_id, 'WARN')

endif

C Close tire property file:

    CALL RTO_CLOSE_FILE_F2C ( CHTDST, NCHTDS, RETURN_VAL )

    ERRFLG = RETURN_VAL .EQ. 0
    MESSAG = 'exa_fiaini: Error closing tire property file.'
    CALL ERRMES( ERRFLG, MESSAG, TIR_ID, 'STOP' )

    RETURN
    END

```

### Sample .TIR file

```

$-----
-----MDI_HEADER
[MDI_HEADER]
FILE_TYPE      = 'tir'
FILE_VERSION   = 2.0
FILE_FORMAT    = 'ASCII'
(COMMENTS)
{comment_string}
'Tyre          - Dunlop 100/90 19 D401'
'Pressure      - Unknown'
'Test Date     - Estimated DAH 2004'

```



```

'Harty Tire Model 2013'
'New File Format v2.1'
$-----
-----units
[UNITS]
LENGTH = 'mm'
FORCE  = 'newton'
ANGLE  = 'radians'
MASS   = 'kg'
TIME   = 'sec'
$-----
-----model
[MODEL]
$          use mode   1   2
$          -----
$          smoothing      X
$
PROPERTY_FILE_FORMAT = 'USER'
FUNCTION_NAME         = 'HTire501_2013::TYR501'
USE_MODE              = 2.0
$-----
-----dimension
[DIMENSION]
UNLOADED_RADIUS = 341
WIDTH = 100.0
$-----
-----parameter
[PARAMETER]
VERTICAL_STIFFNESS = 146.0
VERTICAL_DAMPING   = 0.2
ROLLING_RESISTANCE = 0.02
CMX = 1.70
CGAMMA = 1.00
UMIN = 1.40
UMAX = 1.30
RELAXATION_LENGTH = 100.0
ALPHA_CRITICAL = 10.0
CURVATURE_FACTOR_ANGLE = 2.70
SCALE_FACTOR_LATERAL = 1.6417
RATED_LOAD = 1662
SCALE_FACTOR_DIM = -1.0E-4
SLIP_RATIO_CRITICAL = 20.0
CURVATURE_FACTOR_RATIO = 5.5
PNEUM_TRAILING_SCALING = 1.25
PNEUMATIC_LEAD_CAMBER = 20.0
LIMIT_CAMBER_ONSET_FRIC = 0.80
$

```

**Sample build file**

```
@echo off
rem for Intel Fortran 2013 and ADAMS 2013.1

dir /b *.f > build.lst

call \msc.software\adams_x64\2013_1\common\mdi cr-us n @build.lst
HTire501_2013_for_Adams2013_1.dll

del build.lst
copy HTire501_2013_for_Adams2013_1.dll
"C:\MSC.Software\Adams_x64\2013_1\win64\HTire501_2013.dll"
```

# Glossary of Terms

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## Agility

Ground vehicles are free roaming devices (with the exception of railed vehicles) in which the driver/rider seeks to influence path curvature and speed in order to follow an arbitrary course. The ease, speed and accuracy with which the operator can alter path curvature is agility. For motorcycles, the primary dictator of path curvature is roll angle, and therefore motorcycle agility discussions focus on roll behaviour; the handlebar is a roll acceleration demand. For finite track vehicles, path curvature arises from yaw rate; the handwheel is a yaw rate demand. Measures such as maximum speed through a given slalom crudely discern agility but not in a way that provides information on how to improve the design if more agility is desired.

---

## Anti-aliasing

Anti-aliasing is analogue filtering applied to a signal before it is digitally sampled. Digital data should not be collected without an appropriate anti-aliasing filter. Aliasing is when something happens between subsequent samples that is simply ‘missed’ by the sampling process.

Consider [Figure C.1](#); a 5 Hz process has been sampled at 4.762 Hz. Nyquist’s theory predicts that spectral content is wrapped around according to multiples of half the sampling frequency. For the above example, the wrapping occurs at  $5 - (2 \times 4.762 / 2) = 0.238$  Hz. Once data have been allowed to alias, they cannot be ‘unaliaised’.

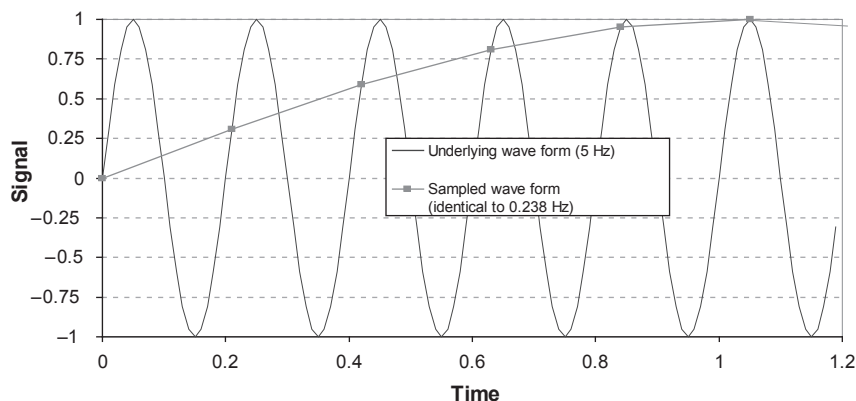
---

## Anti-lift

Anti-lift is a geometric property of the suspension which means the reaction of traction-induced pitch moment is reacted entirely by tension in the mechanical suspension members (100% anti-lift), entirely by the roadspring(s) (0% anti-lift) or some combination of the two.

Anti-lift of less than 0% is described as ‘pro-lift’. The concept is applicable only to a driven front wheel/axle and is thus generally inapplicable to motorcycles. Anti-lift is only applicable to a driven front wheel/axle. For a rear wheel/axle, the appropriate measure is anti-squat (q.v.)

For the behaviour of rear suspension geometry under braking this behaviour is sometimes referred to as ‘anti-pitch’.

**FIGURE C.1**

Aliasing, which can only be prevented with an anti-aliasing filter.

## Anti-pitch

Anti-pitch is a geometric property of the suspension which means the reaction of brake-induced pitch moment is reacted entirely by compression in the mechanical suspension members (100% anti-pitch), entirely by the roadspring(s) (0% anti-pitch) or some combination of the two. Anti-pitch of less than 0% is described as 'pro-pitch'.

Road motorcycles and comfort-oriented passenger cars have a typical inclination of the front suspension elements that provides pro-pitch behaviour. Rear suspension elements are often inclined to provide some anti-pitch under braking, but the effectiveness of this reduces with rear brake apportioning — little brake effort is apportioned to the rear of motorcycles braking hard and so it is generally ineffective at preventing pitch.

The term 'anti-dive' is not preferred since it has been used as a proprietary description for systems that modify front suspension damper calibration in order to resist brake-induced pitch in motorcycles.

For finite track vehicles, excessive anti-pitch geometry degrades refinement by forcing the wheel into any obstacles it encounters instead of allowing it to retreat from them. Brake-induced pitch behaviour is substantially modified by the use of inboard brakes.

## Anti-roll

Anti-roll is a geometric property of the suspension that means the reaction of roll moment is reacted entirely by compression in the mechanical suspension members

(100% anti-roll), entirely by the suspension calibration (0% anti-roll) or some combination of the two. Less than 0% anti-roll is described as ‘pro-roll’. This concept is meaningless for motorcycles.

One effect of anti-roll geometry is to speed the transfer of load into the tyre for a given roll moment. By modifying this load transfer differentially from front to rear strong changes in character can be wrought, though they rarely result in fundamental changes in vehicle behaviour. (This is not to say they are of no import; race performance hinges on ‘character’ since it leads towards or away from a confident driver.)

Different anti-roll geometry from front to rear also acts to provide a yaw/roll coupling mechanism. For typical saloons that coupling is such that roll out of a turn produces a yaw moment out of the turn. This is given by more anti-roll at the rear than at the front axle. Vehicles with less anti-roll at the rear have a distinctive impression of sitting ‘down and out’ at the rear when driven aggressively; it rarely results in confidence.

---

## Anti-squat

Anti-squat is a geometric property of the suspension where the traction-induced pitch moment is reacted entirely by compression in the mechanical suspension members (100% anti-squat), entirely by the roadspring(s) (0% anti-squat) or some combination of the two. Less than 0% anti-squat is described as ‘pro-squat’.

The calculation of anti-squat on a chain-driven motorcycle is nontrivial and must include chain tension reaction components in the free-body diagram for the swing-arm. Anti-squat is only applicable to a driven rear wheel/axle. For a front wheel/axle, the appropriate measure is anti-lift (q.v.).

Anti-squat is not a preferred term for the behaviour of front suspension geometry under braking; the preferred term for this behaviour is ‘anti-pitch’. The term ‘anti-dive’ is not preferred since it has been used as a proprietary description for systems that modify front suspension damper calibration in order to resist brake-induced pitch.

---

## Articulated

An articulated vehicle is one in which two significant bodies are present and must pivot with respect to one another in order for the vehicle to follow a curved path. Motorcycles and tractor–trailer combinations are articulated.

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## Beta dot

See body slip rate.

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## Body slip angle

Body slip angle is the angle measured in the ground plane between the heading angle and the path of the vehicle. Heading may be thought of as what would be read by an onboard compass calibrated to the frame of reference in use. Path is the vector extending in the direction of travel at the current instant. Note that for all vehicles using pneumatic tyres, some body slip angle is inevitable to allow the rear tyres to generate a slip angle (q.v.) when travelling in a curved path.

---

## Body slip rate

Body slip rate is the first time derivative of body slip angle (q.v.). Body slip rate is an angular velocity associated solely with creating or removing body slip angle. The total vehicle yaw rate (q.v.) is made up of the no-slip yaw rate (q.v.) — that part necessary for travelling in a curved path — plus the body slip rate. Body slip rate is zero in the steady state; the relationship between body slip rate and handwheel rate is crucial in modifying driver confidence. Unfortunately, for vehicles other than road cars on high friction surfaces it is difficult to discern experimentally since it relies on ‘the small difference of two large numbers’ and is thus prone to sensor noise.

---

## Bump

Bump is a term used specifically to describe a motion of the suspension arrangement in which the wheel travels closer to the body. Its opposite is rebound (q.v.). Bump is also known as jounce.

---

## Camber

Camber is the angle between a vertical line passing through the wheel centre and the lateral projection of that line onto the wheel plane (the projection being in the lateral direction). Strictly, it is absolute relative to the ground. A totally upright wheel has zero camber. A wheel lying flat on the ground has 90° camber.

Confusion can occur with finite track vehicles as to whether camber is measured with reference to the body coordinate system or a coordinate system on the ground. To ease this, the angle relative to ground can be referred to as the ‘inclination angle’ and that relative to the body as camber angle. A clear and unambiguous statement of the reference frame as often as is necessary is recommended to preclude ambiguity.

---

## Castor

Castor is often referred to as rake (q.v.) for motorcycles. For finite track vehicles, it is the angle that the steering axis makes with the vertical when projected onto the plane of symmetry of the vehicle. For motorcycles, care needs to be taken that the angle of the fork legs, if fitted, may not be the same as the angle of the steering axis. In these conditions, 'rake' may be used for the fork legs and 'castor' for the steering axis.

Among the car community, there is frequent surprise at the notion that the steering axis need not pass through the wheel centre but may be offset from it when viewed the side; thus it is possible to have trail (q.v.) but no castor, or vice versa.

Some sugars are formulated to be thrown ('cast') from a device specially made for the purpose (a 'caster'). They are not connected to steering systems in any way.

---

## Centre of percussion

See inertial conjugate.

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## Centripetal force

Centripetal forces are the unbalanced forces that result in acceleration of a body towards the centre of its path curvature. Without them there is no curvature of path. A body travelling in a curved path is not in equilibrium.

---

## Cepstrum

Cepstrum is a contrived word. It is an anagram of spectrum and is used to describe the Fourier transform of something already expressed in the frequency domain. The technique is used widely in sonar to identify patterns of frequency content such as that generated by rotating machinery. It is slowly creeping into industrial usage and may yet become a tool in the ground vehicle industry.

---

## Coherence

Coherence is an unambiguously defined calculated quantity that describes the consistency of phase relationship between two spectral estimates. Whenever a cross-spectrum or transfer function is calculated, coherence should be calculated and used to assess the credibility of the results. Cross-spectra with a coherence of less than 0.9 should be regarded cautiously and those below about 0.8 should be rejected. Note that these judgements are made on a spectral-line-by-spectral-line basis and it is acceptable for data at 5Hz to be believable while that at 7 Hz is discarded.

---

## Complex numbers

Complex numbers are numbers containing real and imaginary components. The imaginary component of a complex number is a real number multiplied by the square root of  $-1$ . ‘Complex’ thus has a specific meaning and should be avoided to represent ‘complicated’ or ‘elaborate’ in general discussion.

Eigenvalues (q.v.) calculated using complex numbers are ‘complex eigenvalues’; those calculated without are simply ‘real eigenvalues’, as calculated in many finite element (q.v.) packages. Undamped eigenvalues for oscillatory systems as calculated in MBS systems are often purely imaginary.

---

## Computational fluid dynamics

Computational fluid dynamics is a tool for addressing fluid dynamic problems. Closed form (classical) fluid dynamics solutions quickly become extremely cumbersome for any but the simplest problems and boundary layer complexity render closed form fluid dynamics forms inapplicable for any object of interest except perhaps for artillery shells and the like.

In a manner similar to finite element analysis (q.v.), computational fluid dynamics uses a large number of finite volumes, known as cells, for which closed form solutions are known. The equations are coupled by imposing conditions of compatibility between adjacent cells and solving the resulting problem numerically. The solution is a time domain integration across the whole problem region, which is extremely time-consuming for even quite simple bodies.

---

## Contact patch (tyre)

The contact patch for a pneumatic tyre is that part of the tyre in contact with the ground.

---

## Couple

A couple is one or more forces acting at a distance from some point of interest such that a moment is exerted at that point.

In rigorous usage it is distinct from a torque, which is sometimes referred to as a ‘pure moment’ to differentiate it from a couple. Pure moments are unusual; couples are much more common.

This distinction is of little importance in day-to-day use, and unless the distinction between couples and torques is germane to the discussion it is best left unmade; it usually adds more confusion than it avoids. ‘Torque’ is commonly used for both, and ‘pure torque’ as a distinction is acceptable.



---

## Damper

A damper is a device that produces a force opposing a motion applied to it. Gyroscopic torques are not dampers although they are velocity-dependent, since they do not act to oppose a motion applied but instead act at  $90^\circ$  to that motion.

The term damper is frequently applied to mean a hydraulic device to the exclusion of other devices, but this is merely habit rather than preference. Dampers are not shock absorbers — they transmit shocks, not absorb them. The spring and damper assembly together could be described as a shock absorber. The term shock absorber, or ‘shock’ for short, is common in US usage and is often used to mean only the damper; a combined spring and damper unit is sometimes referred to as a ‘coilover’ in US usage. It is wise to check exactly which components are being discussed as often as necessary.

A dampener is something that adds moisture and has nothing to do with dynamics.

---

## Dynamics

The Greek word ‘δύναμις’ (pronounced dunn-a-miss) means energy and is where we get our word dynamic from. In the sense frequently used today by mechanical engineers it is used to signify a time-varying exchange of energy between kinetic (motion), strain (stretching) and/or potential (height) states. Similar phenomena exist with electrical circuits and are crucial to the operation of radios and other elaborate equipment.

When this exchange occurs easily the system is said to be at resonance. It happens at characteristic frequencies (speeds of a repeated motion) for any structure. If an event happens at a speed substantially similar to or greater than the speed (frequency) of the slowest resonance, it will be dynamic or fast. If an event happens at less than that speed, it will be static or slow. For a full system, such as a motorcycle, the same is true. The correct understanding of the dynamic behaviour of vehicle systems at the design stage is vital to avoid costly mistakes being carried forward to the prototype stage or, worse, to production.

---

## Dynamic absorber

A dynamic absorber is an additional spring-mass-damper system added to a mechanical system to take a single resonance and modify it by forming two resonances, each of which is more damped than the single resonance in the unmodified system. One of the two resonances is at a lower frequency than the original and the other is at a higher frequency.

Mass-damper systems can also be added to dissipate energy uniformly across a wide frequency range, and these devices are also sometimes referred to as

‘dynamic absorbers’. Finally, discrete lumped masses can be firmly attached to change resonant frequencies in order to decouple an excitation from a structural resonance. Such a device is frequently and confusingly referred to as a ‘mass damper’ although it would be less ambiguous to call it an inertial attenuator.

---

## Eigensolution, eigenvalues, eigenvectors

There is a clear mathematical description of characteristic solutions (‘eigen’ is German for ‘characteristic’) of matrix problems, but it is unhelpful for this glossary.

Eigensolutions, which consist of eigenvalues and eigenvectors, are quite simply free vibration solutions — that is to say resonances, or ‘natural solutions’ for the system described by the matrix problem. If the system is heavily damped (more than around 5% damping ratio) then complex numbers are necessary for useful eigensolutions of the system. If it is lightly damped then only real numbers are necessary.

Eigensolutions are also sometimes referred to as ‘modal solutions’, ‘normal modes’ and a variety of other titles. Thus a ‘real normal modes’ solution is one that solves the eigenvalue problem for a lightly damped system, using only real numbers. Note that it is a property of modes of vibration calculated in this manner, that they are normal (orthogonal) to one another; they produce a dot product of zero if the vectors are multiplied together. The concept of orthogonality requires at least two items for it to be meaningful.

---

## Expected and unexpected response

Expected response is not the subjective vagary it might be supposed. Expected response is defined clearly for dynamics usage as the product of the system inputs and the idealised characteristics of the system. For example, in yaw for a finite track vehicle the expected response is a yaw rate, the value of which is the forward speed multiplied by the steer angle and divided by the wheelbase. This might also be called ‘idealised yaw rate’. ‘Idealised’ does not imply ‘ideal’ in the sense of most optimum.

The reason for wishing to capture the expected response is to compare it with the actual response. The difference between actual and expected is then logically termed ‘unexpected response’ and is important in quantifying operator interpretation of the vehicle. One vehicle might display a yaw response that differs from another by only a few percent; for example, a comparison of the unexpected components of response might show that one has double the unexpected response of the other and explain the strong preference of operators for one over the other even though the objective overall response is very similar.

If the terms ‘expected’ and ‘unexpected’ cause consternation then the control theory terms ‘reference’ (demanded) and ‘actual’ could be substituted.

---

## Finite element method

Simple engineering structures can be represented using a closed-form equation derived using differential calculus. The beam equation is the most widely used of these forms, being familiar to most engineers. These might be thought of as infinitesimal (infinitely small) element solutions, being derived by considering an infinitesimal slice then summing (integrating) the results.

The derivation of closed form solutions for more complex structures subject to complex loading patterns rapidly becomes cumbersome and impractical. The finite element method uses small but not infinitesimal chunks of the structure, each of which has a closed form solution, and assembles them imposing conditions of force and displacement continuity at the boundaries between elements. The resulting set of simultaneous equations is converted to matrix form and is well suited to being solved using a digital computer.

The finite element method was invented in principle during the latter stages of World War II and immediately thereafter, but did not achieve widespread use until the Apollo programme in the 1960s in America. In the last 30 years it has become more and more popular, with FE tools now available for use with home computers.

---

## Forced response

Forced response is the response of a system under some external excitation. Excitation is usually time-varying; if it is not then the problem is a static one.

If the input excitation has been established for some time then the behaviour of the system will have achieved ‘steady state’ (q.v.); the response will be stationary (q.v.) in character. Such solutions can be calculated using the ‘harmonic forced response’ method. A transfer function is calculated based on the free vibration (natural solution) response of the system then multiplied with the frequency spectrum of the input excitation to provide a response spectrum. Refinement (q.v.) problems are frequently addressed using the harmonic forced response method.

If the input excitation has not been established for some time, then a solution method is required that can capture the developing phase relationships in the system. This is a ‘transient forced response’. There is typically a computational resource penalty of an order of magnitude when switching to transient solution methods from harmonic ones.

---

## Gain

Gain is often used when discussing control systems to describe the relationship between output(s) and input(s). A system with a higher gain produces larger outputs for a given input than a system with a lower gain. Gains may or may not be linear and often vary with input frequency.

In vehicle dynamics, quantities like yaw rate and lateral acceleration are regarded as the output when compared to handwheel angle as an input. For a motorcycle, roll acceleration is the output and steer torque is the input.

Occasionally gain is used when talking about the rate of increase of a quantity, for example when discussing the development of roll angle during a transient manoeuvre. Such confusing usage is unhelpful when clearer terms like “increase” and “rate” are available.

---

## Gyroscope, gyroscopic torques

Much mystique surrounds gyroscopes and gyroscopic torques. Gyroscopes do not produce forces, only torques (moments). Gyroscopic torques are a logical consequence of Newton’s third law in its correct and full form. Sometimes expressed succinctly as ‘applied force is equal to rate of change of momentum’, if a full 6x6 formulation is pursued then differentiating the product of the inertia tensor and velocity vector yields some off-diagonal terms if the inertia tensor varies with time, as it does with a rotating body that is not spherical.

---

## Handwheel

Handwheel is the preferred term for what is popularly called the steering wheel. The reason for this distinction is that a typical car has three steering wheels — two are road wheels and the other is the handwheel. Using handwheel avoids ambiguity although it may seem a little cumbersome.

---

## Harmonic

Harmonic in dynamic terms means ‘consisting of one or more sine waves’.

When the phrase ‘assume harmonic solutions’ is used, it means ‘assume the solution consists of one or more sine waves’. There is no implication of fixed frequency relationships; the term harmonic is sometimes confused in careless usage with ‘harmonies’ as used in music.

---

## Heave

Heave is one of three motions performed by the whole vehicle on its suspension, referred to collectively as ‘primary ride’. It is a motion whereby the whole vehicle rises and falls evenly, with no rotation about any axis. The other two ride motions are pitch and roll. For motorcycles the roll motion is not a primary ride motion but the fundamental degree of freedom for the vehicle. Heave is also known as bounce, or jounce.

In reality, ride motions are never pure heave or pitch but always some combination of the two. This fact is a frequent source of confusion between development staff and analysis staff; the two groups use the terms differently. The problem is even more acute when including roll, with confusion around the notion of the ‘roll centre’ as a motion centre for the vehicle.

---

### **Inertial conjugate (centre of percussion)**

Inertial conjugate is the preferred term for the location at which no translation occurs when a free body with finite mass and inertia properties is loaded in a direction that does not pass through its centre of gravity. No translation occurs at this point during load application, whether or not the load is percussive (an impact). This phenomenon gives rise to the behaviour of ‘rigid’ bats in some sports, particularly baseball and the like, that is described as the ‘sweet spot’. This is not to be confused with the use of the same term for strung rackets.

The term ‘centre of percussion’ is not preferred to describe this location since it implies a percussive loading must be present for the concept to be useful.

---

### **Jounce**

Jounce is another term for heave. It is of US origin.

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### **Kinematics**

Kinematics is the study of motion. In a mechanism it is the study of the motion of the individual motions of the components and how they relate to and constrain each other.

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### **Modes, modal analysis**

Modes is shorthand for ‘modes of vibration’. See ‘eigensolution’ for a description. ‘Modal’ is an adjective meaning ‘of or relating to a mode or modes’. Modal analysis is thus a fairly loose term; its preferred use is as part of the expression, Experimental Modal Analysis, or better still Modal Test, in which experimental methods are used to discern the modes of vibration of a structure. Analysis to predictively calculate modal behaviour is distinguished by the expression ‘modal solution’ (solution being a general word for the submission, retrieval and interpretation of a set of results to a problem using a digital computer).

---

## Multibody system analysis, multibody codes

Mechanical systems can be viewed as a connection of separate items, or bodies, connected by various means. Such a system, comprised of multiple bodies, can be analysed by the application of Newtonian or Lagrangian methods to formulate the equations of motion, which may then be interrogated in a variety of ways — integrated through time, solved for an eigensolution and so on. This is multibody system analysis.

A group of software packages, or codes as they are informally referred to, have emerged that greatly ease the task of formulating and solving the equations of motion. The best known is called MSC ADAMS — Automated Dynamic Analysis of Mechanical Systems — developed by MSC Software.

Until recently, an implicit assumption in this type of analysis has been that the elements comprising the system are rigid, but this limitation is being removed by the elegant integration of multibody methods with structural dynamics methods.

---

## No-slip yaw rate

No-slip yaw rate is the yaw rate required to support a vehicle travelling in a curved path, allowing it to change heading in order to have the correct orientation when it leaves the corner.

If the vehicle did not yaw when travelling in, say, a 90° corner then it would be travelling sideways upon exiting the corner. If it is to be travelling forwards (in a vehicle-centred sense — i.e. as noted by the operator) when it leaves the corner then it must have rotated in plan during the corner.

Consideration of basic physics leads to the observation that no-slip yaw rate is centripetal acceleration divided by forward velocity.

---

## Non-holonomic constraints

*‘Non-holonomic constraints involve nonintegrable relationships between velocities. In vehicle dynamics they arise typically if wheels are assumed to roll without slip in problems of more than one dimension. Suppose a car is parked in an open, flat, high-friction area and radial line marks are appended to the tyre sidewalls and to the points on the ground nearest to them. The car is then driven slowly, without tyre slip, round the area and eventually returned to the precise location where it started. Although the car body can be repositioned precisely the tyre marks will not, in general, align now’.*

Sharp, R.S. Multibody Dynamics Applications in Vehicle Engineering (1998).

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## NVH

See Refinement.

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## Objective

Objective is unfortunately ambiguous. In one sense (as a noun) it is similar to ‘target’ but that is not its preferred usage. The preferred usage is in contrast with ‘subjective’ (q.v.) and is as an adjective. It refers to measurements or conclusions that are independent of the person who observes. The term arises from basic English sentence construction which is ‘subject-verb-object’. If A observes B then B is the object of A’s observation; B is always B whether A or C observes.

Objectivity is essential to dynamic activities; without it work cannot be credible, reproducible or professional. These three are prerequisite for any scientific activity.

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## Operating Shape

When harmonically (q.v.) excited, a system will respond with a characteristic motion that varies over time and for lightly damped systems may often be represented as a scaled sine wave for each point in the system with a fixed phase relationship.

Examining the system at the maxima of these individual sine waves gives a characteristic shape at which the system operates. When operating at resonance, the operating shape strongly resembles the mode shape or eigenvector (q.v.) associated with that resonance.

For heavily damped systems the scaling is usually complex (q.v.), meaning that operating shapes cannot readily be assimilated without animation.

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## Oversteer, understeer

Oversteer is strictly the condition in which the slip angle of the rear tyres exceeds that of the fronts. Understeer is strictly the reverse.

Before debating this matter, a review of Segel and Milliken’s papers from 1956 are in order, where all this was laid down as fact with test work and mathematical development. The definitions arrived at in there are applicable in the linear region only. The persistence of their usage for limit behaviour is an extension beyond their validity. Sometimes, drivers may refer to oversteer and mean a yaw overshoot after turn-in, which is somewhat confusingly a consequence of an understeering car. Equally, a car that has genuine oversteer behaviour will have non-oscillatory roots to its characteristic equation, resulting in excessive yaw damping that manifests itself as a reluctance to turn in and is reported as transient understeer.

Some modern cars have steady state oversteer behaviour on low grip surfaces since it dulls response and makes the vehicle manageable, even though it guarantees the need for driver correction.

Preferred alternative forms of reference are such subjective terms as 'push on' or 'loose rear' (that is loose, not lose) or more technical descriptions, such as 'real roots for the characteristic equation', 'rear slip angle exceeds front' and so on.

The importance of both oversteer and understeer as steady state phenomena in road vehicles in the linear region is greatly exaggerated. Its prime effect in the steady state is to encourage the driver to adjust the handwheel slightly, since it results in a path error due to the difference between demanded [idealised, expected (q.v.)] and no-slip yaw rates (q.v.). In normal road driving this is completely trivial; fitting a 'faster' steering rack will mask the perception of path error related to understeer completely.

For transient handling, the development of body slip angle (q.v.) is of much more importance; the yaw rate associated with changes in body slip angle is acutely sensed by the driver and is used to give warning of an impending spin even though it is only a small fraction of the total yaw rate. If the vehicle manages its body slip rate poorly, then it can give the impression of an impending spin even if no such event was likely. Even at low lateral accelerations, body slip rate is a strong modifier on perceived transient performance.

---

## Path error

Path error is a preferred term for steady state understeer. When the vehicle 'runs wide' from the driver's intended path, the normal driver response is to add more steer angle. If the lateral acceleration is high and the path error is large, then more steer angle may not help reduce path error. This situation is reported by many drivers as 'pushing on' and is common limit handling behaviour for safe road vehicles.

Path error is strictly the difference between idealised [demanded, expected (q.v.)] yaw rate and no-slip yaw rate (q.v.). It is a numerical quantity suitable for discernment from logged data or mathematical models.

---

## Pitch

Pitch is one of three motions performed by the whole vehicle on its suspension, referred to collectively as 'primary ride'. It is a motion whereby the front of the vehicle rises and falls in opposition to the rear of vehicle, rotating about the lateral axis of the vehicle. The other two ride motions are heave and roll. For powered two wheelers the roll motion is not a primary ride motion but the fundamental degree of freedom for the vehicle.

Sometimes compound heave (q.v.) and pitch motions are referred to as 'front end heave' or 'rear end heave' in a verbal effort to describe the operating shape (q.v.).



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## Predictive methods

Predictive methods is an umbrella term for all forms of mathematical modelling, from statistical to explicit, from numerical to algebraic.

‘The only relevant test of the validity of a hypothesis is comparison of prediction with experience’ (Milton Friedman)

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## PTW

PTW is an acronym for ‘powered two wheeler’ — a generic term encompassing mopeds, scooters, motorcycles, enclosed motorcycles, feet firsts (FFs) and any other thing which has two wheels and is self-powered.

---

## Rake

Rake is the angle between the steering axis and the vertical for an upright motorcycle. Typical values for rake vary from 23° for a sports machine and over 30° for a cruiser. Rake is also known as castor (q.v.), and is occasionally quoted as an angle from the ground plane. Some designs of motorcycles have a different angle for the steering axis and the telescopic fork legs; in this case rake is sometimes used for the fork leg angle and castor reserved for the steering stem angle.

---

## Rate

Rate means, clearly and unambiguously, the first time derivative. It is shorthand for ‘rate of change with time’. Using rate for derivatives other than time is not preferred. In particular, using rate as in ‘spring rate’ is undesirable; ‘stiffness’ is preferred in this instance, though the authors make this error frequently.

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## Rebound

Rebound is a term used specifically to describe a motion of the suspension arrangement in which the wheel travels away from the body. Its opposite is bump (q.v.).

---

## Refinement

Refinement is a general term that refers to the ability of a vehicle to isolate its operator and other occupants from external disturbances and from disturbances generated by the vehicle itself — for example engine vibration.

Refinement is frequently referred to as 'NVH'. This stands for 'Noise, Vibration and Harshness'. The distinction between the three phenomena is not well defined. It is suggested that noise refers to audible phenomena from 30 Hz upward, vibration to dynamic phenomena below about 10 Hz and harshness to tactile sensations of vibration of intermediate frequency, for example, steering column shake at idle.

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## Segment

'Segment' is industry shorthand for 'market segment' — the term used to define the objectives for the vehicle in terms of who will buy how many copies of it.

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## Shock absorber

See Damper.

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## Slip, slip angle (of tyres)

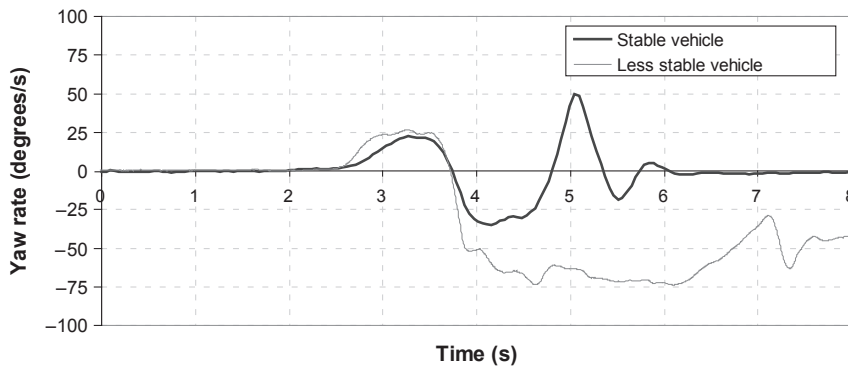
Slip is perhaps the poorest word possible to describe the behaviour of a tyre under lateral loading. A rolling tyre will not support a lateral load due to the repeated relaxation of the sidewalls just behind the contact patch. This action causes the tyre to 'walk' sideways under an applied lateral load. Combined with the forward motion, this results in an angle between the wheel plane and the direction of motion. That angle is dubbed the 'slip angle'. The presence of a slip angle does not imply sliding friction at the contact patch, despite the linguistic link between slip and slide. This is a frequent source of confusion for newcomers to the field.

To further add confusion, the behaviour of a tyre under a tractive or braking torque is also referred to as slip, and is often quantified in percentage. A free-rolling wheel is at zero slip, a completely locked wheel is at 100% brake slip. 100% tractive slip is when the wheel is spinning twice as fast as the free-rolling wheel. The presence of tractive/braking slip does not imply sliding friction at the contact patch, though this is the case at high slip percentages (greater than 20%), as it is at high slip angles (greater than about 6°).

---

## Stability

Stability means the response to a disturbance is bounded (non-infinite). In practical terms for a motorcycle this would mean it would not fall over if nudged. This is clearly not the case when the machine is at rest but is generally true of the motorcycle/rider combination at all speeds above zero. For running vehicles, the notion of stability is generally associated with the control of body slip angle (q.v.) and

**FIGURE C.2**

Time Domain Comparison of Stable and Less Stable Vehicle during 120 kph Entry Chicane Manoeuvre.

hence implicitly the yaw rate behaviour of the vehicle. In practice a response that is simply ‘finite’ is insufficient to guarantee stability in any useful sense; vehicles are not simply ‘stable’ or ‘unstable’ but have degrees of stability that are often related to the nature of the nonlinearity of the system. For example, the front wheel wobble (q.v.) problem on a motorcycle is unstable on-centre but has a ‘bounded instability’ in the sense that the increasingly nonlinear behaviour of the tyres dissipates more and more energy as the amplitude of the cycle rises. This has the effect of ‘containing’ the instability in the system within bounded limits.

Figure C.2 shows a comparison between a vehicle that has a passively stable response and another with a less stable response during a chicane manoeuvre. Both vehicles have comparable mass, wheelbase, inertia properties and tyre fitments. The manoeuvre is completed off-throttle and it can be seen that the less stable vehicle produces a very aggressive yaw acceleration at the steer reversal (3.8 s) and a high yaw rate persists for some time, resulting in a half spin of the vehicle.

It can be seen for a well-controlled manoeuvre like this one, performed under identical conditions with the same driver, it would be easy to formulate some measure of stability based on area under the yaw rate curve. To compare vehicles on an absolute basis requires a well-controlled test that is tester-, venue- and weather-independent — a more difficult task but addressed in part by the ISO manoeuvres.

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## Stationary

In dynamics, stationary does not mean immobile. It is a property of a time-domain signal and is such that lengthening the observation period does not alter the fundamental characteristics (mean, RMS, Standard Deviation, etc.) of the signal.

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## Steady state

In general, steady state refers to the condition where phase relationships are fixed and no longer vary with time.

In the sense used in vehicle dynamics work, it is that definition applied to a vehicle cornering. The phase relationship between all motions of the vehicle is fixed; there is neither yaw nor roll acceleration that is anything other than established and harmonic. In the simplest sense the vehicle is travelling steadily on a curved path. Note that steady state does not imply equilibrium.

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## Steering offset

Steering offset is a fairly general term that has specific meanings when considering the steering system of a finite track vehicle. It is the lateral distance between the steer axis and the wheel plane. It can be seen that if the steer axis is inclined when viewed from the front of the vehicle (as is generally the case) then this offset has different values at different heights above ground.

Wherever it is measured, positive steering offset is when the wheel plane is outboard of the steer axis. Negative steering offset is the reverse.

Steering offset is of interest at ground level since it determines steering system loads (and hence handwheel torque) under differential longitudinal loads. Excessive ground level offset promotes nervousness under braking and sensitivity to ABS operation. However, it also assists the driver in retaining control under split mu braking conditions, with or without ABS. Negative ground level offset makes the handwheel and vehicle insensitive to ABS operation.

The steering offset at wheel centre height, usually called ‘hub level offset’ dominates the sensitivity of the vehicle to tractive effort imbalance — usually called ‘torque steer’. It also strongly colours the amount of vibration passed through to the driver from the road surface under normal driving — so-called ‘texture feel’. With a driven axle, steering offset is normally the outcome of drive shaft plunge calculations and not an independent design variable in its own right.

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## Subjective

Subjective is in contrast with ‘objective’ (q.v.) and is an adjective. It refers to measurements or conclusions that are dependent on the person who observes.

It would be unfair to suggest that subjectivity has no place, but excessive reliance on unreproducible observations is unhelpful, unproductive and unprofessional.

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## Symbolic codes (multibody system analysis)

*‘Conceptually, the simplest approach to solving...(a multi-body system problem) is to give the whole set of differential and algebraic equations to a powerful and appropriate solver... Simulations tend to be painfully slow...’ ‘Symbolic preparation of the problem for numerical solution implies that the symbol manipulations, which are performed only once ahead of numerical processing, can be devoted to making the (later repeated) numerical operations minimal. It also implies independence of the symbolic equation building and the numerical solution’.*

Sharp, R.S. *Multibody Dynamics Applications in Vehicle Engineering* (1998). Sumpack is one of several symbolic codes on the market at the time of writing.

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## Traction, tractive

Traction for ground vehicles is the process of deploying power via a torque at the road wheels in order to provide a forward force on the vehicle. Such force is sometimes referred to as ‘tractive effort’. Multiplying that force by the vehicle velocity gives ‘tractive power’.

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## Trail

Trail is the distance between the point at which the steering axis intercepts the ground plane and the front wheel contact patch centre. Positive trail implies the steering axis intercepts the ground ahead of the front wheel contact patch. Trail can be measured at two different points; the geometric centre of the contact patch and the centre of lateral load of the contact patch. The difference between the two measurements is often referred to as ‘pneumatic trail’. The measurement to the geometric centre of the contact patch is then ‘mechanical’.

With finite track vehicles, trail is an important contributor to the self-aligning behaviour of the steering system and handwheel efforts.

With motorcycles, negative trail machines existed into the 1920s but the destabilising effects with speed rendered them impractical for high-speed use. Negative trail occasionally appears on mountain bicycles today and does not guarantee instability of the machine/rider system. At low speeds, high roll angles and significant steer angles, trail is substantially modified by the wheel geometry.

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## Transient (cornering)

In general, transient refers to the condition where phase relationships are not fixed and are varying with time. In the sense used in this work, it is that definition applied

to a vehicle cornering. Transiently, the phase relationship between all motions of the vehicle is developing and changing. Rates of change are generally decaying with time and the transient motion becomes the steady state motion if sufficient time is allowed. For the transient aspects of the motion to decay completely, something around 3 s is required for cars and motorcycles. This is rarely achieved in practice and so reality consists of a connected series of transient events.

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## Understeer

See Oversteer.

See Expected Response.

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## Vehicle dynamics

For the purpose of this work, a vehicle is defined as a wheeled device capable of propelling itself over the ground. Dynamics, from the Greek ‘dunamis (δύναμις)’ meaning power or the capability to achieve something, is the study of time-varying phenomena. Most dynamic phenomena involve the shuffling or transforming of energy between one item in a system and another. The phrase ‘vehicle dynamics’ is generally used to refer to phenomena of interest to the rider or driver of the vehicle in guiding its path — roll, turn-in, and so on. It includes consideration of elements such as tyres and suspension dampers, and is very much a system level activity.

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## Vehicle programme

‘Vehicle programme’ is industry shorthand for ‘vehicle design, development and sign-off programme’. It is that defined activity that takes as input a desire to have a vehicle in a certain market segment and produces as output a fully mature design, ready to be mass produced. The start of vehicle programmes is shrouded in obfuscation in order that each manufacturer can claim their vehicle programmes are shorter than others; the end of the vehicle programme is frequently referred to as ‘job 1’ or ‘launch’.

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## Weave

Weave is one of the lowest fundamental modes of vibration of a motorcycle. It is a combination of roll, yaw and lateral motion that results in the machine following a sinuous path. Modern road machines rarely display weave behaviour poorly damped enough to hinder the rider’s control of the machine but even current machines can

display a low enough level of weave damping to diminish the rider's confidence, particularly when laden. The highest performance machines typically employ an aluminium beam frame concept and have excellent weave damping, to the point where weave mode cannot readily be discerned with a typical 'rider rock' manoeuvre (a hands-off hip flick) but must instead be found using a swept sine handlebar input.

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## Wheelbase

Wheelbase is the distance between the front and rear wheel centres of a wheeled vehicle.

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## Wheel hop

A description of the system resonance that consists of the unsprung mass (wheel, tyre, brake components, hub, bearings and some proportion of suspension members, springs and dampers) as the dominant kinetic energy storage and the tyre as the dominant strain energy storage, with the tyre providing the dominant energy dissipation route. Typical frequencies are usually around 12 Hz on a finite track vehicle, 15 Hz on a road motorcycle.

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## Wheel trajectory map

A collective name for the characteristics that describe the orientation and location of the wheel plane in relation to the vehicle body. The characteristics are all defined with respect to bump (q.v.) motion. In a finite track vehicle, the wheel trajectory map includes bump steer, bump camber, bump recession and bump track change. The interaction of each of these characteristics with the vehicle is subtle and complicated. For motorcycles it is generally only bump recession, also known as anti-pitch or anti-squat, that is of interest.

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## Wobble

Wobble is a faintly comical term used to describe a distinctly serious trait of motorcycle behaviour. Originally referred to as 'speedmans's wobble' this was shortened to just wobble by frequent usage. It describes the mode of vibration of a motorcycle in which the steered mass is in motion more-or-less independently of the non-steered mass and rider. It is typically around 8 Hz and generally well damped enough that it does not present a problem. However, when ill-damped it can present a terrifying experience whereby the amplitude of the motion set-off by some small disturbance

grows until it is only restrained by the steered mass colliding with some part of the motorcycle at each extreme. It is also referred to as flutter, shimmy or headshake.

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### **Yaw, yaw rate**

Yaw is simply rotation when viewed in plan view. Yaw rate is the first time derivative and is often referred to as yaw velocity. Yaw is measured in degrees or radians and yaw rate in degrees/second or radians/second.



# Standards for Proving Ground Tests

Vehicle handling simulations are intended to recreate the manoeuvres and tests that vehicle engineers carry out using prototype vehicles on the test track or proving ground. Some are defined by the International Standards Organization (ISO), which outlines recommended tests in order to substantiate the handling performance of a new vehicle:

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ISO 3888-1:1999	Passenger cars – Test track for a severe lane-change manoeuvre – Part 1: Double lane-change
ISO 3888-2:2002	Passenger cars – Test track for a severe lane-change manoeuvre – Part 2: Obstacle avoidance
ISO 4138:1996	Passenger cars – Steady-state circular driving behaviour – Open-loop test procedure
ISO 7401:2003	Road vehicles – Lateral transient response test methods – Open-loop test methods
ISO 7975:1996	Passenger cars – Braking in a turn – Open-loop test procedure
ISO/TR 8725:1988	Road vehicles – Transient open-loop response test method with one period of sinusoidal input
ISO/TR 8726:1988	Road vehicles – Transient open-loop response test method with pseudo-random steering input
ISO 9815:2003	Road vehicles – Passenger-car and trailer combinations – Lateral stability test
ISO 9816:1993	Passenger cars – Power-off reactions of a vehicle in a turn – Open-loop test method
ISO 12021-1:1996	Road vehicles – Sensitivity to lateral wind – Part 1: Open-loop test method using wind generator input
ISO 13674-1:2003	Road vehicles – Test method for the quantification of on-centre handling – Part 1: Weave test
ISO 14512:1999	Passenger cars – Straight-ahead braking on surfaces with split coefficient of friction – Open-loop test procedure
ISO 15037-1:1998	Road vehicles – Vehicle dynamics test methods – Part 1: General conditions for passenger cars
ISO 15037-2:2002	Road vehicles – Vehicle dynamics test methods – Part 2: General conditions for heavy vehicles and buses
ISO 17288-1:2002	Passenger cars – Free-steer behaviour – Part 1: Steering-release open-loop test method
ISO/TS 20119:2002	Road vehicles – Test method for the quantification of on-centre handling – Determination of dispersion metrics for straight-line driving

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